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A BOUNDARY INTEGRAL EQUATION APPROACH TO INCREMENTAL NONLINEAR ELASTICITY

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SOMMARIO
Nel lavoro si considera un corpo illimitato, elastico ed incompribile, in uno stato di deformazione piana ed omogenea. La risposta incrementale del materiale è lineare e governata da due moduli istantanei, dipendenti dallo stretch. Ad uno stadio generico di tale percorso di deformazione, quando il materiale è ancora in regime ellittico, si considera applicato un carico incrementale concentrato e, risolvendo il problema corrispondente, si ottiene la funzione di Green per il mezzo illimitato. Tale soluzione è utilizzata per ottenere una rappresentazione integrale della velocità e della tensione idrostatica incrementale nel contesto delle deformazioni elastiche incrementali sovrapposte ad una assegnata deformazione omogenea.

ABSTRACT
An elastic incompressible infinite body, is considered in a state of homogeneous in-plane deformation whose current configuration is characterised by two in-plane stretches. The incremental response of the material is linear and governed by two instantaneous moduli, which are functions of the in-plane stretches. At a generic stage of this deformation path, an incremental point load is superimposed so that the plane strain constraint is not violated. The corresponding problem is solved and the Green's function for the infinite body is obtained. The solution is used to obtain a boundary integral formulation for velocity and hydrostatic stress rate in the framework of incremental elastic deformation superimposed upon a given homogeneous strain

1. INTRODUCTION

An infinite-body Green's function is obtained in the present article for incremental, nonlinear elastic, isochoric deformation. To this purpose, the simplest setting is chosen, corresponding to the Biot problem in which an infinite medium is homogeneously and biaxially deformed an arbitrary amount within the elliptic range. The current configuration is plane strain and characterized by two in-plane stretches. The incremental response of this incompressible solid is linear and governed by the two Biot [1] moduli, which are functions of the in-plane stretches. At a generic stage of this deformation path, an incremental point load is superimposed so that the plane strain constraint is not violated. The corresponding velocity problem is solved and the nominal stress rate distribution is obtained. The singular solution is worked out by using the plane-wave expansion method [2] in the stream function formulation of Hill and Hutchinson [3]. The solution can be employed to investigate the effects of self-equilibrated loading systems and it can be shown that, approaching the elliptic boundary, the incremental solution tends to self-organize along well defined shear band patterns [4]. Here, the infinite-body Green's function is used to obtain a boundary integral formulation for velocity and hydrostatic stress rate in the framework of incremental elastic deformation superimposed upon a given homogeneous strain. Starting from the integral representations, a boundary element technique can be developed; a main advantage of this technique is in dealing in dealing with the incompressibility constraint which is inherent in our formulation. To provide an example of the capabilities of the proposed approach, the problem of transversal incremental loading of an elastic block constituted by a Mooney-Rivlin material and subjected to a compressive axial load, is considered. In this case, the bifurcation load, corresponding to a symmetric mode, is obtained through a perturbative, numerical approach.

2. GOVERNING EQUATIONS

Under plane strain conditions, the most general constitutive equations for an hyperelastic, initially isotropic, incompressible solid have been given by Biot [1] and can conveniently be expressed in the principal reference system of Cauchy stress (here denoted by indices 1 and 2). In this system, using a Lagrangean formulation of the field equations with the current state taken as reference, the relation between material time derivative of nominal stress t_{ij} and velocity gradient $v_{i,j}$ can be written as

$$\dot{t}_{ij} = \mathrm{K}_{ijkl} v_{l,k} + \dot{p}\, \delta_{ij} , \qquad (1)$$

where a comma denotes partial differentiation, repeated indices are summed and range between 1 and 2, δ_{ij} is the Kronecker delta, $\dot{p}$ the in-plane hydrostatic stress rate and K_{ijkl} are the instantaneous moduli. These possess the major symmetry $\mathrm{K}_{ijkl} = \mathrm{K}_{klij}$ and are linear functions of the components of Cauchy stress σ_1 and σ_2 and of two incremental moduli μ and μ_*, denoting respectively the moduli corresponding to shearing parallel to, and at 45° to, the principal stress axes. The components of K_{ijkl} different from zero are

$$\mathrm{K}_{1111} = \mu_* - \frac{\sigma}{2} - p, \quad \mathrm{K}_{1122} = \mathrm{K}_{2211} = -\mu_* , \quad \mathrm{K}_{2222} = \mu_* + \frac{\sigma}{2} - p,$$

$$\mathrm{K}_{1212} = \mu + \frac{\sigma}{2}, \quad \mathrm{K}_{1221} = \mathrm{K}_{2112} = \mu - p, \quad \mathrm{K}_{2121} = \mu - \frac{\sigma}{2}, \qquad (2)$$

with

$$\sigma = \sigma_1 - \sigma_2, \qquad p = \frac{\sigma_1 + \sigma_2}{2} . \qquad (3)$$

In addition to (1), incompressibility requires that the velocity field v_i be solenoidal:

$$v_{i,i} = 0. \tag{4}$$

The constitutive framework described by the above equations is quite broad and includes, for instance, the relevant cases of Mooney-Rivlin, Ogden materials and J_2-deformation theory material, introduced by Hutchinson and Neale [5]. In the Mooney-Rivlin case, the incremental moduli and the deviatoric stress σ depend on the maximum current stretch $\lambda > 1$,

$$\sigma = \mu_0 \,(\lambda^2 - \lambda^{-2}), \qquad \mu_* = \mu = \frac{\mu_0}{2}(\lambda^2 + \lambda^{-2}), \tag{5}$$

through the ground-state shear modulus μ_0.

It may be important to note that constitutive equations (1)-(2) describe also the incremental behaviour of materials which are *initially orthotropic* with respect to directions 1 and 2. In the interest of generality, no specific assumptions will be introduced on the dependence of μ_* and μ on the current state.

At an arbitrary stage of a homogeneous plane deformation of an infinite medium, we consider an incremental force (a line loading extending orthogonally to the plane of deformation) acting at the point $\mathbf{x} = \mathbf{0}$ and with components $\dot{f}_1$, $\dot{f}_2$ along the principal stress axes. The incremental equilibrium equations are given by

$$\dot{t}_{ij,i} + \dot{f}_j \delta(\mathbf{x}) = 0, \tag{6}$$

where δ is the two-dimensional Dirac delta function and $\mathbf{x}$ denotes the generic material point. Using the constitutive equations and assuming a homogeneous current state, eqns. (6) become:

$$\begin{aligned} &(2\mu_* - p)v_{1,11} + (\mu - p)v_{2,12} + (\mu - \frac{\sigma}{2})v_{1,22} + \dot{f}_1\,\delta(\mathbf{x}) = -\dot{\pi}_{,1}, \\ &(2\mu_* - p)v_{2,22} + (\mu - p)v_{1,21} + (\mu + \frac{\sigma}{2})v_{2,11} + \dot{f}_2\,\delta(\mathbf{x}) = -\dot{\pi}_{,2}, \end{aligned} \tag{7}$$

where

$$\dot{\pi} = \frac{\dot{t}_{11} + \dot{t}_{22}}{2} = \dot{p} - \frac{\sigma}{2} v_{1,1}. \tag{8}$$

It is expedient to introduce a stream function $\psi(x_1, x_2)$ defining a solenoidal but otherwise arbitrary velocity field

$$v_1 = \psi_{,2}, \quad v_2 = -\psi_{,1}, \tag{9}$$

so that by differentiating eqns. $(8)_1$ and $(8)_2$ with respect to x_2 and x_1, respectively, and subtracting the results, the following equation is obtained:

$$L(\psi) + \left(\dot{f}_1 \frac{\partial \cdot}{\partial x_2} - \dot{f}_2 \frac{\partial \cdot}{\partial x_1} \right) \delta(\mathbf{x}) = 0, \tag{10}$$

where L is the linear differential operator, with constant coefficients, defined as

$$L(\cdot) = \left(\mu + \frac{\sigma}{2}\right) \frac{\partial^4 \cdot}{\partial x_1^4} + 2(2\mu_* - \mu) \frac{\partial^4 \cdot}{\partial x_1^2 \partial x_2^2} + \left(\mu - \frac{\sigma}{2}\right) \frac{\partial^4 \cdot}{\partial x_2^4}. \tag{11}$$

The standard regime classification is performed on the basis of the characteristic equation associated to (10):

$$\mu\, \omega_2^4 \left[(1+k) \frac{\omega_1^4}{\omega_2^4} + 2\left(2\frac{\mu_*}{\mu} - 1 \right) \frac{\omega_1^2}{\omega_2^2} + (1-k) \right] = 0 , \tag{12}$$

where

$$k = \frac{\sigma}{2\mu}, \tag{13}$$

which, without loss of generality, may always be taken to be non-negative (simply orienting axes 1 and 2 in a proper way).

Equation (12) admits no real solution ω_1/ω_2 in the elliptic (E) regime, four real solutions ω_1/ω_2 in the hyperbolic (H) regime, two real solutions ω_1/ω_2 in the parabolic (P) regime. In particular, the elliptic regime, where $k < 1$, may be further subdivided into elliptic complex (EC) and elliptic imaginary (EI) regimes. In the EC regime, eqn (12) has two conjugate pairs of complex solutions whereas in the EI regime eqn (12) has four purely imaginary solutions (in conjugate pairs).

If we explicitly introduce the roots γ_1 and γ_2 for ω_1^2/ω_2^2 of (12), as

$$\left.\begin{matrix}\gamma_1 \\ \gamma_2\end{matrix}\right\} = \frac{1 - 2\frac{\mu_*}{\mu} \pm \sqrt{\Delta}}{1+k}, \qquad \Delta = k^2 - 4\frac{\mu_*}{\mu} + 4\left(\frac{\mu_*}{\mu}\right)^2, \tag{14}$$

it can be seen that γ_1 and γ_2 are both real and negative in the (EI) regime and are a conjugate pair in the EC regime. Therefore, Δ is positive in EI and negative in EC. Note that the Mooney-Rivlin material corresponds to $\Delta = k$ and $\gamma_1 = (k-1)/(k+1)$, $\gamma_2 = -1$.

There are two ways to exit the (E) regime, either crossing the EI/P boundary (this corresponds to $k = 1$, i.e. γ_1 must vanish) or crossing the EC/H boundary (this corresponds to $\Delta = 0$, i.e. the two γ_i's must coincide). It is worth noting that, when the (EC)/(H) boundary is approached from (E), $\Delta < 0$ and $\mu > 2\mu_*$, whereas $\Delta > 0$ and $\mu < 2\mu_*$, when the (EI)/(P) boundary is approached.

We recall from Biot [1] and Hill and Hutchinson [3] that incompressible, elastic materials deformed in plane strain, which are *isotropic in the initial state,* cannot penetrate the (P) regime, so that the (EI)/(P) boundary can be reached only at the limit of infinite stretch.

Finally, it is important to note that in the following *we will always assume to remain within the elliptic regime*.

3. THE GREEN'S FUNCTION SET

We follow here the general procedure proposed by Willis [6] to solve singular problems in the infinitesimal theory of elasticity. Since the incremental problem is linear, the solution pertaining to a generic point load can be obtained as the superposition of the solutions for two forces, one acting along axis 1 and the other along axis 2. With this reference, we may take $\dot{f}_i = \delta_{ig}$ and rewrite eqn. (10) as

$$L\ \psi^g + \left(\delta_{1g} \frac{\partial \cdot}{\partial x_2} - \delta_{2g} \frac{\partial \cdot}{\partial x_1}\right) \delta(\mathbf{x}) = 0\ . \tag{15}$$

The plane wave expansion of the δ function is [2]:

$$\delta(\mathbf{x}) = -\frac{1}{4\pi^2} \int_{|\omega|=1} \frac{d\omega}{(\omega \cdot \mathbf{x})^2}\ . \tag{16}$$

Hence, defining the analogous transform $\tilde{\psi}^g(\omega \cdot \mathbf{x})$ of $\psi^g(\mathbf{x})$ as

$$\psi^g(\mathbf{x}) = -\frac{1}{4\pi^2} \int_{|\omega|=1} \tilde{\psi}^g(\omega \cdot \mathbf{x})\, d\omega, \tag{17}$$

the transform of (15) yields

$$L(\omega)\,(\tilde{\psi}^g)'''' = 2\,\frac{\delta_{1g}\,\omega_2 - \delta_{2g}\,\omega_1}{(\omega \cdot \mathbf{x})^3}\,, \tag{18}$$

where a prime denotes differentiation with respect to the scalar $\omega \cdot \mathbf{x}$, and

$$L(\omega) = \mu\, \omega_2^4\, (1+k)\left[\frac{\omega_1^2}{\omega_2^2} - \gamma_1\right]\left[\frac{\omega_1^2}{\omega_2^2} - \gamma_2\right]\ . \tag{19}$$

Since the function $L(\omega)$ is always strictly positive in the elliptic regime, integration with respect to the variable $\omega \cdot \mathbf{x}$ gives

$$\tilde{\psi}^g = \frac{\delta_{1g}\,\omega_2 - \delta_{2g}\,\omega_1}{L(\omega)}\, (\omega \cdot \mathbf{x})(\log|\omega \cdot \hat{\mathbf{x}}| - 1), \tag{20}$$

a formula where cubic, quadratic and linear terms in $\omega \cdot \mathbf{x}$, representing inessential contributions, have been disregarded. In eqn. (20), $\hat{\mathbf{x}}$ represents a dimensionless measure of distance, i.e. $\mathbf{x}$ is divided by any characteristic length. The antitransform (17) of eqn (20) determines the stream function:

$$\psi^g = -\frac{r}{2\pi^2\mu(1+k)}\left[(\log\hat{r}-1)\int_0^{\pi}\frac{\sin[\alpha+\vartheta+(1-g)\pi/2]\cos\alpha}{\Lambda(\alpha+\vartheta)}d\alpha\right.$$

$$+\int_0^{\pi/2}\frac{\sin[\alpha+\vartheta+(1-g)\pi/2]\cos\alpha\log(\cos\alpha)}{\Lambda(\alpha+\vartheta)}d\alpha$$

$$\left.-\int_0^{\pi/2}\frac{\cos[\alpha+\vartheta-(g-1)\pi/2]\sin\alpha\log(\sin\alpha)}{\Lambda(\alpha+\vartheta+\pi/2)}d\alpha\right], \tag{21}$$

where the distance r = |**x**| and the angle ϑ are polar coordinates, $\hat{r}$ is a dimensionless measure of distance, and

$$\Lambda(\alpha) = \sin^4\alpha\,[\cot^2\alpha - \gamma_1]\,[\cot^2\alpha - \gamma_2] > 0. \tag{22}$$

The Green's tensor for the infinite body represents the velocity field associated with the stream function (21) and, according to eqn. (9), is given by

$$v_1^g = \frac{\partial\psi^g}{\partial x_2}, \qquad v_2^g = -\frac{\partial\psi^g}{\partial x_1}. \tag{23}$$

In polar coordinates, the components of the Green's tensor take the expression [4]:

$$v_1^1 = \frac{\log\hat{r}}{2\pi\mu(1+k)}\,\frac{1}{\gamma_1\sqrt{-\gamma_2}+\sqrt{-\gamma_1}\gamma_2} - \frac{1}{2\pi^2\mu(1+k)}\int_0^{\pi/2}\frac{\log(\cos\alpha)\,\sin^2(\alpha+\vartheta)}{\Lambda(\alpha+\vartheta)}d\alpha$$

$$-\frac{1}{2\pi^2\mu(1+k)}\int_0^{p/2}\frac{\log(\sin a)\;\cos^2(a+J)}{\Lambda(a+J+p/2)}\,da\,,$$

$$v_2^2 = -\frac{\log\hat{r}}{2\pi\mu(1+k)}\,\frac{1}{\sqrt{-\gamma_1}+\sqrt{-\gamma_2}} - \frac{1}{2\pi^2\mu(1+k)}\int_0^{\pi/2}\frac{\log(\cos\alpha)\cos^2(\alpha+\vartheta)}{\Lambda(\alpha+\vartheta)}d\alpha \tag{24}$$

$$-\frac{1}{2\pi^2\mu(1+k)}\int_0^{\pi/2}\frac{\log(\sin\alpha)\sin^2(\alpha+\vartheta)}{\Lambda(\alpha+\vartheta+\pi/2)}d\alpha,$$

$$v_2^1 = v_1^2 = \frac{1}{2\pi^2\mu(1+k)}\int_0^{\pi/2}\cos(\alpha+\vartheta)\,\sin(\alpha+\vartheta)\left(\frac{\log(\cos\alpha)}{\Lambda(\alpha+\vartheta)}-\frac{\log(\sin\alpha)}{\Lambda(\alpha+\vartheta+\pi/2)}\right)d\alpha\,.$$

In eqn. (33), the dependence on r is explicit and the integrals are improper Riemann integrals which may easily be shown to converge. Therefore, the numerical treatment of (33) is straightforward, at least for material parameters not too close to the (E) boundary.

Once the velocity field is known, the part of nominal stress linearly related to the velocity gradient can be obtained from eqn (1), but the in-plane hydrostatic stress rate $\dot{p}$ (or, equivalently, $\dot{\pi}$) remains unknown. In order to determine $\dot{\pi}$, eqns. $(7)_1$ and $(7)_2$ can be differentiated with respect to x_1 and x_2, respectively, and summed to give

$$\dot{\pi}_{,11} + \dot{\pi}_{,22} = -2(\mu_* - \mu)(v_{1,111} + v_{2,222}) + \frac{\sigma}{2}(v_{1,111} - v_{2,222}) - \dot{f}_1\delta_{,1} - \dot{f}_2\delta_{,2}. \tag{25}$$

Using the relations for the Green velocity field, the Green hydrostatic nominal stress rate can be given the expression [4]:

$$\dot{\pi}^1 = -\frac{\cos\vartheta}{2\pi r} + \frac{1}{2\pi^2 r(1+k)}\int_0^\pi \frac{1}{\cos\alpha}\left(\frac{\sin^2(\alpha+\vartheta)\cos(\alpha+\vartheta)\Gamma(\alpha+\vartheta)}{\Lambda(\alpha+\vartheta)} + \frac{\cos^2\vartheta\sin\vartheta\Gamma(\vartheta+\pi/2)}{\Lambda(\vartheta+\pi/2)}\right)d\alpha,$$

$$\dot{\pi}^2 = -\frac{\sin\vartheta}{2\pi r} - \frac{1}{2\pi^2 r(1+k)}\int_0^\pi \frac{1}{\cos\alpha}\left(\frac{\sin(\alpha+\vartheta)\cos^2(\alpha+\vartheta)\Gamma(\alpha+\vartheta)}{\Lambda(\alpha+\vartheta)} - \frac{\cos\vartheta\sin^2\vartheta\Gamma(\vartheta+\pi/2)}{\Lambda(\vartheta+\pi/2)}\right)d\alpha \tag{26}$$

where

$$\Gamma(\alpha) = 2\left(\frac{\mu_*}{\mu} - 1\right)(2\cos^2\alpha - 1) - k\,. \tag{27}$$

The numerical treatment of eqns. (26) does not present difficulties, at least for material parameters sufficiently distant from the (E) boundary.

The hydrostatic nominal stress rate (26), toghether with the velocity field (24), represent the Green's function set { v_i^g , $\dot{\pi}^g$ } for the homogeneously stretched, infinite elastic body.

4. BOUNDARY INTEGRAL REPRESENTATIONS

Let us consider now a generic hyperelastic solid subjected to certain boundary conditions preserving homogeneous strain until the current state, assumed as the reference configuration. Superimposed infinitesimal deformation (generally inhomogeneous) is produced by incremental mixed boundary conditions in the usual form

$$v = \bar{v} \quad \text{on} \quad \partial B_v \qquad \text{and} \qquad \dot{t}_{ij} n_i = \dot{\tau}_j \quad \text{on} \quad \partial B_\tau\,, \tag{28}$$

where ∂B_v and ∂B_σ are the two non-overlapping portions of the boundary where velocities and nominal traction rates are respectively prescribed. If we assume for simplicity null incremental body forces, superimposed nominal stress rates satisfy equilibrium

$$\dot{t}_{ij,i} = 0\,. \tag{29}$$

Moreover, the nominal stress rate $\dot{t}^g_{ij}(\mathbf{x},\mathbf{y})$ associated to the Green's function set { v^g_i , $\dot{\pi}^g$ } given by (24) and (26) satisfies

$$\dot{t}^g_{ij,i} + \delta_{gj}\ \delta(\mathbf{x}-\mathbf{y}) = 0\,, \tag{30}$$

where **x** is the generic material point and **y** denotes the place where the force is applied, so that $r = |\mathbf{x} - \mathbf{y}|$ and $\vartheta = \tan^{-1}[(x_2-y_2)/(x_1-y_1)]$. Let us consider a disk C_ε of radius ε centered at **y**. We may write:

$$\int_{B-C_\varepsilon} [\dot{t}^g_{ij,i}(\mathbf{x},\mathbf{y})\, v_j(\mathbf{x}) - \dot{t}_{ij,i}(\mathbf{x})\, v^g_j(\mathbf{x},\mathbf{y})]\, d\mathbf{x} = 0\,. \tag{31}$$

By applying the divergence theorem and taking the the major symmetry $K_{ijkl} = K_{klij}$ into account, eqn (31) reduces in the limit $\varepsilon \to 0$ to

$$v_j(\mathbf{y})\, C^g_j = \int_{\partial B} [\dot{t}_{ij}\, n_i v^g_j(\mathbf{x},\mathbf{y}) - \dot{t}^g_{ij}(\mathbf{x},\mathbf{y})\, n_i v_j]\, dl_x\ , \tag{32}$$

where

$$C^g_j = \lim_{\varepsilon\to 0} \int_{\partial C_\varepsilon} \dot{t}^g_{ij}(\mathbf{x},\mathbf{y})\, n_i\ dl_x \tag{33}$$

is the so-called **C**-matrix. For interior points of B, on integration of eqn. (30) over C_ε and use of the properties of the delta function gives $C^g_j = \delta_{jg}$, so that

$$v_g(\mathbf{y}) = \int_{\partial B} [\dot{t}_{ij}\, n_i v^g_j(\mathbf{x},\mathbf{y}) - \dot{t}^g_{ij}(\mathbf{x},\mathbf{y})\, n_i v_j]\quad dl_x\ . \tag{34}$$

Eqn. (34) represents an integral equation relating the velocity in interior points of the body to the boundary values of nominal traction rates and velocities. Eqn. (34) is formally similar to the analogous boundary integral equation in the infinitesimal theory.

In addition to eqn. (34), a boundary integral equation for the in-plane hydrostatic stress rate $\dot{p}$ is needed, to complete the boundary integral representation of field quantities. To this purpose, substituting the constitutive equation (1) into (29) we may express the gradient of $\dot{p}$ as a function of the second gradient of velocity. This may be obtained deriving (34) twice with respect to **y**, thus giving

$$\dot{p}_{,h}(\mathbf{y}) = -\int_{\partial B} K_{nhsg}[\dot{t}_{ij}\, n_i v^g_{j,sn}(\mathbf{x},\mathbf{y}) - \dot{t}^g_{ij,sn}(\mathbf{x},\mathbf{y})\, n_i v_j]\, dl_x\ . \tag{35}$$

For interior points, $\mathbf{x} \neq \mathbf{y}$, rate equilibrium requires that

$$K_{nhsg}\, v^g_{j,sn} = K_{nhsg}\, v^j_{g,sn} = -\,\dot{p}^j_{,h}\,, \tag{36}$$

where

$$\dot{p}^g = \dot{\pi}^g + \frac{\sigma}{2} v^g_{1,1}\,. \tag{37}$$

Taking eqns. (36) and (37) into account, eqn. (35) becomes

$$\dot{p}_{,h}(\mathbf{y}) = \int_{\partial B} \dot{t}_{ig}\, n_i\, \dot{p}^{g}_{,h}(\mathbf{x},\mathbf{y})\, dl_x + \int_{\partial B} \mathrm{K}_{nhsg}\; \dot{t}^{g}_{ij,sn}(\mathbf{x},\mathbf{y})\, n_i v_j\, dl_x\,. \tag{38}$$

Using the constitutive relation (1) for $t^{g}_{ij,sn}$ and the derivative of (36) into 63, gives

$$\dot{p}_{,h}(\mathbf{y}) = \int_{\partial B} \dot{t}_{ig}\, n_i\, \dot{p}^{g}_{,h}(\mathbf{x},\mathbf{y})\, dl_x - \int_{\partial B} n_i v_j\; \mathrm{K}_{ijkg}\; \dot{p}^{g}_{,hk}(\mathbf{x},\mathbf{y})\, dl_x \;+ \int_{\partial B} n_i v_i\; \mathrm{K}_{nhsg}\; \dot{p}^{g}_{,sn}(\mathbf{x},\mathbf{y})\, dl_x\,. \tag{39}$$

Moreover, it can be proved [4] that

$$\mathrm{K}_{nhsg}\; \dot{p}^{g}_{,sn} \;= [(4\mu\,\mu_* - 4\,\mu_*^2 + \mu\sigma - 2\,\mu_*\,\sigma - \frac{\sigma^2}{2})\, v^{1}_{1,11} - \sigma(\mu + \frac{\sigma}{2})\, v^{2}_{2,11}\,]_{,h}\;, \tag{40}$$

so that eqn (39) can be integrated with respect to y_h, yielding

$$\dot{p}\,(\mathbf{y}) = -\int_{\partial B} \dot{t}_{ig}\; n_i\, \dot{p}^{\,g}(\mathbf{x},\mathbf{y})\, dl_x + \int_{\partial B} n_i v_j\; \mathrm{K}_{ijkg}\; \dot{p}^{g}_{,k}(\mathbf{x},\mathbf{y})\, dl_x$$

$$- \int_{\partial B} n_i v_i\; [(4\mu\,\mu_* - 4\,\mu_*^2 + \mu\sigma - 2\,\mu_*\,\sigma - \frac{\sigma^2}{2})\, v^{1}_{1,11}(\mathbf{x},\mathbf{y}) - \sigma(\mu + \frac{s}{2})\, v^{2}_{2,11}(\mathbf{x},\mathbf{y})\,]\;\, dl_x\,, \tag{41}$$

Eqn. (41) represents an integral equation relating the in-plane hydrostatic stress rate in the interior points of the body to the boundary values of nominal traction rates and velocities.

5. APPLICATION

The boundary integral equation (32) may be used as the starting point for developing a boundary element technique to solve problems of incremental deformations superimposed upon a given homogeneous strain. An application to the solution of an incremental boundary value problem is developed in this section.

In the current configuration, a square elastic block in plane strain conditions is considered, subject to uniaxial, compressive stress. For simplicity, the analysis is restricted to Mooney-Rivlin material, for which the axial stress and incremental moduli are provided by eqn. (5). Planar, smooth, rigid constraints maintain the uniform compression [1, 3, 7] and, starting from this configuration, a symmetric perturbation is assigned as an incremental nominal traction, acting orthogonally to a portion of the free edges of the block. In order to determine the nominal traction rates on the constrained ends and the velocities on the free edges of the block, the boundary element procedure is applied.

The boundary is discretized using linear shape functions for velocities and nominal stress rates. The loaded portion has been taken equal to 1/9 of the edge length. Two meshes having 72 and 144 elements of equal length have been employed. The results of numerical investigation are reported in Fig. 1. The velocity v_C at the middle point of the edge (nondimensionalized as $\bar{v}_C = \mu v_C/(b\dot{\tau})$, where b is the half-length of the edge and $\dot{\tau}$ is the applied nominal traction rate) is plotted versus the pre-stress k. Comparison is also made with results obtained using ABAQUS-Standard (Ver. 5.8-Hibbitt, Karlsson and Sorensen Inc, Pawtucket, RI), with plane-strain, 4-node, bilinear, hybrid elements (CPE4H). A progressive degradation of stiffness in the response may be noted from Fig. 1, when the level of pre-stress is increased. The curves relative to two meshes exhibit asymptotes at critical values of k between 0.849 and 0.850 for the fine mesh, and between 0.880 and 0.881 for the coarse mesh. In the same figure, the value $k \approx 0.839$ corresponding to the surface instability limit given by the bifurcation analysis [1, 3, 7] is also reported.

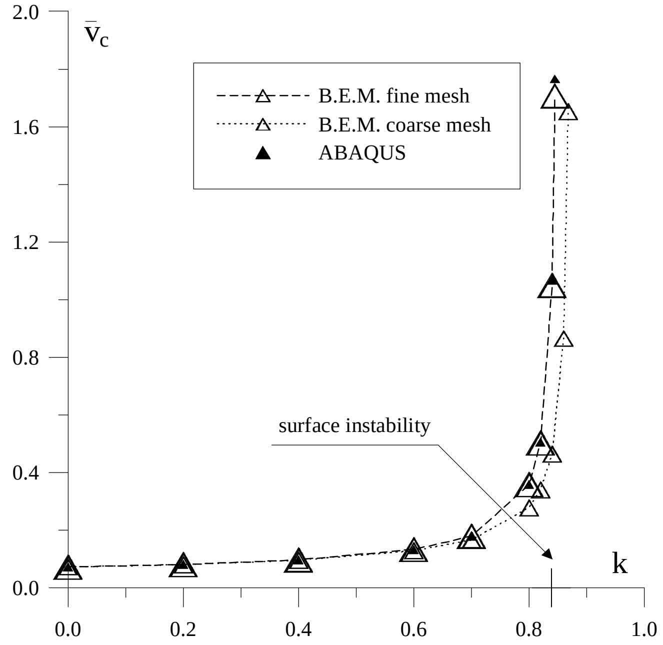

Fig. 1: Velocity versus pre-stress k at the middle point of the block edge.

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