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INTEGRAL REPRESENTATIONS AT THE BOUNDARY FOR STOKES FLOW

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SOMMARIO
Viene proposta una formulazione simmetrica del metodo degli elementi di contorno per problemi bidimensionali e stazionari riguardanti fluidi viscosi incomprimibili. La formulazione richiede la deduzione di rappresentazioni integrali al contorno per pressione e gradiente di velocità, in se rilevanti. Tali rappresentazioni risultano essere accoppiate in corrispondenza dei punti angolosi del contorno.

ABSTRACT
A symmetric Galerkin boundary element formulation is given for two-dimensional, steady and incompressible flow. The formulation requires the derivation of certain – *per se* meaningful – integral representations at the boundary for velocity gradient and pressure; these turn out to be coupled at angular points of the contour profile.

1. INTRODUCTION

Integral equations and related numerical techniques are classical in linear fluid mechanics. Among these numerical techniques, the boundary element method permits a successful treatment of flow incompressibility constraint and has been therefore thoroughly employed. However, direct boundary element methods involve an unsymmetrical coefficient matrix of the final solving algebraic system. This is certainly a drawback of the method, becoming particularly evident when boundary elements are coupled to finite elements. Following an approach known in solid mechanics, symmetry is recovered in the so-called symmetric Galerkin boundary element method (see BONNET *et al*. (1998) and references quoted therein), a technique apparently not developed in hydrodynamics.

In the present paper, integral representations of pressure and stress tensor at the boundary are derived, representing generalizations of formulations valid in interior points given by LADYZHENSKAYA (1963). In particular, integral representations at corner points of the boundary are obtained, providing explicit expression for the so-called "free term matrices". Interestingly, the integral representations for pressure and velocity gradient turn out to be coupled at corner points and –in the hypersingular form– also involve terms depending on the boundary curvatures at the corner. Analysis of corner points is believed to be relevant in relation to a number of problems in Stokes flow: for instance, flow past non-smooth rigid particles or cusp formation at fluid interfaces. In the special case of smooth boundary (the so-called "Lyapunov surfaces"), the equations de-couple and the integral formulations obtained by POZRIKIDIS (2001) and by LIRON & BARTA (1992) are recovered when the boundary conditions model a gas bubble in a viscous liquid and when the boundary conditions pertain to flow past a rigid particle of arbitrary shape. The obtained integral equations are finally employed to establish a symmetric Galerkin formulation for Stokes flow.

2. BASIC EQUATIONS

Two-dimensional viscous flow at small Reynold number is considered, so that unsteady and inertial forces are negligible. For simplicity, body forces are also not included and reference is made to a two-dimensional domain Ω of closed boundary $\partial\Omega$. This boundary may correspond to a particle or a bubble of arbitrary shape in an infinite flow or to a domain, such as for instance a rectangular cavity, confining the flow. The boundary $\partial\Omega$ is divided into two non-overlapping portions $\partial\Omega_u$ and $\partial\Omega_f$ where velocities and tractions are prescribed.

Denoting with $\mathbf{u}$ and p the velocity and the pressure in the fluid, the linear equations for Stokes flow are

$$\operatorname{div} \mathbf{u} = 0, \quad -\nabla p + \mu\, \Delta\mathbf{u} = \mathbf{0}, \tag{1}$$

and the stress tensor is given by (the symbol $(\cdot)_{\mathrm{sym}}$ stands for the symmetric part of a tensor)

$$\sigma = -\, p\, \mathbf{I} + 2\mu\, (\nabla\mathbf{u})_{\mathrm{sym}}. \tag{2}$$

With reference to two orthogonal coordinate axes singled out by the unit vectors $\mathbf{e}^g$, the two-dimensional free-space Green's function set, collecting the Stokeslet and the associated pressure $\{\mathbf{u}^g, p^g\}$, is

$$\mathbf{u}^g = -\frac{1}{4\pi\mu}\left(\mathbf{e}^g \ln r - \frac{r_g}{r^2}\mathbf{r}\right), \qquad p^g = \frac{1}{2\pi}\frac{r_g}{r^2}, \tag{3}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$ and $r = |\mathbf{r}\,|$. The Green's function set satisfies the equations

$$\operatorname{div} \mathbf{u}^g = 0, \qquad \mu\, \Delta\mathbf{u}^g(\mathbf{x}, \mathbf{y}) - \nabla p^g(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y})\, \mathbf{e}^g = \mathbf{0}, \tag{4}$$

where $\delta(\mathbf{x} - \mathbf{y})$ is the Dirac delta function and all differentiations are carried out with respect to the variable $\mathbf{x}$, with the point $\mathbf{y}$ (where the point force is applied) playing the role of a parameter. Note that –unless otherwise stated– differentiations are to be understood always in this sense throughout this paper. According to eqn (2), the stress tensor $\boldsymbol{\sigma}^g$ associated to the fundamental solution is given by:

$$\sigma^g = -\frac{1}{\pi}\frac{r_g}{r^4}\mathbf{r}\otimes\mathbf{r} \tag{5}$$

where $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i\, b_j$, for every vector **a** and **b**.

The boundary integral equations for velocity and pressure are (LADYZHENSKAYA, 1963)

$$u_g(\mathbf{y}) = \alpha\, u_g^{\infty}(\mathbf{y}) - \int_{\partial\Omega} \mathbf{u}^g(\mathbf{x},\mathbf{y}) \bullet \sigma\mathbf{n}\, dl_x + \int_{\partial\Omega} \mathbf{u} \bullet \sigma^g(\mathbf{x},\mathbf{y})\mathbf{n}\, dl_x \;, \tag{6}$$

$$p(\mathbf{y}) = \alpha\, p^{\infty}(\mathbf{y}) + \int_{\partial\Omega} (\sigma\mathbf{n})_g\, p^g(\mathbf{x},\mathbf{y})\, dl_x - 2\mu \int_{\partial\Omega} u_g \nabla p^g(\mathbf{x},\mathbf{y}) \bullet \mathbf{n}\, dl_x \;, \tag{7}$$

where a dot denotes scalar product between vectors, $(\sigma\mathbf{n})_g = \sigma_{gk} n_k$, in which **n** is the unit inward normal to $\partial\Omega$ (pointing into the fluid). The fields u_g^{∞} and p^{∞} are the velocity and pressure of the incident flow, so that $\alpha = 0$ for a flow in a bounded domain or $\alpha = 1$ for flow past a bubble or a rigid particle.

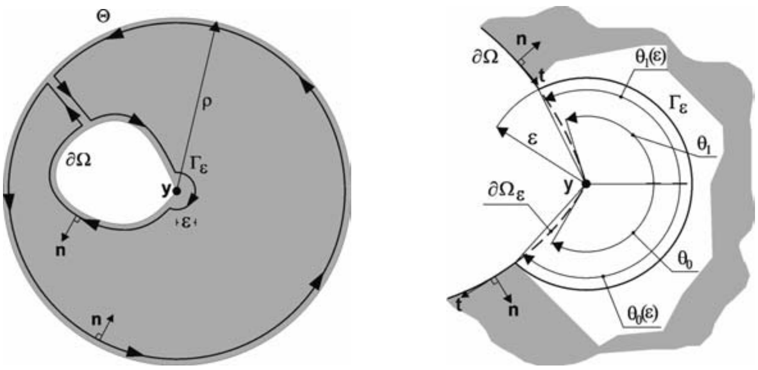

Fig. 1: Geometry of the problem and integration contours

3. INTEGRAL REPRESENTATIONS AT THE BOUNDARY

The velocity field at the boundary can be obtained from eqn (6) moving the source point **y** on the boundary $\partial\Omega$, considering the integration contours sketched in Fig. 1 and taking the limit for $\varepsilon \to 0$ (and for $\rho \to \infty$ in the case of flow past a particle). Correspondingly,

$$C_i^g\, u_i(\mathbf{y}) \;=\; \alpha\, u_g^{\infty}(\mathbf{y}) - \int_{\partial\Omega} \mathbf{u}^g(\mathbf{x},\mathbf{y}) \bullet \sigma\mathbf{n}\, dl_x + \overset{PV}{\int_{\partial\Omega}} \mathbf{u} \bullet \sigma^g(\mathbf{x},\mathbf{y})\mathbf{n}\, dl_x \;, \tag{8}$$

where PV denotes the Cauchy principal value and the **C**-matrix is defined as

$$C_i^g = -\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} (\sigma^g \mathbf{n})_i\, dl_x = \alpha\, \delta_{gi} + \overset{PV}{\int_{\partial\Omega}} (\sigma^g \mathbf{n})_i\, dl_x \;, \tag{9}$$

in which Γ_ε is the intersection of the circle of radius ε centered at **y** with the region occupied by the fluid and the inward unit normal **n** points towards the fluid. Using eqn. (5) we get

$$\mathbf{C} = \frac{1}{2\pi}[(\theta_1 - \theta_0)\,\mathbf{I} - (\mathbf{n}(\theta_1) \otimes \mathbf{t}(\theta_1))_{\mathsf{sym}} + (\mathbf{n}(\theta_0) \otimes \mathbf{t}(\theta_0))_{\mathsf{sym}}], \tag{10}$$

where ($\mathbf{t}$, $\mathbf{n}$) are the unit vectors of a right-handed Cartesian reference system (Fig. 1) and θ_0 and θ_1 are the angular coordinates of the half-tangents to the boundary at point $\mathbf{y}$. Note that at a smooth point of the boundary, where $\theta_1 = \theta_0 + \pi$, $\mathbf{C} = ½\,\mathbf{I}$.

In order to obtain a boundary integral representation for the pressure p, we note that, for a closed contour not enclosing the singularity point $\mathbf{y}$, the following condition holds

$$\oint \nabla p^g(\mathbf{x},\mathbf{y}) \bullet \mathbf{n}(\mathbf{x})\, dl_x = 0. \tag{11}$$

The derivation here and in the following will be restricted to the case of a domain in an infinite flow (Fig. 1), whereas the easier situation of the flow confined to a closed domain will be simply stated. An infinite flow $\mathbf{u}$ and its corresponding pressure p are decomposed into unperturbed components $\mathbf{u}^\infty$ and p^∞, that would prevail in the absence of any disturbance, and perturbed components $\mathbf{u}^{\mathrm{D}}$ and p^{D}, so that $\mathbf{u} = \mathbf{u}^\infty + \mathbf{u}^{\mathrm{D}}$ and $p = p^\infty + p^{\mathrm{D}}$. As for the perturbed field, applying eqn. (7) together with condition (11) to the contour shown in Fig. 1 yields

$$\int\limits_{\partial\Omega - \partial\Omega_\varepsilon + \Gamma_\varepsilon + \Theta_\rho} \left[\left(\sigma^{\mathrm{D}}\mathbf{n}\right)\!(\mathbf{x})_g\, p^g(\mathbf{x},\mathbf{y}) - 2\mu\, \nabla p^g(\mathbf{x},\mathbf{y}) \bullet \mathbf{n}(\mathbf{x})\left(u_g^{\mathrm{D}}(\mathbf{x}) - u_g^{\mathrm{D}}(\mathbf{y})\right)\right] dl_x = 0, \tag{12}$$

where $\partial\Omega_\varepsilon$ is the boundary length intercepted by Γ_ε and Θ_ρ is a circle of radius ρ centered at $\mathbf{y}$ (Fig. 1). Taking the limit for ρ tending to infinity, considering the relation $\mathbf{r} = r\,\mathbf{n}$ on Γ_ε and assuming that σ^{D} decays and that $\mathbf{u}^{\mathrm{D}}$ does not grow faster than ρ^n, with $n<1$, the integral along Θ_ρ turns out to vanish. In order to evaluate the integral along Γ_ε in the limit $\varepsilon \to 0$, we introduce the first-order and zeroth-order series expansions for the velocity $\mathbf{u}^{\mathrm{D}}(\mathbf{x})$ and the stress $\sigma^{\mathrm{D}}(\mathbf{x})$

$$\mathbf{u}^{\mathrm{D}}(\mathbf{x}) - \mathbf{u}^{\mathrm{D}}(\mathbf{y}) = \nabla\mathbf{u}^{\mathrm{D}}(\mathbf{y})\,(\mathbf{x} - \mathbf{y}) + o(|\mathbf{x} - \mathbf{y}|^2), \qquad \sigma^{\mathrm{D}}(\mathbf{x}) = \sigma^{\mathrm{D}}(\mathbf{y}) + o(|\mathbf{x} - \mathbf{y}|). \tag{13}$$

Using eqns (13) and taking the limit $\varepsilon \to 0$, eqn (12) can be re-written as

$$\frac{\theta_1 - \theta_0}{2\pi}\, p^{\mathrm{D}}(\mathbf{y}) - 2\mu\, \mathbf{C} \bullet \nabla\mathbf{u}^{\mathrm{D}}(\mathbf{y}) = \overset{PV}{\int\limits_{\partial\Omega}} \left(\sigma^{\mathrm{D}}\mathbf{n}\right)_g p^g\, dl_x - 2\mu \overset{PV}{\int\limits_{\partial\Omega}} \nabla p^g \bullet \mathbf{n}\left(u_g^{\mathrm{D}}(\mathbf{x}) - u_g^{\mathrm{D}}(\mathbf{y})\right) dl_x\,. \tag{14}$$

The unperturbed field p^∞ is now analyzed, considering a contour enclosing the inclusion and excluding the point $\mathbf{y}$ through a circle of radius ε. Writing the analogous of the (12) for the unperturbed fields and for the considered contour, gives in the limit $\varepsilon \to 0$

$$\left(\frac{\theta_1 - \theta_0}{2\pi} - 1\right) p^\infty(\mathbf{y}) - 2\mu\, \mathbf{C} \bullet \nabla\mathbf{u}^\infty(\mathbf{y}) = \overset{PV}{\int\limits_{\partial\Omega}} \left(\sigma^\infty\mathbf{n}\right)_g p^g\, dl_x - 2\mu \overset{PV}{\int\limits_{\partial\Omega}} \nabla p^g \bullet \mathbf{n}\left(u_g^\infty(\mathbf{x}) - u_g^\infty(\mathbf{y})\right) dl_x \tag{15}$$

so that summing to (14) yields

$$\frac{\theta_1-\theta_0}{2\pi}\,p(\mathbf{y})-2\mu\,\mathbf{C}\bullet\nabla\mathbf{u}(\mathbf{y})=\alpha\,p^{\infty}(\mathbf{y})+\overset{PV}{\underset{\partial\Omega}{\int}}(\sigma\mathbf{n})_g\,p^g\,dl_x-2\mu\overset{PV}{\underset{\partial\Omega}{\int}}\nabla p^g\bullet\mathbf{n}\big(u_g(\mathbf{x})-u_g(\mathbf{y})\big)dl_x \quad (16)$$

a formula that has been obtained for $\alpha = 1$ and that, for $\alpha = 0$, reduces to the case of flow confined to a finite domain.

Eqn (16) is the integral equation representing the pressure p at points of the boundary. In this equation, the boundary values of p are coupled with those of the velocity gradient but, at smooth points of the contour, where $\theta_1 - \theta_0 = \pi$ and $\mathbf{C} = ½\,\mathbf{I}$, the integral equation simplifies to

$$\frac{1}{2}\,p(\mathbf{y})=\alpha\,p^{\infty}(\mathbf{y})+\overset{PV}{\underset{\partial\Omega}{\int}}(\sigma\mathbf{n})_g\,p^g\,dl_x-2\mu\overset{PV}{\underset{\partial\Omega}{\int}}\nabla p^g\bullet\mathbf{n}\big(u_g(\mathbf{x})-u_g(\mathbf{y})\big)dl_x\,. \quad (17)$$

Note that when $\partial\Omega$ represents the boundary of a bubble, the traction is given by

$$\sigma\mathbf{n} = (-p_B + \gamma\kappa)\,\mathbf{n}, \quad (18)$$

where p_B is the pressure, γ the surface tension and κ is the curvature of the bubble, so that condition (17) becomes equivalent to the integral equation obtained by POZRIKIDIS (2001). Moreover, when $\partial\Omega$ represents the boundary of a rigid inclusion, the term due to the gradient of the Stokeslet pressure vanishes and a boundary equation implicitly derived by LIRON & BARTA (1992) is recovered.

An alternative form of (16) involving hypersingular integrals will be useful later and is simply obtained as follows. Considering the contour reported in Fig. 1, we obtain when $\rho \to \infty$

$$\int_{\partial\Omega-\partial\Omega_\varepsilon+\Gamma_\varepsilon+\Theta_\rho}\nabla p^g(\mathbf{x},\mathbf{y})\bullet\mathbf{n}(\mathbf{x})\,dl_x = \int_{\partial\Omega-\partial\Omega_\varepsilon}\nabla p^g(\mathbf{x},\mathbf{y})\bullet\mathbf{n}(\mathbf{x})\,dl_x-\frac{1}{2\pi\varepsilon}\int_{\theta_0(\varepsilon)}^{\theta_1(\varepsilon)}n_g(\theta)\,d\theta=0, \quad (19)$$

where $\theta_0(\varepsilon)$ and $\theta_1(\varepsilon)$ are the angles singling out the initial and final edges of the arc Γ_ε (see the detail in Fig. 1). A Taylor series expansion of the integral yields

$$\int_{\theta_0(\varepsilon)}^{\theta_1(\varepsilon)}\mathbf{n}(\theta)\,d\theta=\mathbf{n}(\theta_1)+\mathbf{n}(\theta_0)-\varepsilon\,[\theta_1'(0)\,\mathbf{t}(\theta_1)+\theta_0'(0)\,\mathbf{t}(\theta_0)]+o(\varepsilon^2), \quad (20)$$

where $\theta_0 = \theta_0(0)$ and $\theta_1 = \theta_1(0)$ are the angular coordinates of the half-tangents to the boundary at point $\mathbf{y}$ (Fig. 1) and a prime denotes differentiation with respect to the argument. In the limit $\varepsilon \to 0$, we obtain

$$\overset{FP}{\underset{\partial\Omega}{\int}}\nabla p^g\bullet\mathbf{n}\,dl_x=-\frac{1}{2\pi}\,[\theta_1'(0)\,t_g(\theta_1)+\theta_0'(0)\,t_g(\theta_0)], \quad (21)$$

where *FP* denotes the Hadamard finite part of the integral. Employing (21) in (16) we arrive at the expression

$$\frac{\theta_1-\theta_0}{2\pi}p(\mathbf{y})-2\mu\,\mathbf{C}\bullet\nabla\mathbf{u}(\mathbf{y})+\frac{\mu}{\pi}[\,\theta_1'(0)\,\mathbf{t}(\theta_1)+\theta_0'(0)\,\mathbf{t}(\theta_0)]\bullet\mathbf{u}(\mathbf{y}),$$

$$=\alpha\,p^{\infty}(\mathbf{y})+\int_{\partial\Omega}^{PV}(\sigma\mathbf{n})_g\,p^g\,dl_x-2\mu\int_{\partial\Omega}^{FP}\nabla p^g\bullet\mathbf{n}\,u_g\,dl_x\,. \tag{22}$$

Note that in the case of piecewise linear boundary $\theta_1'(0)=\theta_0'(0)=0$ and for smooth boundary, where $\theta_1-\theta_0=\pi$, $\theta_1'(0)=-\theta_0'(0)$ and $\mathbf{t}(\theta_1)=\mathbf{t}(\theta_0)$, we get

$$\frac{1}{2}\,p(\mathbf{y})=\alpha\,p^{\infty}(\mathbf{y})+\int_{\partial\Omega}^{PV}(\sigma\mathbf{n})_g\,p^g\,dl_x-2\,\mu\int_{\partial\Omega}^{FP}\nabla p^g\bullet\mathbf{n}\,u_g\,dl_x\,, \tag{23}$$

which is an alternative to (17), involving hypersingular kernel.

An integral equation for the velocity gradient at boundary points can be obtained starting from eqn (6). In particular, we begin noting that for a closed contour not enclosing the singularity point $\mathbf{y}$, the following condition holds

$$\oint\sigma_{,k}^{g}(\mathbf{x},\mathbf{y})\mathbf{n}(\mathbf{x})\,dl_x=0. \tag{24}$$

Restricting the derivation to the case of infinite flow past a particle, we get for the perturbed and unperturbed fields

$$\int_{\partial\Omega-\partial\Omega_\varepsilon+\Gamma_\varepsilon+\Theta_\rho}\left[\sigma^{\mathrm{D}}\mathbf{n}\bullet\mathbf{u}_{,k}^{g}-\sigma_{,k}^{g}\mathbf{n}\bullet\left(\mathbf{u}^{\mathrm{D}}(\mathbf{x})-\mathbf{u}^{\mathrm{D}}(\mathbf{y})\right)\right]dl_x\;=\;0, \tag{25}$$

$$\int_{\partial\Omega-\partial\Omega_\varepsilon+\Gamma_\varepsilon}\left[\sigma^{\infty}\mathbf{n}\bullet\mathbf{u}_{,k}^{g}-\sigma_{,k}^{g}\mathbf{n}\bullet\left(\mathbf{u}^{\infty}(\mathbf{x})-\mathbf{u}^{\infty}(\mathbf{y})\right)\right]dl_x\;=\;0, \tag{26}$$

where $\mathbf{u}_{,k}{}^{g}$ and $\boldsymbol{\sigma}_{,k}{}^{g}$ are singular as $1/r$ and $1/r^2$ respectively, when r tends to zero. Following a procedure analogous to that illustrated for the derivation of pressure representation and taking the limits for $\rho\to\infty$ and $\varepsilon\to0$, the combination of eqns. (25)-(26) gives

$$(\mathbf{A}\nabla\mathbf{u}(\mathbf{y})+\mathbf{B}\,\mathbf{D}(\mathbf{y})+p(\mathbf{y})\mathbf{F}\,)_{gk}=\alpha\,u_{g,k}^{\infty}(\mathbf{y})-\int_{\partial\Omega}^{PV}\sigma_{,k}^{g}\mathbf{n}\bullet\left(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\right)dl_x+\int_{\partial\Omega}^{PV}\sigma\mathbf{n}\bullet\mathbf{u}_{,k}^{g}\,dl_x \tag{27}$$

where

$$\mathrm{A}_{gkim}=\lim_{\varepsilon\to0}\int_{\Gamma_\varepsilon}\sigma_{ij,k}^{g}n_j(x_m-y_m)\,dl_x\,,\quad \mathrm{B}_{gkim}=-\lim_{\varepsilon\to0}2\mu\int_{\Gamma_\varepsilon}n_m u_{i,k}^{g}\,dl_x\,,\quad \mathrm{F}_{gk}=-\lim_{\varepsilon\to0}\int_{\Gamma_\varepsilon}\mathbf{n}\bullet\mathbf{u}_{,k}^{g}\,dl_x \tag{28}$$

and **D** is the rate-of-strain tensor

$$\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}). \tag{29}$$

Substituting the expressions for the velocity and stress gradients of the Green state and taking the relation $\mathbf{r} = r\,\mathbf{n}$ on Γ_ε into account, eqns (28) yield

$$\mathbf{A}\nabla\mathbf{u} + \mathbf{B}\,\mathbf{D} = \frac{1}{4}[\,\nabla\mathbf{u}\mathbf{C} + \nabla\mathbf{u}^{\mathrm{T}}\mathbf{C} - \mathbf{C}\nabla\mathbf{u} - 5\,\mathbf{C}\nabla\mathbf{u}^{\mathrm{T}} - 6(\mathbf{C}\bullet\nabla\mathbf{u})\mathbf{I} + 16\,\mathbf{E}\nabla\mathbf{u}], \tag{30}$$

$$\mathbf{F} = -\frac{\theta_1 - \theta_0}{4\pi\mu}\mathbf{I} + \frac{1}{2\mu}\mathbf{C}, \tag{31}$$

where

$$\mathbf{E} = \frac{1}{\pi}\int_{\theta_0}^{\theta_1} \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\, d\theta. \tag{32}$$

In the special case of a smooth (Lyapunov) boundary

$$\mathbf{B}\,\mathbf{D} = \frac{1}{4}\mathbf{D}, \quad \mathbf{A}\nabla\mathbf{u} + \mathbf{B}\,\mathbf{D} = \frac{1}{2}\nabla\mathbf{u}, \quad \mathbf{F} = \mathbf{0}, \tag{33}$$

so that eqn. (27) reduces to

$$\frac{1}{2}u_{g,k}(\mathbf{y}) = \alpha\, u^{\infty}_{g,k}(\mathbf{y}) - \int_{\partial\Omega}^{PV} \sigma^{g}_{,k}\mathbf{n}\bullet(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\,dl_x + \int_{\partial\Omega}^{PV} \sigma\mathbf{n}\bullet\mathbf{u}^{g}_{,k}\,dl_x\,. \tag{34}$$

By means of eqns. (17) and (34), the constitutive eqn. (2) provides the boundary integral representation for the stress

$$\frac{1}{2}\sigma_{gk}(\mathbf{y}) = \sigma^{\infty}_{gk}(\mathbf{y}) + \int_{\partial\Omega}^{PV} (\sigma^{g}\sigma\mathbf{n})_k\,dl_x\,,$$

$$-\,\mu\int_{\partial\Omega}^{PV}\left[(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\bullet(\sigma^{g}_{,k} + \sigma^{k}_{,g})\mathbf{n} + 2\delta_{gk}\,(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))_i\,\nabla p^{i}\bullet\mathbf{n}\,\right]dl_x\,. \tag{35}$$

In the special case of flow past a particle, the second integral vanishes and eqn (35) reduces to a two-dimensional version of that obtained by LIRON & BARTA (1992) for three-dimensional flow.

An alternative form of (27) involving hypersingular integrals can be obtained considering the contour of Fig. 1 and taking the limit for $\rho \to \infty$, so that

$$\int_{\partial\Omega-\partial\Omega_\varepsilon} \sigma^g_{,k}(\mathbf{x},\mathbf{y})\,\mathbf{n}(\mathbf{x})\,dl_x + \frac{1}{\pi\varepsilon}\int_{\theta_0(\varepsilon)}^{\theta_1(\varepsilon)} \left(n_g\,\mathbf{e}^k + \delta_{gk}\,\mathbf{n} - 3n_g n_k\,\mathbf{n}\right) d\theta = 0. \tag{36}$$

Expanding the second integral in Taylor series, taking the limit for $\varepsilon \to 0$, and substituting into eqn. (27), lead to

$$(\mathbf{H}\,\mathbf{u}(\mathbf{y}) + \mathbf{A}\nabla\mathbf{u}(\mathbf{y}) + \mathbf{B}\,\mathbf{D}(\mathbf{y}) + p(\mathbf{y})\mathbf{F}\,)_{gk} = \alpha\, u^\infty_{g,k}(\mathbf{y}) - \overset{FP}{\int_{\partial\Omega}} \sigma^g_{,k}\mathbf{n}\bullet\mathbf{u}\,dl_x + \overset{PV}{\int_{\partial\Omega}} \sigma\mathbf{n}\bullet\mathbf{u}^g_{,k}\,dl_x\,, \tag{37}$$

where

$$\mathbf{H} = \frac{1}{\pi}\,[\,\theta_1'(0)\,(\mathbf{t}(\theta_1)\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{t}(\theta_1) - 3\,\mathbf{t}(\theta_1)\otimes\mathbf{t}(\theta_1)\otimes\mathbf{t}(\theta_1))$$

$$+\ \theta_0'(0)\,(\,\mathbf{t}(\theta_0)\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{t}(\theta_0) + 3\,\mathbf{t}(\theta_1)\otimes\mathbf{t}(\theta_0)\otimes\mathbf{t}(\theta_0))]. \tag{38}$$

The tensors **H**, **A**, **B** and **F** collect the *free-terms* of the integral equation representing the velocity gradient at points on the boundary. Note that **H** depends on the curvatures of the boundary and vanishes both for smooth and piecewise rectilinear boundary. This dependence on the curvature has a correspondence in elasticity (GUIGGIANI, 1995). At smooth points of the contour, where **F** turns out to be zero, eqn (37) simplifies as follows:

$$\frac{1}{2}\,u_{g,k}(\mathbf{y}) = \alpha\, u^\infty_{g,k}(\mathbf{y}) - \overset{FP}{\int_{\partial\Omega}} \sigma^g_{,k}\mathbf{n}\bullet\mathbf{u}\,dl_x + \overset{PV}{\int_{\partial\Omega}} \sigma\mathbf{n}\bullet\mathbf{u}^g_{,k}\,dl_x\,. \tag{39}$$

Once the velocity gradient and the pressure are given, the integral representation of the stress tensor on smooth points of the boundary follows from eqn. (2):

$$\frac{1}{2}\,\sigma_{gk}(\mathbf{y}) = \alpha\,\sigma^\infty_{gk}(\mathbf{y}) + \overset{PV}{\int_{\partial\Omega}} \left(\sigma^g\sigma\mathbf{n}\right)_k dl_x - \mu \overset{FP}{\int_{\partial\Omega}} \left[\mathbf{u}\bullet\left(\sigma^g_{,k} + \sigma^k_{,g}\right)\mathbf{n} + 2\delta_{gk}\,u_i\nabla p^i\bullet\mathbf{n}\,\right] dl_x\,. \tag{40}$$

In closure of this Section, we note that eqns. (39) and (40) permit the complete determination of velocity gradient and stress at the boundary when tractions and velocities are here known.

4. SIMMETRIC FORMULATION OF THE BOUNDARY ELEMENT METHOD

Let δu_g and δf_g be virtual velocity and traction fields satisfying the boundary conditions

$$\delta u_g = 0, \quad \text{on } \partial\Omega_{\mathbf{u}}, \qquad \delta f_g = 0, \quad \text{on } \partial\Omega_{\mathbf{f}} \tag{41}$$

where $\partial\Omega_\mathbf{u}$ and $\partial\Omega_\mathbf{f}$ are the portions of the boundary where velocities and tractions $\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}$ are assigned, respectively.

Taking the scalar product of velocity $\mathbf{u}$ and traction $\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}$, as given by eqns (8) and (40) on a Lyapunov boundary, by the virtual fields and integrating over the contour, the following equations are obtained:

$$\frac{1}{2}\int_{\partial\Omega_\mathbf{u}} u_g(\mathbf{y})\,\delta f_g(\mathbf{y})\,dl_y - \alpha\int_{\partial\Omega_\mathbf{u}} u_g^\infty(\mathbf{y})\,\delta f_g(\mathbf{y})\,dl_y =$$

$$-\int_{\partial\Omega_\mathbf{u}} \delta f_g(\mathbf{y})\left(\int_{\partial\Omega}\mathbf{u}^g\bullet\mathbf{f}\,dl_x - \overset{PV}{\int_{\partial\Omega}}\mathbf{u}\bullet\mathbf{f}^g\,dl_x\right)dl_y\,, \qquad (42)$$

$$\frac{1}{2}\int_{\partial\Omega_\mathbf{f}} \delta u_g(\mathbf{y})\,f_g(\mathbf{y})\,dl_y - \alpha\int_{\partial\Omega_\mathbf{f}} \delta u_g(\mathbf{y})\;f_g^\infty(\mathbf{y})\,dl_y = \int_{\partial\Omega_\mathbf{f}} \delta u_g(\mathbf{y})\,[\mathbf{n}(\mathbf{y})\bullet\overset{PV}{\int_{\partial\Omega}}\sigma^g\mathbf{f}\,dl_x\,]\,dl_y$$

$$-\,\mu\int_{\partial\Omega_\mathbf{f}} \delta u_g(\mathbf{y})\{\,n_k(\mathbf{y})\overset{FP}{\int_{\partial\Omega}}\left[\mathbf{u}\bullet\left(\sigma^g_{,k}+\sigma^k_{,g}\right)\mathbf{n}+2\delta_{gk}\,u_i\nabla p^i\bullet\mathbf{n}\right]dl_x\,\}\,dl_y. \qquad (43)$$

Eqns (42) and (43) are the starting point to derive a symmetric formulation of the boundary element method. To this purpose, the boundary $\partial\Omega$ is divided into n_e elements $\partial\Omega^\mathrm{e}$ (e = 1,..., n_e), with subsets n_e^u and n_e^t belonging to $\partial\Omega_\mathrm{u}$ and $\partial\Omega_\mathrm{t}$, respectively (i.e. $n_\mathrm{e} = n_\mathrm{e}^\mathrm{u} + n_\mathrm{e}^\mathrm{t}$). Within each boundary element $\partial\Omega^\mathrm{e}$, the following representations for velocities and tractions are chosen:

$$u_g(x) = \sum_{\alpha=1}^{n_\alpha}\varphi^\alpha(x)\,u_g^\alpha\,, \qquad f_g(x) = \sum_{\beta=1}^{n_\beta}\varphi^\alpha(x)\,f_g^\alpha \qquad (44)$$

where u_g^α and f_g^α are the nodal values of velocities and tractions, respectively and φ^α are the relevant shape functions. Assuming for δu_g and δf_g the *same shape functions* as in (44), and taking into account that eqns. (42), (43) hold true for every δu_g and δf_g, the following linear algebraic system can be obtained:

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{D} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{f}\end{bmatrix} = \begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} \qquad (45)$$

representing the governing equations of the discrete model.

In eqn (45), vectors $\mathbf{u}$, $\mathbf{f}$ collect the unknown nodal values of velocity and tractions, the system matrix is obtained by assembling the element sub-matrices

$$A_{gi}^{\alpha\beta} = -\,\mu\int_{\partial\Omega_f^e}\varphi^\alpha(\mathbf{y})\;\{\,n_k(\mathbf{y})\overset{FP}{\int_{\partial\Omega_f^e}}\left[\left(\sigma^g_{ij,k}+\sigma^k_{ij,g}\right)n_j+2\delta_{gk}\,p^i_{,j}n_j\right]\varphi^\beta(\mathbf{x})dl_x\,\}\,dl_y,$$

$$B_{gi}^{\alpha\beta} = \int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y})\,[\,n_j(\mathbf{y}) \overset{PV}{\int_{\partial\Omega_u^e}} \sigma_{ij}^g \varphi^\beta(\mathbf{x})\,dl_x\,]\,dl_y, \tag{46}$$

$$C_{gi}^{\alpha\beta} = -\int_{\partial\Omega_u^e} \varphi^\alpha(\mathbf{y}) \left(\int_{\partial\Omega_u^e} u_i^g\, \varphi^\beta(\mathbf{x})\,dl_x \right) dl_y\,, \quad D_{gi}^{\alpha\beta} = \int_{\partial\Omega_u^e} \varphi^\alpha(\mathbf{y})\,\Big(\overset{PV}{\int_{\partial\Omega_f^e}} \sigma_{ij}^g n_j \varphi^\beta(\mathbf{x})\,dl_x\Big)\,dl_y,$$

and vectors on the right-hand side are given, for node α of each element, by

$$p_g^\alpha = \frac{1}{2}\int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) f_g(\mathbf{y})\,dl_y - \alpha \int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y})\; f_g^\infty(\mathbf{y})\,dl_y - \int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y})\,[\,n_j(\mathbf{y}) \overset{PV}{\int_{\partial\Omega_f}} \sigma_{ij}^g f_i\,dl_x\,]\,dl_y$$

$$+\,\mu \int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y})\,\{\,n_k(\mathbf{y}) \overset{FP}{\int_{\partial\Omega_u}} \big[u_i(\sigma_{ij,k}^g + \sigma_{ij,g}^k)n_j + 2\delta_{gk}\,u_i p_{,j}^i n_j\,\big] dl_x\,\}\,dl_y,$$

$$q_g^\alpha = \frac{1}{2}\int_{\partial\Omega_u^e} u_g(\mathbf{y})\;\varphi^\alpha(\mathbf{y})\,dl_y - \alpha \int_{\partial\Omega_u^e} u_g^\infty(\mathbf{y})\;\varphi^\alpha(\mathbf{y})\,dl_y \tag{47}$$

$$+ \int_{\partial\Omega_u^e} \varphi^\alpha(\mathbf{y}) \left(\int_{\partial\Omega_f} u_i^g f_i\,dl_x - \int_{\partial\Omega_u} u_i f_i^g\,dl_x \right) dl_y\,.$$

Simple algebra, omitted for brevity, shows that $\mathbf{D} = \mathbf{B}^T$ so that system matrix appearing in (45) turns out to be symmetric, a feature characteristic of the Galerkin boundary element formulation.

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