Flutter instability in elastoplastic solids with nonassociative flow rule: a dynamical interpretation

Davide Bigoni, Andrea Piccolroaz
Dipartimento di Ingegneria Meccanica e Strutturale
Università di Trento
Via Mesiano 77 – 38050 Trento
E-mail: *bigoni@ing.unitn.it; andrea.piccolroaz@ing.unitn.it*

John R. Willis
Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences,
Cambridge University
Wilberforce Road, Cambridge CB3 OWA U.K.

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SUMMARY: Flutter instability in an infinite medium is a form of material instability corresponding to the occurrence of complex conjugate squares of the acceleration wave velocities. Although its occurrence is known to be possible in elastoplastic materials with nonassociative flow law and to correspond to some dynamically growing disturbance, its mechanical meaning has to date eluded a precise interpretation. This is provided here by constructing the infinite-body, time-harmonic Green's function for the loading branch of an elastoplastic material in flutter conditions. Used as a perturbation, it reveals that flutter corresponds to a spatially blowing-up disturbance, exhibiting well-defined directional properties, determined by the wave directions for which the eigenvalues become complex conjugate. Flutter is shown to be connected to the formation of localized deformations, a dynamical phenomenon sharing geometrical similarities with the well-known mechanism of shear banding occurring under quasi-static loading. Differently from the latter phenomenon, flutter may occur much earlier in a process of continued plastic deformation.

1. INTRODUCTION

Several micromechanisms acting at a microscale during deformation of granular and rock-like materials involve Coulomb friction. As a consequence, the flow rule becomes nonassociative and the phenomenological rate elastoplastic constitutive equations for these materials become unsymmetric. Due to this lack of symmetry, two squares of the propagation velocity of acceleration waves or, in other words, two eigenvalues of the acoustic tensor, may become a complex conjugate pair. That this situation might correspond to a form of material instability particularly relevant in granular material was clear since J.R. Rice (1977) coined for it the term “flutter instability”, but neither examples of constitutive equations displaying this instability nor a mechanical interpretation for it were given at that time. Consequently, research was initially focused on the determination of situations in which flutter was possible (see Bigoni, 2000; Loret et al. 2000 for a review). In particular, it was shown that flutter instability may occur more often than one might expect, not satisfying any hierarchical relation to other instabilities (such as for instance shear banding), possibly at an early stage of a hardening process and typically triggered by noncoaxiality (of the flow rule or induced by elastic or plastic anisotropy). However, the problem of finding a mechanical interpretation for the instability remained almost completely unexplored (with the exceptions of Bigoni and Willis, 1992 and Simões, 1997). This has been a major problem retarding further progress in research since, though generically believed to correspond to a dynamically growing disturbance, only the knowledge of the precise mechanical features of the instability can permit its identification for real materials.

To shed light on this problem, a perturbative approach is developed, following the methodology proposed by Bigoni and Capuani (2002; 2005) to investigate shear banding and other forms of material instabilities. In more detail, the analysis is limited to the loading branch [see Bigoni and Petryk (2002) for a discussion on this delicate assumption] of an elastoplastic constitutive operator (taken from Bigoni and Petryk, 2002) embodying features typical of the behaviour of granular materials and capable of exhibiting flutter instability. An infinite body is considered made up of this material, homogeneously and quasi-statically deformed in two dimensions (plane strain or generalized plane stress). For this configuration, a time-harmonic Green's function is found, representing the first Green's function obtained for a nonsymmetric constitutive equation [A quasi-static Green's function for unsymmetric constitutive equation has been developed by Bertoldi et al. (2005), but this is unsuitable for flutter anaylses, since this instability is essentially dynamic and thus remains unrevealed under the quasi-static assumption. In addition, Bertoldi et al. (2005) also derive boundary integral equations under the unsymmetric constitutive assumption, which are shown to possess certain typical features although not directly connected to the present discussion]. The Green's function is employed to form a pulsating dipole (two equal and opposite forces having a magnitude varying sinusoidally with time) to be used as a dynamic perturbation revealing effects of flutter.

Results demonstrate the following features of flutter instability *occurring in a material for which the tangent constitutive operator is positive definite* (so that negative second-order work and shear bands are excluded at the considered stress level).

i. Differently from shear bands, becoming already evident when the boundary of the region of ellipticity is approached from its interior (Bigoni and Capuani, 2002; 2005), flutter instability remains undetected while the eigenvalues of the acoustic tensor lie in the real range, appearing only after two real eigenvalues have coalesced and then become a complex conjugate pair;

ii. flutter instability corresponds to a disturbance blowing-up in space from the perturbing dipole and *self-organizing along well-defined plane waves*.

iii. the normals to the above plane waves lie within the fan of directions corresponding to flutter and have been found to have an inclination remarkably different from that corresponding to shear bands, occurring later in the hardening process.

It should be noted that the blow-up found in our solution will occur rapidly and nonlinearities neglected in our analysis (such as for instance the possibility of elastic unloading and plastic reloading) may soon become important, possibly changing the overall mechanical response. However, our results suggest that flutter instability should induce a layering in an initially homogeneous material, inducing a localization of strain in a form somehow similar –though occurring much earlier in a hardening process– to that pertaining to shear bands occurring in a dynamical context (Bigoni and Capuani, 2005). Our hope is that this feature revealed by our results has now been made accessible to experimental investigation.

2. THE OCCURRENCE OF FLUTTER

The occurrence of flutter instability is analyzed for the constitutive model presented in Bigoni and Petryk (2002); see also Bigoni and Loret (1999). For a given set of constitutive parameters, it is possible to study flutter for all the propagation directions **n** while varying the plastic modulus H/μ, by use of inequalities obtained by Bigoni and Loret (1999). Therefore, the ranges in which flutter occurs can be plotted in the plane H/μ versus ϑ_n, where the latter defines the direction of **n**. Restricting the analysis to the infinitesimal theory, analyses have been performed assuming different values of the Lode parameter ϑ_L = {60°, 30°, 0°, –30°, –60°} in the deviatoric stress space, as shown in Fig. 1.

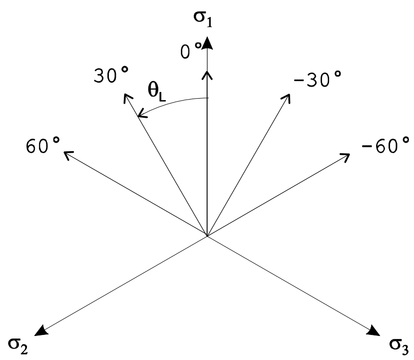

Fig. 1 Stress directions in the deviatoric plane, defined by the Lode angle, considered for flutter analysis

Results are reported in Figs. 2 and 3, the latter giving more detail for four of the cases reported in the former figure. Different stress paths defined by the values of the Lode angle reported in Fig. 1 are considered for different anisotropy inclination ϑ_σ (see Bigoni and Loret, 1999 or Bigoni and Petryk, 2002 for a definition) in Fig. 2 at given values of values of pressure sensitivity and dilatancy parameters, respectively, ψ = 30° and χ = 0° (Bigoni and Loret, 1999 or Bigoni and Petryk, 2002). In the graphs reported in Figs. 2 and 3, the closed contours denote regions where flutter occurs in the plane defined by the normalized critical plastic modulus H/μ and the inclination of propagation direction ϑ_n.

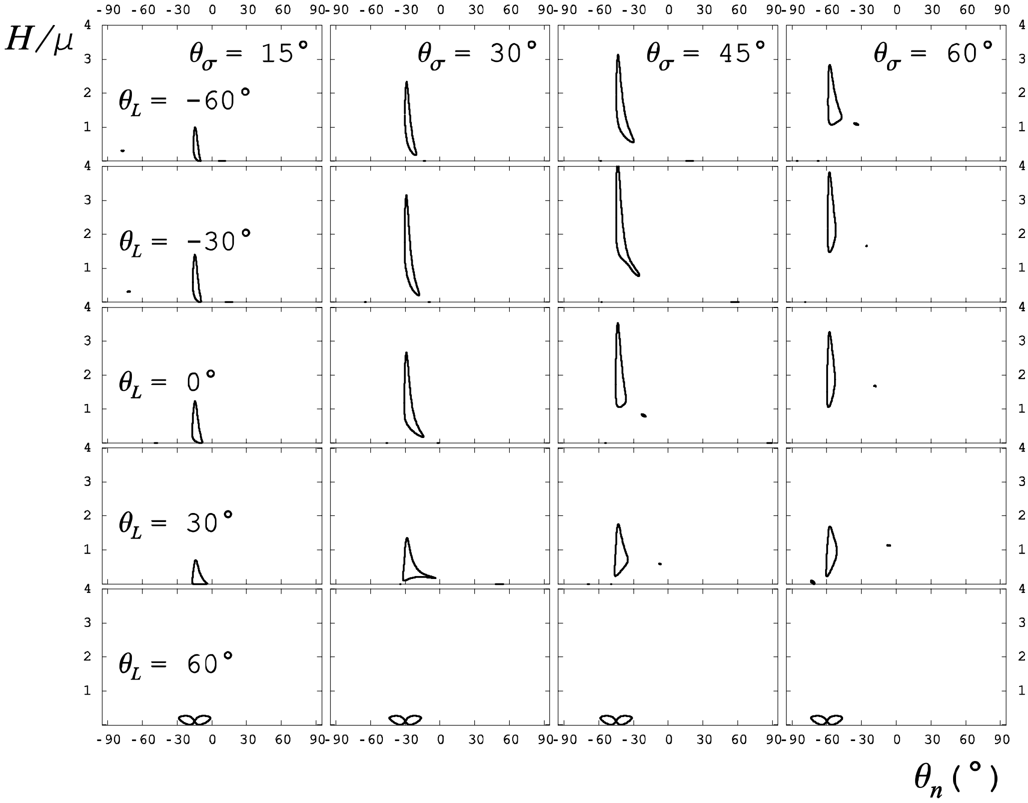

Fig. 2 Regions of flutter instability (occurring for internal points) in the H/μ vs. ϑ_n plane, for the stress paths shown in Fig. 1 at various anisotropy inclinations ϑ_σ.

Four details of Fig. 2 are reported in Fig. 3, where $\lambda/\mu = 1$ (elastic moduli playing a role similar to the usual Lamé constants), $\hat{b} = 80°$ (Bigoni and Loret, 1999 or Bigoni and Petryk, 2002), $\psi = 30°$ and $\chi = 0°$, as in Fig. 2. The six regions in Fig. 3 correspond to the four cases:

$\vartheta_L = 0°$ and $\vartheta_\sigma = 15°$ (Case 1),
$\vartheta_L = \vartheta_\sigma = 30°$ (Case 2),
$\vartheta_L = 0°$ and $\vartheta_\sigma = 45°$ (Case 3),
$\vartheta_L = 0°$ and $\vartheta_\sigma = 60°$ (Case 4).

With reference to the Cases 1,2,3 and 4, detailed in Fig. 3, we note that the critical values of hardening modulus for loss of positive definiteness of the constitutive operator H_{cr}^{PD} and for loss of ellipticity H_{cr}^{E} permitting shear bands with normal inclined at ϑ_{nE} are

$$
\begin{array}{llll}
\text{Case 1:} & H_{cr}^{PD}/\mu = 0.42, & H_{cr}^{E}/\mu = 0.19, & \vartheta_{nE} = -28.0° \\
\text{Case 2:} & H_{cr}^{PD}/\mu = 1.22, & H_{cr}^{E}/\mu = 0.18, & \vartheta_{nE} = -16.4° \\
\text{Case 3:} & H_{cr}^{PD}/\mu = 1.03, & H_{cr}^{E}/\mu = 0.74, & \vartheta_{nE} = -32.0° \\
\text{Case 4:} & H_{cr}^{PD}/\mu = 1.81, & H_{cr}^{E}/\mu = 1.57, & \vartheta_{nE} = -33.9°
\end{array}
\tag{1}
$$

so that in all cases flutter may initiate when the constitutive operator is positive definite (therefore at an early stage of a deformation process) and may extend in a region possibly involving loss of ellipticity.

Note that thresholds (1) have been graphically represented in Fig, 3, where light grey regions correspond to regions where flutter may occur with the constitutive operator still positive definite, while in the dark grey regions ellipticity is lost (horizontal lines marking ellipticity loss are denoted with “E (case i)”, where i = 1,..,4 stands for the number of the relevant Case). In the same figure, three black spots and a white spot (referred to Case 2) indicate the inclinations of shear bands at first loss of ellipticity. Note that the small flutter regions of cases 3 and 4 are beyond the positive definiteness threshold, but still in the elliptic region.

It may be important to remark that *the initial inclinations of propagation normals for flutter and shear bands are unrelated and result remarkably different*.

From the above analysis it can be deduced that the constitutive model allows one to approach flutter starting from a well-behaved state. Moreover, it may be interesting to note from Fig. 3 that there are overlapping regions corresponding to different stress states (Cases 1 and 2). In these zones the flutter may have identical characteristics even if the stress state is different.

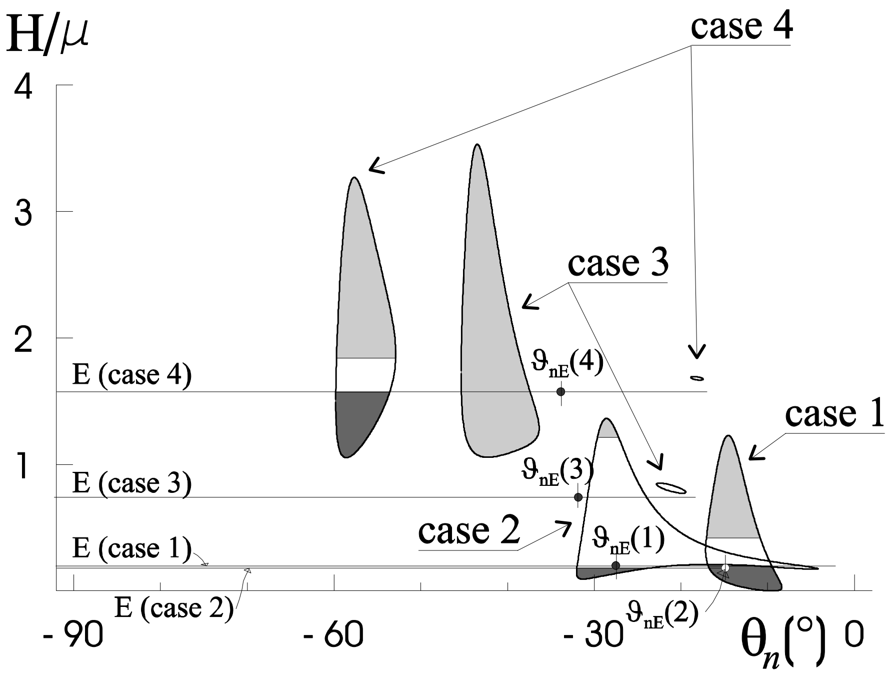

Fig. 3 Regions of flutter instability (occurring for internal points) in the H /μ vs. ϑ_n plane, for the stress paths shown in Fig. 1 at various anisotropy inclinations ϑ_σ. The regions of positive definiteness of the constitutive operator are marked in light grey, while (E) denotes loss of ellipticity into shear bands (regions shaded in dark grey) inclined at ϑ_{nE}(i), where i=1,..,4 denotes the relevant Case, see (1).

3. THE DYNAMIC TIME-HARMONIC GREEN'S FUNTION FOR GENERAL NONSYMMETRIC CONSTITUTIVE EQUATIONS

An initial static homogeneous deformation of an infinite body is considered, satisfying equilibrium in terms of first Piola-Kirchhoff stress and taken as the reference state in an updated Lagrangian formulation. A dynamic perturbation is superimposed upon this state, defined by an incremental displacement **u** satisfying the equations of incremental motion, written with reference to the constitutive equation employed by Bigoni and Petryk (2002), in which "dots" over symbols are to be interpreted now as incremental quantities rather than rates. Thus

$$C_{ijkl}\, u_{k,lj} + f_i = \rho\, u_{i,tt} \qquad (2)$$

where $,t$ denotes material time derivative and f_i and ρ are the incremental body forces and the mass density, respectively. Equations (2) look like ordinary elastodynamics, except that C_{ijkl} has neither the usual major nor the minor symmetries. Note that tensor C_{ijkl} can be identified (and will be in the examples) with that provided by Bigoni and Petryk (2002), but can also be thought completely arbitrary in the following.

To investigate the properties of eqn. (2), outside and inside the flutter region we follow the Bigoni and Capuani (2005) approach, based on the determination of the dynamic Green's function, sought for simplicity under the time-harmonic assumption

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x})\, e^{-i\,\omega t}, \qquad \mathbf{f}(\mathbf{x}, t) = \hat{\mathbf{f}}(\mathbf{x})\, e^{-i\,\omega t}, \tag{3}$$

where ω is the circular frequency and t and $\mathbf{x}$ denote time and space variables, respectively, so that the time dependence can be removed from eqn. (2) and consequently

$$C_{ijkl}\, \hat{u}_{k,lj} + \rho\,\omega^2\, \hat{u}_i + \hat{f}_i = 0\,. \tag{4}$$

The Green's tensor $G_{ip}(\mathbf{x})$ is obtained by solving eqn. (4) under the hypothesis $\hat{f}_i = \delta_{ip}\,\delta(\mathbf{x})$, with $\delta(\mathbf{x})$ denoting the Dirac delta. In order to approach the flutter condition, we exploit a plane strain deformation analysis, in which only two relevant components of the Green's function appear.

The Green's function is determined employing a Radon transform technique, as proposed by Willis (1991). In particular, having introduced polar coordinates (so that the position vector $\mathbf{x}$ has modulus $r = |\mathbf{x}|$ and is inclined at angle ϑ to the x_1–axis), the two-dimensional, time-harmonic Green's function corresponding to a generic, completely non-symmetric constitutive (tangent) fourth-order tensor, can be written in the form

$$\mathbf{G}(\mathbf{x}) = -\frac{1}{4\pi^2} \tag{5}$$

$$\sum_{N=1}^{2}\int_0^{\pi}\left[2\cos(rk_N\,|\cos\alpha|)Ci(rk_N\,|\cos\alpha|) + 2\sin(rk_N\,|\cos\alpha|)Si(rk_N\,|\cos\alpha|) - i\pi\cos(rk_N\,|\cos\alpha|)\right]\frac{\mathbf{v}_N\otimes\mathbf{w}_N}{\rho\, c_N^2}\,d\alpha$$

where Ci and Si denote the cosin integral and the sin integral functions, $k_N = \omega/c_N$ and c_N are the eigenvalues of the acoustic tensor $\mathbf{A}(\mathbf{n})$ (relative to the constitutive incremental tensor) with corresponding left and right eigenvectors $\mathbf{w}_N$ and $\mathbf{v}_N$, all quantities depending on the propagation inclination $\mathbf{n}$ (which means on $\alpha + \vartheta$).

4. A DYNAMICAL INTERPRETATION OF FLUTTER INSTABILITY

The behaviour of the Green's function, eqn. (5), is analyzed here, outside and inside the flutter region. As a reference, we consider Case 3 shown in Fig. 3, in which the material is subject to the radial stress path corresponding to $\vartheta_L = 0$ in Fig. 1 and the direction of the axis of elastic symmetry is taken inclined at $\vartheta_\sigma = 45°$ with respect to the principal stress direction $\mathbf{k}_1$.

The employed material parameters are $\lambda/\mu = 1$, $\hat{b} = 80°$, $\psi = 30°$, and $\chi = 0°$. Dimensionless Green's tensor components have been computed for

$$\varpi = a\,\omega\sqrt{\frac{\rho}{\mu}} = 1,$$

where a is a characteristic length, and for several values of the plastic modulus H/μ, including the values 3.53 and 1.5. These correspond, respectively, to situations near and inside the flutter region (see Fig. 3), but still in a situation where the second order work is positive definite. The values of the components are plotted in Fig. 4 as functions of the distance from the singularity along a radial line inclined at $-45°$ with respect to the x_1–axis, normalized through division by a.

The real (imaginary) parts of the Green's function components are plotted left (right) in the figure and the plots having been obtained starting from $x_1 = 1/10$ to exclude the singularity (in the real components of the Green's tensor).

Commenting the results:

- first, we note from the figure that the Green's tensor is not symmetric (since the acoustic tensor is not), so that $G_{12} \neq G_{21}$;
- second, results referring to values of plastic mudulus H/μ higher than 3.53 and up to 7 (not reported here for conciseness) produce curves practically coincident to those pertaining to H/μ = 3.53, we can therefore conclude that there is not much difference between the situations in which the material is far from and very near to the flutter region. This feature has been confirmed by us with several calculations (not reported here) and distinguishes flutter from shear banding, the latter becoming already visible when the condition of loss of ellipticity is approached from the interior of the elliptic range (Bigoni and Capuani, 2002; 2005);
- third, a blow-up of the solution with the space variable, clearly visible in all components of the Green's tensor is the characteristic feature of instability inside the flutter region, H/μ = 3.53. This blow-up is similar to that evidenced by Bigoni and Willis (1994), but in a constitutive setting including viscosity, which is now absent.

Further exploration of flutter instability requires plotting of incremental displacement maps produced by a pulsating dipole, in a way similar to (Bigoni and Capuani, 2005). This (not reported here for conciseness) reveals the above-determined spatial blow-up corresponds to the formation of deformation patterns exhibiting well-defined directional properties, determined by the wave directions for which the eigenvalues become complex conjugate.

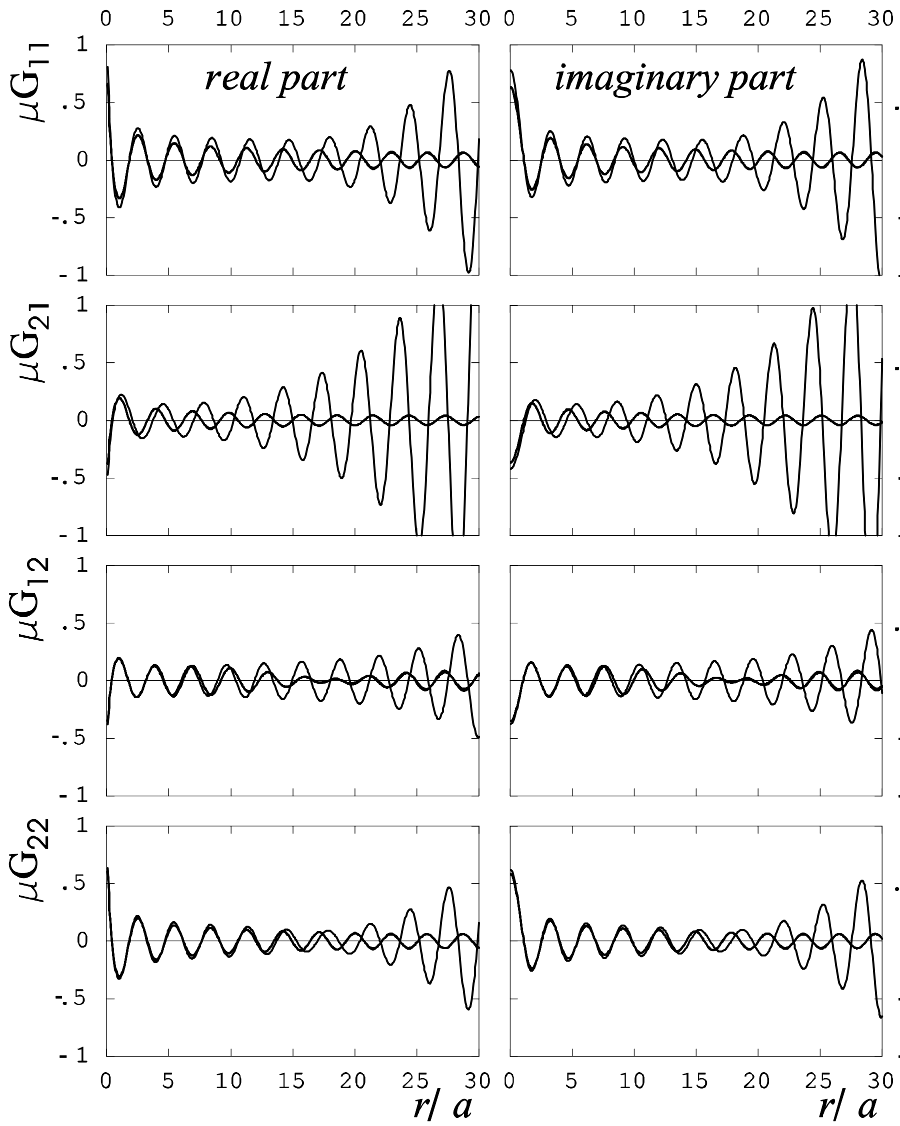

Fig. 4 Dimensionless Green's tensor components (real part left, imaginary part right in the figure) along a radial line inclined at –45° with respect to the x_1–axis for Case 3 of Fig. 3 and $\varpi = 1$. Two values of the plastic modulus $H/\mu = \{3.53, 1.5\}$ are considered, corresponding, respectively, to situations near and inside the flutter region. The blow-up of all components of the Green's tensor is evident in the flutter region, $H/\mu = 1.5$.

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