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ON EFFECTS OF PRE-STRESS ON CRACK-TIP FIELDS IN ELASTIC, INCOMPRESSIBLE SOLIDS

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SOMMARIO
Nella presente memoria viene presentata una soluzione analitica incrementale dei campi di tensione e velocità in prossimità dell'apice di una frattura per un ampia classe di materiali elastici incomprimibili. In particolare, si considerano gli effetti di una sollecitazione di Modo I o Modo II sovrapposta ad uno stato di tensione preesistente nella direzione della frattura. Per descrivere il comportamento meccanico del materiale viene impiegata la teoria elastica incrementale sviluppata da Biot. La soluzione ottenuta mostra che i campi incrementali di tensione nominale mostrano la stesso tipo di singolarità della teoria infinitesima.

ABSTRACT
A closed-form asymptotic solution is provided for the nominal stress rates, hydrostatic pressure and velocity fields near the tip of a stationary crack in a homogeneously pre-stressed configuration for a broad class of incompressible elastic materials undergoing small superimposed deformations, under Mode I or Mode II loading conditions. The Biot theory of incremental elasticity is assumed to govern the mechanical behavior of the medium. The obtained asymptotic solution reveals that the nominal stress rate fields near the crack-tip displays the square root singularity as for the infinitesimal deformation theory.

1. INTRODUCTION

The problem of a semi-infinite plane crack in a homogeneously pre-stressed infinite elastic medium is considered in this work, within the framework of finite deformations. A Lagrangean formulation is adopted, taking the current state as the reference configuration, which is assumed in a homogeneous state defined by a constant stress component collinear with the crack. The effects of small superimposed deformations on the crack-tip rate fields are then investigated under Mode I and Mode II symmetry conditions. As well known [1], for linear elastic fracture mechanics the presence of a collinear stress only influences the higher order terms of the asymptotic crack-tip fields. However, we show that in the context of finite elastic deformation the lowest order terms result to be strongly influenced by the collinear pre-stress.

The constitutive equations used in this study refer to the formalism adopted by HILL & HUTCHINSON [2] and YOUNG [3] to describe the BIOT [4] model for incompressible and orthotropic incremental elasticity. Next, by following the approach used by PIVA & RADI [5] and LORET & RADI [6] for the analysis of crack propagation in linear elastic and poroelastic materials, a complex variable approach is adopted to solve the field equations in term of a stream function. Two different cases are analyzed, corresponding to elliptic imaginary and elliptic complex regimes, as defined by the values of the material parameters and collinear pre-stress. An asymptotic analysis is then performed to describe the lower order term of the stream function close to the crack-tip. The lower order terms of the stress and velocity fields are thus obtained in closed form for the case of elliptic imaginary and elliptic complex regimes, both for Mode I and Mode II loading conditions.

The results obtained show that the leading order nominal stress rates display a square-root singularity near to the crack-tip and can be defined to within an amplitude factor (stress intensity factor rate), in agreement with the theory of linear elastic fracture mechanics. However, it must be noted that as the stress in the current configuration approaches the critical value of the instability of the crack surfaces (which may occur in the elliptic regime) the asymptotic crack-tip rate fields become unbounded.

2. GOVERNING EQUATIONS

Let us consider an incompressible elastic material under finite plane strain deformation, thus obeying the condition $v_{1,1} = - v_{2,2}$, where $\mathbf{v}$ is the velocity vector. Then, the most general constitutive equation for an hyperelastic orthotropic material can be conveniently expressed by adopting a Lagrangean formulation with the current state taken as reference, so that the components of the nominal stress rate tensor $\dot{\mathbf{t}}$ may be written in the form [2]:

$$\dot{t}_{11} = 2\mu\,(\xi - k)\, v_{1,1} + \dot{p}\,, \qquad \dot{t}_{22} = 2\mu\,\xi\, v_{2,2} + \dot{p}\,,$$
$$\dot{t}_{12} = \mu\,[(1 + k)\, v_{2,1} + (1 - k)\, v_{1,2}], \qquad \dot{t}_{21} = \mu\,(1 - k)\,(v_{2,1} + v_{1,2}). \tag{1}$$

where $\xi = \mu_*/\mu$, being μ and μ_* two incremental moduli (corresponding to shearing parallel to the principal stress axes and inclined at 45°), $k = \sigma_1/2\mu$ (being σ_1 the Cauchy stress component collinear with the crack) and $\dot{p}$ is the hydrostatic stress rate. Since a homogeneous current state is considered, by using (1) the equilibrium equation in rate form, namely $\dot{t}_{ij,i} = 0$, imply:

$$\dot{p}_{,1} = \mu\,[(1 + k - 2\,\xi)\, v_{1,11} - (1 - k)\, v_{1,22}], \tag{2}$$
$$\dot{p}_{,2} = \mu\,[(1 - k - 2\,\xi)\, v_{2,22} - (1 + k)\, v_{2,11}].$$

The above formulation describes a broad class of constitutive relations, including the relevant cases of Mooney-Rivlin, Ogden and J_2-deformation [7] materials. By introducing a stream function $\psi(x_1, x_2)$ such that the incompressibility condition is automatically satisfied, namely $v_1 = \psi_{,2}$ and $v_2 = -\psi_{,1}$, it follows that [2]:

$$(1 + k)\, \psi_{,1111} + 2\,(2\xi - 1)\, \psi_{,1122} + (1 - k)\, \psi_{,2222} = 0. \tag{3}$$

A Cartesian co-ordinate system (0, x_1, x_2, x_3) and a cylindrical co-ordinate system (0, r, ϑ, x_3) both centered at the crack-tip are considered, with the out-of-plane x_3-axis along the straight crack front. The current configuration is assumed to be in a homogeneous state defined by a constant stress component σ_1 collinear with the crack. Now, it is instrumental to assume the stream function in the form $\psi(x_1, x_2) = A\, F(x_1 + \Omega\, x_2)$, where A and Ω are complex constants and F is an analytic function of its complex argument. Then, a substitution of the assumed stream function into (3) yields the following biquadratic equation for Ω:

$$(1 - \mathrm{k})\, \Omega^4 + 2\, (2\xi - 1)\, \Omega^2 + 1 + \mathrm{k} = 0. \tag{4}$$

The roots Ω_j (j =1,2,3,4) of equation (4) will be shown to be real or complex, depending on the parameters ξ and k. Let us restrict the analysis to the case $\xi > 0$ and $\mathrm{k}^2 < 1$, so that only three cases may arise, which are classified as follows.

The *Elliptic Imaginary regime* (EI) occurs for $2\xi > 1+\sqrt{1-\mathrm{k}^2}$. In this case equation (4) admits four purely imaginary roots, namely $\Omega_1 = i\beta_1$, $\Omega_2 = i\beta_2$, $\Omega_3 = -\, i\beta_1$ and $\Omega_4 = -\, i\beta_2$, being $i = \sqrt{-1}$ and:

$$\left.\begin{matrix}\beta_1\\ \beta_2\end{matrix}\right\} = \sqrt{\frac{2\xi - 1 \pm \sqrt{4\xi^2 - 4\xi + \mathrm{k}^2}}{1-\mathrm{k}}}\,. \tag{5}$$

The *Elliptic Complex regime* (EC) occurs for $1-\sqrt{1-\mathrm{k}^2} < 2\xi < 1+\sqrt{1-\mathrm{k}^2}$. In this case equation (4) admits four complex conjugate roots, namely $\Omega_1 = -\alpha + i\beta$, $\Omega_2 = \alpha + i\beta$, $\Omega_3 = \overline{\Omega}_1$ and $\Omega_4 = \overline{\Omega}_2$, where:

$$\left.\begin{matrix}\beta\\ \alpha\end{matrix}\right\} = \sqrt{\frac{\sqrt{1-\mathrm{k}^2} \pm (2\xi - 1)}{2\,(1-\mathrm{k})}}\,. \tag{6}$$

The *Hyperbolic regime* (H) occurs for $2\xi < 1-\sqrt{1-\mathrm{k}^2}$. In this case equation (4) admits four real roots, namely $\Omega_1 = \alpha_1$, $\Omega_2 = \alpha_2$, $\Omega_3 = -\, \alpha_1$ and $\Omega_4 = -\, \alpha_2$, being

$$\left.\begin{matrix}\alpha_1\\ \alpha_2\end{matrix}\right\} = \sqrt{\frac{1 - 2\xi \pm \sqrt{4\xi^2 - 4\xi + \mathrm{k}^2}}{1-\mathrm{k}}}\,. \tag{7}$$

Finally, we remark that the condition $\mathrm{k}^2>1$ (not considered here) defines the *Parabolic regime* (P), where equation (4) admits two real roots only. All these regimes are labelled in Fig. 1, in the plane k-ξ. In the following, reference is made to the four complex variables:

$$\mathrm{z}_j = x_1 + \Omega_j\, x_2 = x_1 + \alpha_j\, x_2 + i\beta_j\, x_2, \qquad (j = 1, 2, 3, 4), \tag{8}$$

being $\alpha_j = \mathrm{Re}[\Omega_j]$ and $\beta_j = \mathrm{Im}[\Omega_j]$, where the operators Re and Im define the real and imaginary parts of a complex number. The complex variable z_j defined in (8) also admits the polar representation $\mathrm{z}_j = \mathrm{r}_j \exp(i\, \vartheta_j)$, where:

$$\mathrm{r}_j = \mathrm{r}\, \sqrt{(\cos\vartheta + \alpha_j \sin\vartheta)^2 + \beta_j^2 \sin^2\vartheta}\,, \qquad \tan\vartheta_j = \frac{\beta_j \sin\vartheta}{\cos\vartheta + \alpha_j \sin\vartheta}. \tag{9}$$

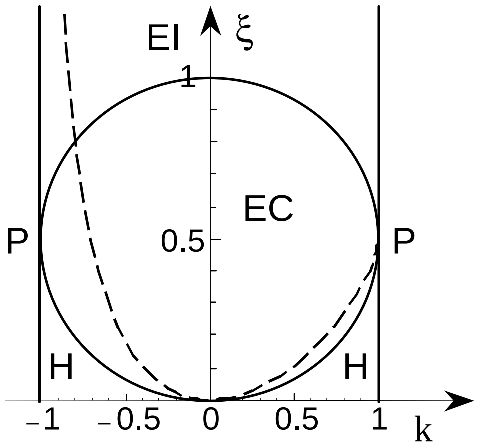

Figure 1. Elliptic imaginary, elliptic complex, hyperbolic and parabolic regimes in the k-ξ plane. Equation (18) is shown as the dashed curve in the elliptic regimes.

The general solution of equation (3) may be represented in the form:

$$\psi(x_1, x_2) = \frac{2\dot{\mathrm{K}}}{3\mu\sqrt{\pi}} \sum_{j=1}^{4} A_j\, F_j\,(\mathrm{z}_j), \tag{10}$$

where $\dot{\mathrm{K}}$ denotes the stress-intensity factor rate, namely, $\dot{\mathrm{K}} = \dot{\mathrm{K}}_{\mathrm{I}}$ for Mode I and $\dot{\mathrm{K}} = \dot{\mathrm{K}}_{\mathrm{II}}$ for Mode II, being:

$$\dot{\mathrm{K}}_{\mathrm{I}} = \lim_{\mathrm{r}\to 0} \sqrt{2\pi \mathrm{r}}\; \dot{\mathrm{t}}_{22}\,(\mathrm{r}, \vartheta{=}0), \qquad \dot{\mathrm{K}}_{\mathrm{II}} = \lim_{\mathrm{r}\to 0} \sqrt{2\pi \mathrm{r}}\; \dot{\mathrm{t}}_{21}\,(\mathrm{r}, \vartheta{=}0). \tag{11}$$

In a neighborhood of the crack-tip, only the lower order terms of an asymptotic expansion of the stress rate and velocity fields are retained. Within this context, the function F_j may be sought in a power form, $F_j\,(\mathrm{z}_j) = \mathrm{z}_j^{\,\gamma}$, being γ a real number to be obtained, together with the constants A_j, by imposing the following conditions:

i) the stream function (10) must assume real values;

ii) the traction rates $\dot{\mathrm{t}}_{22}$ and $\dot{\mathrm{t}}_{21}$ must vanish on the crack surface at $x_2 = 0$ and $x_1 < 0$;

iii) the velocity fields must obey Mode I or Mode II symmetry conditions:

$$\mathrm{v}_1(x_1, x_2) = \mathrm{v}_1(x_1, -x_2), \qquad \mathrm{v}_2(x_1, x_2) = -\mathrm{v}_2(x_1, -x_2), \tag{12}$$

$$\mathrm{v}_1(x_1, x_2) = -\mathrm{v}_1(x_1, -x_2), \qquad \mathrm{v}_2(x_1, x_2) = \mathrm{v}_2(x_1, -x_2). \tag{13}$$

3. ASYMPTOTIC CRACK-TIP FIELDS IN THE EI REGIME

In the EI regime $\alpha_j = 0$ and, thus, from (8) it follows that $\mathrm{z}_n = x_1 + i\beta_n x_2$, for $n = 1, 2$; $\mathrm{z}_3 = \bar{\mathrm{z}}_1$ and $\mathrm{z}_4 = \bar{\mathrm{z}}_2$. Noting that $\bar{\mathrm{z}}^{\gamma} = \overline{\mathrm{z}^{\gamma}}$, it follows that the condition *i*) implies $A_3 = \overline{A_1}$ and $A_4 = \overline{A_2}$, so that the stream function (10) may be equivalently written as

$$\psi(x_1, x_2) = \frac{4\dot{\mathrm{K}}}{3\mu\sqrt{\pi}} \sum_{n=1}^{2} \mathrm{Re}[A_n\, \mathrm{z}_n^{\gamma}], \tag{14}$$

where $A_n = a_n + i\, b_n$, for $n = 1,2$; being a_n and b_n real constants. The velocity components may be obtained by taking the partial derivatives of the stream function, namely:

$$v_1 = -\frac{4\gamma\dot{K}}{3\mu\sqrt{\pi}} \sum_{n=1}^{2} \beta_n \,\mathrm{Im}[A_n z_n^{\gamma-1}], \qquad v_2 = -\frac{4\gamma\dot{K}}{3\mu\sqrt{\pi}} \sum_{n=1}^{2} \mathrm{Re}[A_n z_n^{\gamma-1}]. \tag{15}$$

A substitution of (15) into the nominal stress rate components (1), by using conditions (2), after some manipulations yields the following expressions of the leading order terms of the nominal and hydrostatic stress rates:

$$\dot{t}_{11} = -\frac{4\gamma\dot{K}}{3\sqrt{\pi}}(\gamma-1)\sum_{n=1}^{2}\varepsilon_n\,\beta_n\,\mathrm{Im}[A_n z_n^{\gamma-2}], \qquad \dot{t}_{22} = \frac{4\gamma\dot{K}}{3\sqrt{\pi}}(\gamma-1)\sum_{n=1}^{2}\chi_n\,\beta_n\,\mathrm{Im}[A_n z_n^{\gamma-2}],$$

$$\dot{t}_{12} = -\frac{4\gamma\dot{K}}{3\sqrt{\pi}}(\gamma-1)\sum_{n=1}^{2}\chi_n\,\beta_n^2\,\mathrm{Re}[A_n z_n^{\gamma-2}], \qquad \dot{t}_{21} = -\frac{4\gamma\dot{K}}{3\sqrt{\pi}}(\gamma-1)\sum_{n=1}^{2}\varepsilon_n\,\mathrm{Re}[A_n z_n^{\gamma-2}], \tag{16}$$

$$\dot{p} = \frac{4\gamma\dot{K}}{3\sqrt{\pi}}(\gamma-1)\sum_{n=1}^{2}(\chi_n - 2\xi)\,\beta_n\,\mathrm{Im}[A_n z_n^{\gamma-2}],$$

where: $\varepsilon_n = (1-k)(1+\beta_n^2)$, $\qquad \chi_n = 4\xi - 1 - k - (1-k)\beta_n^2$.

Note that definition (5) implies $\varepsilon_1 = \chi_2$ and $\varepsilon_2 = \chi_1$.

3.1 Mode I symmetry conditions

Under Mode I loading conditions, in view of (12), the velocity fields (15) should be endowed with the symmetry properties $v_1(z_1, z_2) = v_1(\bar{z}_1, \bar{z}_2)$ and $v_2(z_1, z_2) = -v_2(\bar{z}_1, \bar{z}_2)$, which imply $a_1 = a_2 = 0$, so that $A_n = i\, b_n$.

When the boundary conditions *ii*) are imposed on the stress rate $(16)_{2,4}$, the following homogeneous system for the real constants b_1 and b_2 may be obtained by using relations (9):

$$\Big(\sum_{n=1}^{2} b_n\,\chi_n\,\beta_n\Big)\cos\gamma\pi = 0, \qquad \Big(\sum_{n=1}^{2} b_n\,\varepsilon_n\Big)\sin\gamma\pi = 0. \tag{17}$$

If the critical condition $\varepsilon_2^2\beta_1 = \varepsilon_1^2\beta_2$ is excluded, then the system (17) admits a non-trivial solution for the constants b_1 and b_2 if and only if the condition $\sin 2\gamma\pi = 0$ is verified. The set of values of γ satisfying the latter condition and allowing the strain energy density to be integrable in a neighborhood of the crack-tip is $\gamma = 3/2, 2, 5/2,\ldots$. The lower admissible value $\gamma = 3/2$ leads to the square root singularity of the local crack-tip fields for the nominal and hydrostatic stress rates. After some algebraic manipulations, it can be shown that the critical condition $\varepsilon_2^2\beta_1 = \varepsilon_1^2\beta_2$ occurs when

$$\xi = \frac{k}{2}\left(1-\sqrt{\frac{1-k}{1+k}}\right), \tag{18}$$

corresponding to the dashed curve in the EI regime plotted in Fig. 1. It must be noted that (18) coincides with the surface instability of a semi-infinite block under plane tension or compression, as found by HILL & HUTCHINSON [2] and YOUNG [3], respectively.

If the critical condition (18) is excluded, the condition $(17)_2$ yields a constraint between the constants b_1 and b_2, for $\gamma = 3/2$. Therefore, the lowest order contributions of the asymptotic fields can be expressed in terms of an amplitude constant, to be obtained from the far-field loading rate. This observation matches with the results of asymptotic analysis of the crack-tip fields in linear elastic materials. Then, the lowest order terms of the velocity and stress rate fields can be written in the form:

$$\mathbf{v}(\mathrm{r}, \vartheta) = \frac{\dot{\mathrm{K}}_{\mathrm{I}}}{\mu}\sqrt{\frac{\mathrm{r}}{2\pi}}\ \boldsymbol{\nu}(\vartheta), \qquad \dot{\mathbf{t}}\,(\mathrm{r}, \vartheta) = \frac{\dot{\mathrm{K}}_{\mathrm{I}}}{\sqrt{2\pi \mathrm{r}}}\ \boldsymbol{\tau}(\vartheta), \qquad \dot{p}\,(\mathrm{r}, \vartheta) = \frac{\dot{\mathrm{K}}_{\mathrm{I}}}{\sqrt{2\pi \mathrm{r}}}\ \rho(\vartheta). \tag{19}$$

where the functions $\boldsymbol{\nu}(\vartheta)$, $\boldsymbol{\tau}(\vartheta)$ and $\rho(\vartheta)$ denote the angular variation of the velocity, nominal stress rate and hydrostatic stress rate, respectively. In particular, by introducing the constants $A_n = i\ b_n$ (n =1,2) into the expressions (15) and (16) of the rate fields, the angular functions defined in (19) assume the following analytical expressions:

$$\begin{aligned}
&\nu_1(\vartheta) = -\,2\sum_{n=1}^{2} b_n\,\beta_n \sqrt{g_n(\vartheta)+\cos\vartheta}, && \nu_2(\vartheta) = 2\sum_{n=1}^{2} b_n \sqrt{g_n(\vartheta)-\cos\vartheta}, \\
&\tau_{11}(\vartheta) = -\sum_{n=1}^{2} b_n\,\frac{\varepsilon_n\,\beta_n}{g_n(\vartheta)}\sqrt{g_n(\vartheta)+\cos\vartheta}, && \tau_{22}(\vartheta) = \sum_{n=1}^{2} b_n\,\frac{\chi_n\,\beta_n}{g_n(\vartheta)}\sqrt{g_n(\vartheta)+\cos\vartheta}, \\
&\tau_{12}(\vartheta) = -\sum_{n=1}^{2} b_n\,\frac{\chi_n\,\beta_n^2}{g_n(\vartheta)}\sqrt{g_n(\vartheta)-\cos\vartheta}, && \tau_{21}(\vartheta) = -\sum_{n=1}^{2} b_n\,\frac{\varepsilon_n}{g_n(\vartheta)}\sqrt{g_n(\vartheta)-\cos\vartheta}, \\
&\rho(\vartheta) = \sum_{n=1}^{2} b_n\,(\chi_n - 2\xi)\frac{\beta_n}{g_n(\vartheta)}\sqrt{g_n(\vartheta)+\cos\vartheta}, && \text{being} \quad g_n(\vartheta) = \sqrt{\cos^2\vartheta + \beta_n^2 \sin^2\vartheta}\,.
\end{aligned} \tag{20}$$

Moreover, condition $(17)_2$ together with $(11)_1$, which implies $\tau_{22}(0)=1$, yield the following expressions for b_1 and b_2 (which diverge as the critical condition (18) is approached):

$$b_n = \frac{\varepsilon_m}{\sqrt{2}\,(\varepsilon_m^2\,\beta_n - \varepsilon_n^2\,\beta_m)}, \quad (n, m = 1, 2;\ m \neq n). \tag{21}$$

3.2 Mode II symmetry conditions

Under Mode II loading conditions, in view of (13), the velocity fields (15) must meet the symmetry properties $v_1(z_1,z_2) = -v_1(\bar{z}_1, \bar{z}_2)$ and $v_2(z_1,z_2) = v_2(\bar{z}_1, \bar{z}_2)$, which imply $b_1 = b_2 = 0$, so that $A_n = a_n$.

When the boundary conditions *ii*) are imposed on the stress rate $(16)_{2,4}$, the following homogeneous system for the real constants a_1 and a_2 may be obtained by using relations (9):

$$\Big(\sum_{n=1}^{2} a_n\,\chi_n\,\beta_n\Big)\sin\gamma\pi = 0, \qquad \Big(\sum_{n=1}^{2} a_n\,\varepsilon_n\Big)\cos\gamma\pi = 0. \tag{22}$$

If the critical condition (18) is excluded, the lower admissible value of γ (leading to a non-trivial solution of the system (22) for the constants a_1 and a_2) is 3/2. For $\gamma = 3/2$, the crack-tip fields for the nominal and hydrostatic stress rates display the usual square root singularity. Thus, by introducing the angular functions $\boldsymbol{\nu}(\vartheta)$, $\boldsymbol{\tau}(\vartheta)$ and $\rho(\vartheta)$ the velocity and stress rate fields can be written in the form:

$$\mathbf{v}(\mathrm{r}, \vartheta) = \frac{\dot{\mathrm{K}}_{\mathrm{II}}}{\mu}\sqrt{\frac{\mathrm{r}}{2\pi}}\ \boldsymbol{\nu}(\vartheta), \qquad \dot{\mathbf{t}}\,(\mathrm{r}, \vartheta) = \frac{\dot{\mathrm{K}}_{\mathrm{II}}}{\sqrt{2\pi \mathrm{r}}}\ \boldsymbol{\tau}(\vartheta), \qquad \dot{p}\,(\mathrm{r}, \vartheta) = \frac{\dot{\mathrm{K}}_{\mathrm{II}}}{\sqrt{2\pi \mathrm{r}}}\ \rho(\vartheta), \tag{23}$$

where

$$\nu_1(\vartheta) = -2\sum_{n=1}^{2} a_n\, \beta_n \sqrt{g_n(\vartheta) - \cos\vartheta}\,, \qquad \nu_2(\vartheta) = -2\sum_{n=1}^{2} a_n \sqrt{g_n(\vartheta) + \cos\vartheta}\,,$$

$$\tau_{11}(\vartheta) = \sum_{n=1}^{2} a_n \frac{\varepsilon_n\, \beta_n}{g_n(\vartheta)} \sqrt{g_n(\vartheta) - \cos\vartheta}\,, \qquad \tau_{22}(\vartheta) = -\sum_{n=1}^{2} a_n \frac{\chi_n\, \beta_n}{g_n(\vartheta)} \sqrt{g_n(\vartheta) - \cos\vartheta}\,,$$

$$\tau_{12}(\vartheta) = -\sum_{n=1}^{2} a_n \frac{\chi_n\, \beta_n^2}{g_n(\vartheta)} \sqrt{g_n(\vartheta) + \cos\vartheta}\,, \qquad \tau_{21}(\vartheta) = -\sum_{n=1}^{2} a_n \frac{\varepsilon_n}{g_n(\vartheta)} \sqrt{g_n(\vartheta) + \cos\vartheta}\,, \tag{24}$$

$$\rho(\vartheta) = -\sum_{n=1}^{2} a_n\, (\chi_n - 2\xi) \frac{\beta_n}{g_n(\vartheta)} \sqrt{g_n(\vartheta) - \cos\vartheta}\,,$$

and $g_n(\vartheta)$ has been defined in $(20)_8$. Moreover, condition $(22)_1$ together with $(11)_2$, which implies $\tau_{21}(0)=1$, yield the following expressions for a_1 and a_2 (which diverge as the critical condition (18) is approached):

$$a_n = \frac{\varepsilon_n\, \beta_m}{\sqrt{2}\, (\varepsilon_m^2\, \beta_n - \varepsilon_n^2\, \beta_m)}\,, \quad (n, m = 1,2;\ m \neq n). \tag{25}$$

4. ASYMPTOTIC CRACK-TIP FIELDS IN THE EC REGIME

In the EC regime the complex variables defined in (8) admit the form $z_n = x_1 + (-1)^n \alpha x_2 + i\beta x_2$ for $n = 1, 2$; $z_3 = \bar{z}_1$ and $z_4 = \bar{z}_2$, where α and β have been defined in (6). Moreover, the condition *i*) implies $A_3 = \overline{A_1}$ and $A_4 = \overline{A_2}$, so that the expression (14) still holds for the stream function. In this case, the velocity components may be obtained from the derivatives of (14) as:

$$v_1 = \frac{4\gamma\dot{K}}{3\mu\sqrt{\pi}} \sum_{n=1}^{2} \{(-1)^n\, \alpha\, \mathrm{Re}[A_n z_n^{\gamma-1}] - \beta\, \mathrm{Im}[A_n z_n^{\gamma-1}]\}, \quad v_2 = -\frac{4\gamma\dot{K}}{3\mu\sqrt{\pi}} \sum_{n=1}^{2} \mathrm{Re}[A_n z_n^{\gamma-1}]. \tag{26}$$

A substitution of (26) into the nominal stress rates (1), by using conditions (2), yields the following expressions for the leading order terms of the nominal and hydrostatic stress rates:

$$\dot{t}_{11} = \frac{4\gamma\dot{K}}{3\sqrt{\pi}} (\gamma-1)\chi \sum_{n=1}^{2} \{(-1)^n (\beta\delta + \alpha)\, \mathrm{Re}[A_n z_n^{\gamma-2}] + (\alpha\delta - \beta)\, \mathrm{Im}[A_n z_n^{\gamma-2}]\},$$

$$\dot{t}_{22} = \frac{4\gamma\dot{K}}{3\sqrt{\pi}} (\gamma-1)\chi \sum_{n=1}^{2} \{(-1)^n (\beta\delta - \alpha)\, \mathrm{Re}[A_n z_n^{\gamma-2}] + (\alpha\delta + \beta)\, \mathrm{Im}[A_n z_n^{\gamma-2}]\}, \tag{27}$$

$$\dot{t}_{12} = -\frac{4\gamma\dot{K}}{3\sqrt{\pi}} (\gamma-1)\chi \sum_{n=1}^{2} \{(\beta^2 - \alpha^2 + 2\alpha\beta\delta)\, \mathrm{Re}[A_n z_n^{\gamma-2}] + (-1)^n (\delta\alpha^2 - \delta\beta^2 + 2\alpha\beta)\, \mathrm{Im}[A_n z_n^{\gamma-2}]\},$$

$$\dot{t}_{21} = -\frac{4\gamma\dot{K}}{3\sqrt{\pi}} (\gamma-1)\chi \sum_{n=1}^{2} \{\mathrm{Re}[A_n z_n^{\gamma-2}] + (-1)^n\, \delta\, \mathrm{Im}[A_n z_n^{\gamma-2}]\},$$

$$\dot{p} = \frac{4\gamma\dot{K}}{3\sqrt{\pi}} (\gamma-1) \sum_{n=1}^{2} \{(-1)^n\, \alpha[2(1-k)\beta^2 + k]\, \mathrm{Re}[A_n z_n^{\gamma-2}] + \beta\, [2(1-k)\alpha^2 - k]\, \mathrm{Im}[A_n z_n^{\gamma-2}]\},$$

where:

$$\chi = 2\xi - \mathrm{k}, \qquad \delta = 2(1-\mathrm{k})\alpha\beta/\chi = \sqrt{4\xi - 4\xi^2 - \mathrm{k}^2}\,/\chi. \tag{28}$$

4.1 Mode I symmetry conditions

Under Mode I loading conditions, in view of (12), the velocity fields (26) must meet the symmetry properties $v_1(z_1, z_2) = v_1(\bar{z}_2, \bar{z}_1)$ and $v_2(z_1, z_2) = -v_2(\bar{z}_2, \bar{z}_1)$, which imply $A_2 = -\overline{A_1}$, or equivalently $A_n = -(-1)^n a + ib$ (for n = 1, 2), being a and b real constants.

When the boundary conditions *ii*) are imposed on the stress rate $(27)_{2,4}$, the following homogeneous system for the real constants a and b may be obtained by using relations (9):

$$[(\alpha - \beta\delta)\, a + (\beta + \alpha\delta)\, b] \cos\gamma\pi = 0, \qquad (\delta a + b)\sin\gamma\pi = 0. \tag{29}$$

If the critical condition $(1-\delta^2)\alpha = 2\beta\delta$, is excluded, the lower admissible value of γ (leading to a non-trivial solution of the system (29) for the constants a and b) is 3/2. This value implies the usual square root singularity of the asymptotic stress fields. Moreover, for $\gamma = 3/2$ the condition $(29)_2$ implies $b = -\delta a$. Therefore, the lowest order contributions of the asymptotic fields can be expressed in terms of a single amplitude constant a. It must be noted that the critical condition occurs for the same values of k and ξ defined by (18). The introduction of the angular functions $\mathbf{v}(\vartheta)$, $\boldsymbol{\tau}(\vartheta)$ and $\rho(\vartheta)$ allows us to represent the stress and velocity asymptotic fields in the form (19). These angular functions can be obtained from (26) and (27) in the form:

$$v_1(\vartheta) = 2a\sum_{n=1}^{2}\{(\delta\beta-\alpha)\, c_n(\vartheta) + (-1)^n (\beta+\delta\alpha) s_n(\vartheta)\}, \quad v_2(\vartheta) = 2a\sum_{n=1}^{2}\{(-1)^n c_n(\vartheta) - \delta\, s_n(\vartheta)\},$$

$$\tau_{11}(\vartheta) = -a\,\chi\,(1+\delta^2)\sum_{n=1}^{2}\{\alpha\, \hat{c}_n(\vartheta) + (-1)^n \beta\, \hat{s}_n(\vartheta)\},$$

$$\tau_{22}(\vartheta) = a\,\chi\sum_{n=1}^{2}\{[\alpha(1-\delta^2) - 2\beta\delta]\,\hat{c}_n(\vartheta) + (-1)^n [\beta(1-\delta^2) + 2\alpha\delta]\,\hat{s}_n(\vartheta)\}, \tag{30}$$

$$\tau_{12}(\vartheta) = a\,\chi\sum_{n=1}^{2}\{(-1)^n [(1-\delta^2)(\beta^2-\alpha^2) + 4\alpha\beta\delta]\,\hat{c}_n(\vartheta) + 2[\delta(\beta^2-\alpha^2) - \alpha\beta(1-\delta^2)]\,\hat{s}_n(\vartheta)\},$$

$$\tau_{21}(\vartheta) = a\,\chi\,(1+\delta^2)\sum_{n=1}^{2}(-1)^n\,\hat{c}_n(\vartheta),$$

$$\rho(\vartheta) = -a\sum_{n=1}^{2}\{[(\beta+\alpha\delta)\,\delta\chi + (\alpha-\beta\delta)\mathrm{k}]\,\hat{c}_n(\vartheta) + (-1)^n [(\beta+\alpha\delta)\,\mathrm{k} - (\alpha-\beta\delta)\,\delta\chi]\,\hat{s}_n(\vartheta)]\},$$

being:

$$\hat{c}_n(\vartheta) = \frac{c_n(\vartheta)}{g_n(\vartheta)}, \qquad \hat{s}_n(\vartheta) = \frac{s_n(\vartheta)}{g_n(\vartheta)}, \qquad g_n(\vartheta) = \sqrt{[\cos\vartheta + (-1)^n\alpha\sin\vartheta]^2 + \beta^2\sin^2\vartheta},$$

$$c_n(\vartheta) = \sqrt{g_n(\vartheta) + \cos\vartheta + (-1)^n\alpha\sin\vartheta}, \qquad s_n(\vartheta) = \sqrt{g_n(\vartheta) - \cos\vartheta - (-1)^n\alpha\sin\vartheta}\,. \tag{31}$$

Finally, the definition $(11)_1$ implies the normalization condition $\tau_{22}(0)=1$, and thus, by using $(30)_4$ the amplitude coefficient a must assume the following form (which diverges as the condition (18) is approached):

$$a = \{2\sqrt{2}\,\chi\,[\alpha(1-\delta^2) - 2\beta\delta)]\}^{-1}. \qquad (32)$$

4.2 Mode II symmetry conditions

Under Mode II loading conditions, in view of (13), the velocity fields (26) should be endowed with the symmetry properties $v_1(z_1, z_2) = -v_1(\bar{z}_2, \bar{z}_1)$ and $v_2(z_1, z_2) = v_2(\bar{z}_2, \bar{z}_1)$, which imply $A_2 = \overline{A_1}$, or equivalently $A_n = a - (-1)^n\, ib$, for $n = 1, 2$; being a and b real constants.

When the boundary conditions *ii*) are imposed on the stress rate $(27)_{2,4,}$ the following homogeneous system for the real constants a and b may be obtained by using relations (9):

$$[(\beta + \alpha\delta)\, a + (\beta\delta - \alpha)\, b] \sin\gamma\pi = 0, \qquad (a - \delta b) \cos\gamma\pi = 0. \qquad (33)$$

Again, if the critical condition (18) is excluded, the lower admissible value of γ is 3/2. Moreover, for $\gamma = 3/2$ the condition $(33)_1$ implies that $b = -a\,(\beta+\alpha\delta)/(\beta\delta-\alpha)$. Therefore, the lowest order contributions of the asymptotic fields can be expressed in terms of a single amplitude constant a. The introduction of the angular functions $\mathbf{v}(\vartheta)$, $\boldsymbol{\tau}(\vartheta)$ and $\rho(\vartheta)$ allows to represent the stress and velocity asymptotic fields in the form (23). These angular functions can be obtained from (26) and (27) in the form:

$$v_1(\vartheta) = -2a\,(\alpha^2+\beta^2) \sum_{n=1}^{2} \{\delta\, s_n(\vartheta) + (-1)^n\, c_n(\vartheta)\},$$

$$v_2(\vartheta) = 2a \sum_{n=1}^{2} \{(\beta + \alpha\delta)(-1)^n\, s_n(\vartheta) + (\alpha - \beta\delta)\, c_n(\vartheta)\},$$

$$\tau_{11}(\vartheta) = c\chi\,(\alpha^2 + \beta^2) \sum_{n=1}^{2} \{2\delta\, \hat{s}_n(\vartheta) - (1-\delta^2)\,(-1)^n\, \hat{c}_n(\vartheta)\},$$

$$\tau_{22}(\vartheta) = a\chi\,(1+\delta^2)\,(\alpha^2 + \beta^2) \sum_{n=1}^{2} (-1)^n\, \hat{c}_n(\vartheta), \qquad (34)$$

$$\tau_{12}(\vartheta) = -\,a\chi\,(1+\delta^2)\,(\alpha^2 + \beta^2) \sum_{n=1}^{2} \{\beta\,(-1)^n\, \hat{s}_n(\vartheta) + \alpha\, \hat{c}_n(\vartheta)\},$$

$$\tau_{21}(\vartheta) = -\,a\chi \sum_{n=1}^{2} \{[2\alpha\delta + \beta\,(1-\delta^2)]\,(-1)^n\, \hat{s}_n(\vartheta) + [2\beta\delta - \alpha\,(1-\delta^2)]\, \hat{c}_n(\vartheta)\},$$

$$\rho(\vartheta) = a\,(\alpha^2 + \beta^2) \sum_{n=1}^{2} \{\delta\,(\chi + \mathrm{k})\, \hat{s}_n(\vartheta) + (\chi\delta^2 - \mathrm{k})\,(-1)^n\, \hat{c}_n(\vartheta)\}.$$

The definition $(11)_2$ implies the normalization condition $\tau_{21}(0)=1$, and thus, by using $(34)_6$ it follows that the condition (32) for a, must still hold for Mode II loading condition.

5. RESULTS

The variation of nominal stress rate tensor $\boldsymbol{\tau}$, hydrostatic stress rate ρ and velocity vector $\mathbf{v}$ fields with the polar coordinate ϑ may be obtained, by making use of the analytical results (20), (24), (30) and (34), valid for the EI and EC regimes, under Mode I and Mode II loading conditions. In particular, the results reported in Fig. 2 concern the Mode I loading conditions for the EI regime, with $\xi=1$ and $k=\pm0.5$. Note that, for $k = 0.5$ a tensile pre-stress $\sigma_1 = \mu$ in the direction of the crack is present in the current configuration, whereas for $k = -0.5$ the pre-stress is compressive. These results show that when the pre-stress is negative the nominal stress rate component t_{11} is greater then the opening stress rate t_{22} ahead of the crack-tip at $\vartheta = 0$, and the crack-tip opening displacement increases (at $\vartheta = \pi$).

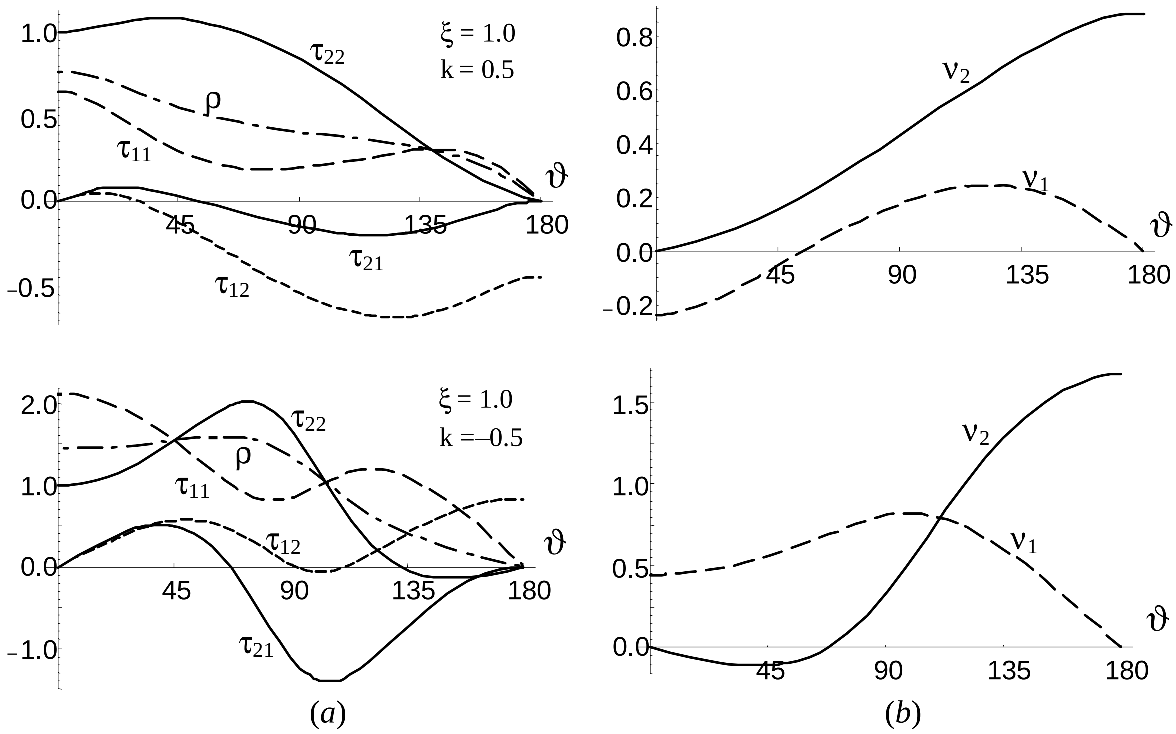

Figure 2. Angular variations of the nominal stress (*a*) and velocity (*b*) components in the EI range, for Mode I.

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