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Cones of localized shear strain in incompressible elasticity with prestress: Green's function and integral representations

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Abstract

Infinite-body three-dimensional Green's function set (for incremental displacement and mean stress) is derived for the incremental deformation of a uniformly strained incompressible, nonlinear elastic body. Particular cases of the developed formulation are the Mooney-Rivlin elasticity and the J_2-deformation theory of plasticity. These Green's functions are used to develop a boundary integral equation framework, by introducing an 'ad hoc' potential, which paves the way for a boundary element formulation of three-dimensional problems of incremental elasticity. Results are used to investigate the behaviour of a material deformed near the limit of ellipticity and to reveal patterns of shear failure. In fact, within the investigated three-dimensional framework, localized deformations emanating from a perturbation are shown to be organized in conical geometries rather than in planar bands, so that failure is predicted to develop through curved and thin surfaces of intense shearing, as can for instance be observed in the cup-cone rupture of ductile metal bars.

Keywords: nonlinear elasticity; shear bands; boundary element method; prestressed material; anisotropy; incompressible elasticity.

1 Introduction

The response of a homogeneously deformed nonlinear elastic solid to a perturbing agent is the key to the investigation of several important problems, such as for instance, bifurcation of an elastic block [1, 2], or layered structures [3–6], wave propagation [7, 8], near-crack stress field determination [9–12], and shear band development [13, 14]. In these investigations, the availability of an infinite-body Green's function allows the treatment of 'complex' problems (for instance, the stress field near a dislocation in a prestressed solid [15]) and permits the development of boundary integral equations and the related boundary element techniques. Despite its importance, the first Green's function set for incompressible homogeneously deformed elastic solids has been provided by Willis [16] and explicitly derived for two-dimensional elasticity by Bigoni & Capuani [17]. Later, Green's functions have been derived for time-harmonic problems [18, 19] and boundary element formulations have been proposed [20–22]. However, all these results are

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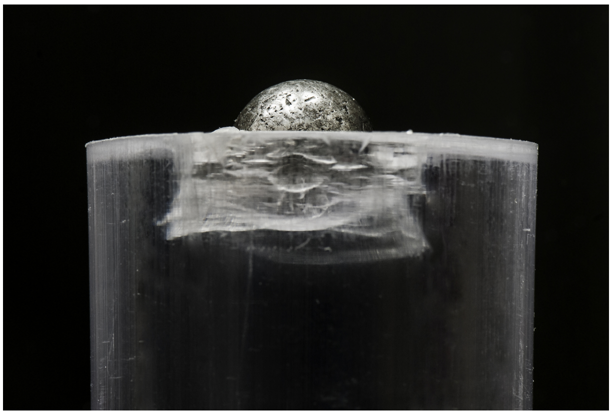

Figure 1: Conical fracture produced by a spherical indenter (14 mm diameter) at an indentation speed of 8 mm s^{-1} in a polycarbonate cylindrical specimen at 0 °C (experiment performed by the authors at the Instabilities Lab. of the University of Trento).

restricted to plane elasticity, so that the only contribution valid for a three-dimensional context still remains that of Willis [16].

The purpose of the present article is to derive infinite-body Green's functions for incremental displacement and incremental mean stress for a nonlinear elastic incompressible solid deformed homogeneously. Based on these Green's functions, the boundary integral equation set for the incremental response of a homogeneously prestressed elastic solid is derived, which provides the basis for boundary element techniques. These results [1] generalize Bigoni & Capuani [17] to three-dimensional elasticity and apply, as particular cases, to Mooney-Rivlin elasticity and J_2–deformation theory of plasticity. The latter case allows for the analysis of localized shear deformation as induced by a perturbing force dipole in a three-dimensional elastic context. In this case, the incremental displacements are shown to be localized along cones of concentrated incremental shear strains, which differ from the 'usual' planar shear band geometry. This result may explain the well-known cup-cone failure of ductile metal bars (see for instance [24]) and the conical failure zone observed by Desrues *et al*. [25] in cylindrical specimen of granular material. Moreover, it may be related to the mechanisms of conical fracturing observed in brittle materials (such as glass, see Lawn [26], and/or polycarbonate, Figure. 1) and rocks subjected to impact (for instance, the shatter cones found in shocked rocks near meteorite impact or underground nuclear test sites, see French [27] and Sagy et al. [28]).

2 The infinite body Green's function set

2.1 Constitutive assumptions and field equations

In a relative Lagrangean description, a prestressed elastic solid is characterized by a linear relation (see Bigoni [29] for details) between the increment (denoted with a superposed dot) of nominal stress t_{ij} and the gradient of incremental displacement $v_{i,j}$

$$\dot{t}_{ij} = \dot{p}\delta_{ij} + \mathbb{K}_{ijkl}\, v_{l,k} \,, \tag{1}$$

where δ_{ij} is the Kronecker delta and $\dot{p}$ represents the incremental mean stress ($p = \operatorname{tr}\boldsymbol{T}/3$, with $\boldsymbol{T}$ being the Cauchy stress), which plays the role of a Lagrangean multiplier, related to the

[1] The results presented in this paper can be applied as a particular case (with a change in notation) to the linearized viscous flow of an incompressible *orthotropic* fluid, for which we give for the first time infinite body Green's function and boundary equation sets. This may open a perspective in the development of boundary element techniques for liquid crystals or nematic elastomers [23].

incompressibility constraint, namely, the requirement that the velocity field v_i be solenoidal

$$v_{k,k} = 0\,. \tag{2}$$

Note that the incremental constitutive tensor $\mathbb{K}_{ijkl}$ does not possess the minor symmetries and the major symmetry follows from an incremental potential $W(\nabla \boldsymbol{v})$, namely,

$$\mathbb{K}_{ijkl} = \frac{\partial^2 W}{\partial v_{j,i} \partial v_{l,k}}\,, \tag{3}$$

which is assumed to exist in the following.

The incremental equilibrium equations are

$$\dot{t}_{ij,i} + \dot{f}_j = 0\,, \tag{4}$$

where $\dot{f}$ is the increment of body force.

2.2 The Green's function set

The Green's function set encloses a Green's function for incremental displacements v_i^g and one for incremental mean stress $\dot{p}^g$, so that the Green's stress can be evaluated as

$$\dot{t}_{ij}^g = \dot{p}^g \delta_{ij} + \mathbb{K}_{ijkl}\, v_{l,k}^g\,, \tag{5}$$

and satisfies the field equation

$$\dot{t}_{ij,i}^g + \delta_{jg} \delta(\boldsymbol{x}) = 0\,, \tag{6}$$

where $\delta(\boldsymbol{x})$ is the three-dimensional delta function, and $\boldsymbol{x}$ is the generic material point.

Taking into account the definition of the Green's function set, equation (5), the equilibrium equation (4), with the incremental body force replaced by a Dirac delta function, can be rewritten as

$$\mathbb{K}_{ijkl}\, v_{l,ki}^g + \dot{p}_{,j}^g + \delta_{jg} \delta(\boldsymbol{x}) = 0\,. \tag{7}$$

The plane-wave expansion on the unit sphere $|\boldsymbol{\omega}| = 1$ of the delta function

$$\delta(\boldsymbol{x}) = -\frac{1}{8\pi^2} \int_{|\boldsymbol{\omega}|=1} \delta''(\boldsymbol{\omega} \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{\omega}\,, \tag{8}$$

and of the Green's incremental displacement and mean stress

$$v_k^g(\boldsymbol{x}) = -\frac{1}{8\pi^2} \int_{|\boldsymbol{\omega}|=1} \hat{v}_k^g(\boldsymbol{\omega} \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{\omega}\,, \tag{9a}$$

$$\dot{p}^g(\boldsymbol{x}) = -\frac{1}{8\pi^2} \int_{|\boldsymbol{\omega}|=1} \hat{p}^g(\boldsymbol{\omega} \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{\omega}\,, \tag{9b}$$

can be used to convert equation (7) into

$$A_{jl}(\boldsymbol{\omega})(\hat{v}_l^g)''(\boldsymbol{\omega} \cdot \boldsymbol{x}) + \omega_j (\hat{p}^g)'(\boldsymbol{\omega} \cdot \boldsymbol{x}) + \delta_{jg} \delta''(\boldsymbol{\omega} \cdot \boldsymbol{x}) = 0\,, \tag{10}$$

where A_{jl} is the acoustic tensor,

$$A_{jl}(\boldsymbol{\omega}) = \omega_i \mathbb{K}_{ijkl}\, \omega_k\,, \tag{11}$$

which is symmetric since the incremental constitutive tensor $\mathbb{K}_{ijkl}$ possesses the major symmetry.

The incompressibility constraint, equation (2), in the transformed domain assumes the form

$$\omega_k (\hat{v}_k^g)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} = 0\,, \tag{12}$$

which can be differentiated with respect to the coordinate x_s to yield the following useful relation

$$\omega_s\omega_k (\hat{v}_k^g)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\omega} = 0\,. \tag{13}$$

In the elliptic range, the acoustic tensor is invertible, so that equation (10) can be written as

$$(\hat{v}_k^g)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_{kj}^{-1}(\boldsymbol{\omega})\,\omega_j(\hat{p}^g)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + A_{kg}^{-1}(\boldsymbol{\omega})\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0\,, \tag{14}$$

but from equation (13) a projection on $\boldsymbol{\omega}$ yields

$$\omega_k A_{kj}^{-1}(\boldsymbol{\omega})\,\omega_j(\hat{p}^g)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) + \omega_k A_{kg}^{-1}(\boldsymbol{\omega})\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = 0\,, \tag{15}$$

an equation which allows to obtain the expression for the derivative of the mean stress

$$(\hat{p}^g)^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = -\frac{\omega_k A_{kg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s}\,\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,, \tag{16}$$

and therefore the mean stress in the transformed domain

$$\hat{p}^g(\boldsymbol{\omega}\cdot\boldsymbol{x}) = -\frac{\omega_k A_{kg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s}\,\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,. \tag{17}$$

A substitution of equation (16) into equation (14) provides the second-order derivative of the velocity in the following form

$$(\hat{v}_k^g)^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \left[\frac{A_{kj}^{-1}(\boldsymbol{\omega})\,\omega_j\,\omega_t A_{tg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s} - A_{kg}^{-1}(\boldsymbol{\omega})\right]\delta^{''}(\boldsymbol{\omega}\cdot\boldsymbol{x})\,. \tag{18}$$

An integration of equations (18) and (16) and a subsequent anti-transformation yield

the Green's function set for an incompressible, elastic, prestressed solid

$$v_k^g(\boldsymbol{x}) = -\frac{1}{8\pi^2 r}\int_{|\boldsymbol{\omega}|=1}\left[\frac{A_{kj}^{-1}(\boldsymbol{\omega})\,\omega_j\,\omega_t A_{tg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s} - A_{kg}^{-1}(\boldsymbol{\omega})\right]\delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega}\,, \tag{19}$$

$$\dot{p}^g(\boldsymbol{x}) = \frac{1}{8\pi^2 r^2}\int_{|\boldsymbol{\omega}|=1}\frac{\omega_k A_{kg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s}\,\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega}\,, \tag{20}$$

where $r = |\boldsymbol{x}|$, $\boldsymbol{e}_r = \boldsymbol{x}/r$, holding for a symmetric and invertible, acoustic tensor $A_{ij}(\boldsymbol{\omega})$.

The Green's incremental nominal stresses can be obtained from equation (5) employing the gradient of Green's incremental displacements

$$v_{k,l}^g(\boldsymbol{x}) = -\frac{1}{8\pi^2 r^2}\int_{|\boldsymbol{\omega}|=1}\omega_l\left[\frac{A_{kj}^{-1}(\boldsymbol{\omega})\,\omega_j\,\omega_t A_{tg}^{-1}(\boldsymbol{\omega})}{\omega_r A_{rs}^{-1}(\boldsymbol{\omega})\,\omega_s} - A_{kg}^{-1}(\boldsymbol{\omega})\right]\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega}\,. \tag{21}$$

Note the following:

- If instead of the constitutive equation (3) and of the equilibrium equation (1), the following

$$\dot{S}_{ij} = \dot{p}\delta_{ij} + \mathbb{G}_{ijkl}\, v_{k,l}\,, \tag{22a}$$

$$\dot{S}_{ij,j} + \dot{f}_i = 0\,, \tag{22b}$$

(where $\dot{S}_{ij} = \dot{t}_{ji}$ is the increment of the first Piola-Kirchhoff stress) are employed, the Green's functions (19) and (20) do not change, but the acoustic tensor changes its definition into

$$A_{ik}(\boldsymbol{\omega}) = \omega_j \mathbb{G}_{ijkl}\, \omega_l\,. \tag{23}$$

- To obtain equations (19) and (20), the well-known property (see, for example, Gel'fand and Shilov [30, p. 213, equation (20)]) of the delta function was used

$$\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{x}) = \frac{1}{r^2}\delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,. \tag{24}$$

- From equation (19), we can note that the following symmetry between indices k and g holds

$$v_k^g = v_g^k\,, \tag{25}$$

  if and only if the acoustic tensor A_{ij} is symmetric, a property following from the major symmetry of $\mathbb{K}_{ijkl}$.

3 Evaluation of the plane wave expansion integrals

The application of equations (19)–(21) requires the evaluation of integrals containing the delta function and its first and second derivatives. To this purpose, it is useful to introduce the two reference systems shown in Figure 2, where the system defined by the unit vectors triad $\{\, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 \,\}$ is centred at $\boldsymbol{x}$ and chosen as

$$\tilde{\boldsymbol{e}}_1 = \frac{x_2\boldsymbol{e}_1 - x_1\boldsymbol{e}_2}{\sqrt{x_1^2+x_2^2}}\,, \tag{26a}$$

$$\tilde{\boldsymbol{e}}_2 = \frac{x_1x_3\boldsymbol{e}_1 + x_2x_3\boldsymbol{e}_2 - \left(x_1^2+x_2^2\right)\boldsymbol{e}_3}{r\sqrt{x_1^2+x_2^2}}\,, \tag{26b}$$

$$\tilde{\boldsymbol{e}}_3 = \frac{x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3}{r} = \boldsymbol{e}_r\,, \tag{26c}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the unit vectors defining the reference system with the origin at the application point of the concentrated force. In the following the components in the reference system $\boldsymbol{e}_i$ (reference system $\tilde{\boldsymbol{e}}_i$) will be denoted with Latin (Greek) letters, so that the unit vector $\boldsymbol{\omega}$ can be written as

$$\{\, \omega_\alpha \,\} = \{\, \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi \,\}\ . \tag{27}$$

Introducing the rotation matrix

$$[\boldsymbol{Q}] = \frac{1}{r\sqrt{x_1^2+x_2^2}}\begin{bmatrix} rx_2 & x_1x_3 & x_1\sqrt{x_1^2+x_2^2} \\ -rx_1 & x_2x_3 & x_2\sqrt{x_1^2+x_2^2} \\ 0 & -x_1^2-x_2^2 & x_3\sqrt{x_1^2+x_2^2} \end{bmatrix}\,, \tag{28}$$

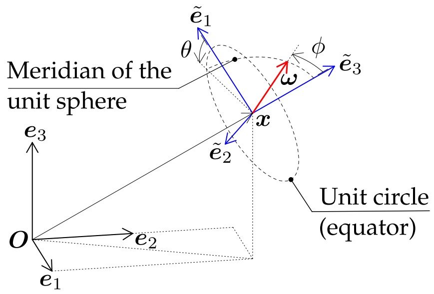

Figure 2: Reference system for the evaluation of the plane wave expansion integrals (19), (20) and (65). The unit vector $\boldsymbol{\omega}$, shown in red, defines a unit spherical surface centred at $\boldsymbol{x}$. The dashed circle and the dashed arc define, respectively, the equator and the meridian related to $\boldsymbol{\omega}$ within the local reference system $\{\, \tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 \,\}$.

the integral in the Green's function for incremental displacement (19) can be expressed as

$$\int_{|\boldsymbol{\omega}|=1} V_{gk}(\boldsymbol{\omega})\, \delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\, \mathrm{d}\boldsymbol{\omega} = Q_{g\alpha}Q_{k\beta}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} V_{\alpha\beta}(\theta,\phi)\, \delta(\cos\phi)\sin\phi\, \mathrm{d}\phi\,, \tag{29}$$

where

$$V_{gk}(\boldsymbol{\omega}) = \frac{A^{-1}_{kj}(\boldsymbol{\omega})\, \omega_j\, \omega_t A^{-1}_{tg}(\boldsymbol{\omega})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega})\, \omega_s} - A^{-1}_{kg}(\boldsymbol{\omega})\,, \tag{30}$$

which, in the reference system centred at $\boldsymbol{x}$, has the components $V_{\alpha\beta}$ with the transformed acoustic tensor

$$A_{\beta\delta} = \omega_\alpha Q_{i\alpha}Q_{j\beta}\mathbb{K}_{ijkl}Q_{k\gamma}Q_{l\delta}\omega_\gamma\,, \tag{31}$$

so that the Green's function for incremental displacements (19) can be expressed as

$$v^g_k(\boldsymbol{x}) = -\frac{1}{8\pi^2 r}Q_{g\alpha}Q_{k\beta}\int_0^{2\pi} V_{\alpha\beta}(\theta,\pi/2)\, \mathrm{d}\theta\,. \tag{32}$$

The integral in the Green's function for incremental mean stress (20) can be rewritten as

$$\int_{|\boldsymbol{\omega}|=1} P_g(\boldsymbol{\omega})\, \delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\, \mathrm{d}\boldsymbol{\omega} = Q_{g\alpha}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} P_{\alpha}(\theta,\phi)\, \delta^{'}(\cos\phi)\sin\phi\, \mathrm{d}\phi\,, \tag{33}$$

where

$$P_g(\boldsymbol{\omega}) = \frac{\omega_k A^{-1}_{kg}(\boldsymbol{\omega})}{\omega_r A^{-1}_{rs}(\boldsymbol{\omega})\, \omega_s}\,, \tag{34}$$

with the change in variable $y = \cos\phi$ and using the property of the derivative of the delta function we arrive at the Green's function for incremental mean stress

$$\dot{p}^g(\boldsymbol{x}) = \frac{1}{8\pi^2 r^2}Q_{g\alpha}\int_0^{2\pi} \left.\frac{\partial P_\alpha(\theta,\phi)}{\partial\phi}\right|_{\phi=\pi/2} \mathrm{d}\theta\,. \tag{35}$$

The integral in the gradient of the incremental displacement field (21) can be written as

$$\int_{|\boldsymbol{\omega}|=1} D_{gkl}(\boldsymbol{\omega})\, \delta^{'}(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\, \mathrm{d}\boldsymbol{\omega} = Q_{g\alpha}Q_{k\beta}Q_{l\gamma}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} D_{\alpha\beta\gamma}(\theta,\phi)\, \delta^{'}(\cos\phi)\sin\phi\, \mathrm{d}\phi\,, \tag{36}$$

so that the Green's function for the gradient of incremental displacements becomes

$$v^g_{k,l}(\boldsymbol{x}) = -\frac{Q_{g\alpha}Q_{k\beta}Q_{l\gamma}}{8\pi^2 r^2}\int_0^{2\pi} \left.\frac{\partial D_{\alpha\beta\gamma}(\theta,\phi)}{\partial\phi}\right|_{\phi=\pi/2} \mathrm{d}\theta\,. \tag{37}$$

3.1 Application to incompressible isotropic elasticity or Stokes flow

As a particular case, the Green's function and boundary integral equation sets are valid for incompressible isotropic elasticity, where equation (22) reduces to

$$\dot{\sigma}_{ij} = \dot{p}\delta_{ij} + \mu\left(v_{i,j} + v_{j,i}\right), \tag{38}$$

so that

$$\mathbb{K}_{ijkl} = \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big), \tag{39}$$

where μ is the shear modulus and

$$A_{jl}(\boldsymbol{\omega}) = \mu(\omega_j\omega_l + \delta_{jl}), \tag{40a}$$

$$A^{-1}_{jl}(\boldsymbol{\omega}) = -\frac{1}{2\mu}\omega_j\omega_l + \frac{1}{\mu}\delta_{jl}\,. \tag{40b}$$

The Green's function set becomes

$$v^g_k(\boldsymbol{x}) = \frac{\delta_{gk}}{4\pi r} - \frac{1}{8\pi^2 r}\int_{|\boldsymbol{\omega}|=1} \omega_g\,\omega_k\,\delta(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega}\,, \tag{41a}$$

$$\dot{p}^g(\boldsymbol{x}) = \frac{1}{8\pi^2 r^2}\int_{|\boldsymbol{\omega}|=1} \omega_g\,\delta'(\boldsymbol{\omega}\cdot\boldsymbol{e}_r)\,\mathrm{d}\boldsymbol{\omega}\,. \tag{41b}$$

In particular, we may write

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{4\pi r}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{1}{8\pi^2 r}\int_0^{2\pi}\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta & 0\\ \cos\theta\sin\theta & \sin^2\theta & 0\\ 0&0&0\end{bmatrix}\mathrm{d}\theta\,, \tag{42}$$

and

$$\dot{\boldsymbol{p}}(\boldsymbol{x}) = \frac{1}{8\pi^2 r}\int_0^{2\pi}\begin{bmatrix}0\\0\\-1\end{bmatrix}\mathrm{d}\theta\,, \tag{43}$$

which, integrated and rotated to the system $\boldsymbol{e}_i$, provide

$$v^g_k(\boldsymbol{x}) = \frac{1}{8\pi\mu\, r}\left(\delta_{kg} + \frac{x_k x_g}{r^2}\right), \tag{44a}$$

$$\dot{p}^g(\boldsymbol{x}) = \frac{x_g}{4\pi\, r^3}\,, \tag{44b}$$

representing, for Stokes flow, the well-known Stokeslet.

4 Boundary integral equations for homogeneously prestressed three-dimensional solids

The boundary integral equation for the incremental displacement of a uniformly prestressed nonlinear elastic body in the absence of body forces and subjected to mixed boundary conditions has been given by Bigoni & Capuani [17], with reference to a two-dimensional deformation. However, their result can be immediately generalized to three-dimensional deformation. In fact, with reference to a uniformly prestressed body subjected to the following incremental boundary conditions holding on non-overlapping parts ∂B_v and ∂B_τ of the boundary ∂B

$$v = \bar{v} \quad \text{on} \quad \partial B_v\,, \qquad \dot{t}_{ij} n_i = \dot{\tau}_j \quad \text{on} \quad \partial B_\tau\,, \tag{45}$$

the Betti identity written on incremental fields yields for the incremental displacement at the interior point $\boldsymbol{y}$

$$v_j(\boldsymbol{y}) C_j^g(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ij} n_i v_j^g(\boldsymbol{x}, \boldsymbol{y}) - \dot{t}_{ij}^g(\boldsymbol{x}, \boldsymbol{y}) n_i v_j \right] \mathrm{d}S_{\boldsymbol{x}}\,, \tag{46}$$

where

$$C_j^g(\boldsymbol{y}) = \lim_{\varepsilon \to 0} \int_{\partial C_\varepsilon} \dot{t}_{ij}^g(\boldsymbol{x}, \boldsymbol{y}) n_i \, \mathrm{d}S_{\boldsymbol{x}}\,, \tag{47}$$

is the C-matrix defined in the limit of vanishing radius ε of the sphere C_ε.

Note that the equilibrium equation (6) yields $C_j^g = \delta_{gj}$, so that the boundary integral equation for incremental displacements is obtained

$$v_g(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ij} n_i v_j^g(\boldsymbol{x}, \boldsymbol{y}) - \dot{t}_{ij}^g(\boldsymbol{x}, \boldsymbol{y}) n_i v_j \right] \mathrm{d}S_{\boldsymbol{x}}\,. \tag{48}$$

For points $\boldsymbol{y}$ on the boundary ∂B, where a corner can be present, we use the Green's stress (5), together with equations (34) and (37), and denote with

$$\boldsymbol{n} = - \begin{bmatrix} \cos\zeta \sin\xi \\ \sin\zeta \sin\xi \\ \cos\xi \end{bmatrix}\,, \tag{49}$$

the components of the inward normal to C_ε (enclosing the point $\boldsymbol{y}$), to obtain the following expression for the C-matrix

$$\begin{aligned} C_j^g(\boldsymbol{y}) = \frac{1}{8\pi^2} \int_0^{\Xi} \mathrm{d}\xi \int_0^{\aleph} \Bigg[& n_j Q_{g\alpha} \int_0^{2\pi} \frac{\partial}{\partial \phi} P_\alpha(\theta, \phi) \Big|_{\phi = \pi/2} \mathrm{d}\theta \\ & - \mathbb{K}_{ijkl} n_i Q_{g\alpha} Q_{k\beta} Q_{l\gamma} \int_0^{2\pi} \frac{\partial}{\partial \phi} D_{\alpha\beta\gamma}(\theta, \phi) \Big|_{\phi = \pi/2} \mathrm{d}\theta \Bigg] \sin\xi \, \mathrm{d}\zeta\,, \end{aligned} \tag{50}$$

where Ξ and $\aleph$ are the angles defining the corner in the boundary ∂B at $\boldsymbol{y}$. For a smooth boundary $\Xi = \pi/2$ and $\aleph = 2\pi$, so that $C_j^g = \delta_{jg}/2$.

Although equations (46)–(48) are formally identical to equations (57)–(59) of Bigoni & Capuani [17], the boundary integral equation for the incremental mean stress $\dot{p}(\boldsymbol{y})$ requires a complex derivation (since the result shown in Appendix B of Bigoni & Capuani is strictly limited to two-dimensional deformation), which is provided in the following through the introduction of an *ad hoc* potential Φ.

4.1 Boundary integral equation for the incremental mean stress

The boundary integral equation for the mean stress increment $\dot{p}(\boldsymbol{y})$ is the necessary complement to the equation for incremental displacements (48). This can be obtained with reference to the incremental mixed boundary conditions (45), through a double differentiation of equation (48) with respect to $\boldsymbol{y}$ and use of the incremental equilibrium equations (4) with null body forces to obtain

$$\dot{p}^g_{,h}(\boldsymbol{y}) = -\int_{\partial B} \mathbb{K}_{nhsg} \left[\dot{t}_{ij} n_i v^g_{j,sn}(\boldsymbol{x},\boldsymbol{y}) - \dot{t}^g_{ij,sn}(\boldsymbol{x},\boldsymbol{y}) n_i v_j \right] \mathrm{d}S_{\boldsymbol{x}} \,. \tag{51}$$

Repeated use of the incremental equilibrium equations (4) yields

$$\dot{p}^g_{,h}(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ig} n_i \dot{p}^g_{,h}(\boldsymbol{x},\boldsymbol{y}) - n_i v_j \mathbb{K}_{ijkg} \dot{p}^g_{,hk}(\boldsymbol{x},\boldsymbol{y}) + n_i v_i \mathbb{K}_{nhsg} \dot{p}^g_{,sn}(\boldsymbol{x},\boldsymbol{y}) \right] \mathrm{d}S_{\boldsymbol{x}} \,, \tag{52}$$

which, introducing the potential Φ defined as

$$\mathbb{K}_{sirg}\, \dot{p}^g_{,rs} = \Phi_{,i} \,, \tag{53}$$

becomes

$$\dot{p}^g_{,h}(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ig} n_i \dot{p}^g_{,h}(\boldsymbol{x},\boldsymbol{y}) - n_i v_j \mathbb{K}_{ijkg} \dot{p}^g_{,hk}(\boldsymbol{x},\boldsymbol{y}) + n_i v_i \Phi_{,h}(\boldsymbol{x},\boldsymbol{y}) \right] \mathrm{d}S_{\boldsymbol{x}} \,, \tag{54}$$

Equation (54) can be integrated to obtain *the boundary integral equation for the incremental mean stress*

$$\dot{p}^g(\boldsymbol{y}) = \int_{\partial B} \left[\dot{t}_{ig} n_i \dot{p}^g(\boldsymbol{x},\boldsymbol{y}) - n_i v_j \mathbb{K}_{ijkg} \dot{p}^g_{,k}(\boldsymbol{x},\boldsymbol{y}) + n_i v_i \Phi(\boldsymbol{x},\boldsymbol{y}) \right] \mathrm{d}S_{\boldsymbol{x}} \,, \tag{55}$$

complementing equation (48) and thus providing the boundary integral equation set for incompressible, prestressed elasticity. Now, the existence of potential Φ has to be proven and its form has to be determined.

4.1.1 The potential Φ

The necessary and sufficient condition for the existence of potential (53) is that

$$e_{jti} \frac{\partial}{\partial x_t} \left(\mathbb{K}_{sirg}\, \dot{p}^g_{,rs} \right) = 0 \,, \tag{56}$$

where e_{jti} is the Ricci alternating tensor, providing the vanishing of the following determinant

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathbb{K}_{s1rg}\, \dot{p}^g_{,rs} & \mathbb{K}_{s2rg}\, \dot{p}^g_{,rs} & \mathbb{K}_{s3rg}\, \dot{p}^g_{,rs} \end{vmatrix} = \boldsymbol{0} \,. \tag{57}$$

Condition (57) is equivalent to

$$\mathbb{K}_{sirg}\, \dot{p}^g_{,rsu} = \mathbb{K}_{surg}\, \dot{p}^g_{,rsi} \,, \tag{58}$$

a condition which can be proven to be true by differentiating the equilibrium equation

$$\mathbb{K}_{tipq}\, v^g_{q,pt} + \dot{p}^g_{,i} + \delta_{ig}\delta(\boldsymbol{x}) = 0 \,, \tag{59}$$

to obtain

$$\mathbb{K}_{tipq}\, v^g_{q,ptrs} + \dot{p}^g_{,irs} + \delta_{ig}\delta(\boldsymbol{x})_{,rs} = 0 \,, \tag{60}$$

so that

$$\mathbb{K}_{surg}\mathbb{K}_{tipq}\, v^g_{q,ptrs} + \mathbb{K}_{surg}\, \dot{p}^g_{,irs} + \mathbb{K}_{suri}\delta(\boldsymbol{x})_{,rs} = 0 \,. \tag{61}$$

The symmetry with respect to indices i and u in equation (61) follows from the identity (25), which is a consequence of the symmetry of the acoustic tensor (which is directly related to the symmetry of the incremental elastic tensor $\mathbb{K}_{ijkl}$). In this way the existence of Φ is proved under the condition that the incremental elastic tensor has the major symmetry, namely, $\mathbb{K}_{ijkl} = \mathbb{K}_{klij}$.

4.1.2 The form of the potential Φ:

Equation (53) in the transformed domain becomes

$$\omega_i \hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) = A_{ig}(\boldsymbol{\omega}) \left[\hat{p}^g(\boldsymbol{\omega} \cdot \boldsymbol{x})\right]'' , \tag{62}$$

so that the scalar product with $\omega_k A^{-1}_{ki}$ yields

$$\hat{\Phi}'(\boldsymbol{\omega} \cdot \boldsymbol{x}) = \frac{\omega_g \left[\hat{p}^g(\boldsymbol{\omega} \cdot \boldsymbol{x})\right]''}{\boldsymbol{\omega} \cdot \boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} , \tag{63}$$

and integration and use of the derivative of the mean stress, equation (16) leads to

$$\hat{\Phi}(\boldsymbol{\omega} \cdot \boldsymbol{x}) = -\frac{\delta''(\boldsymbol{\omega} \cdot \boldsymbol{x})}{\boldsymbol{\omega} \cdot \boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} . \tag{64}$$

Equation (64) can be antitransformed to provide the representation of function Φ

$$\Phi(\boldsymbol{x}) = \frac{1}{8\pi^2 r^3} \int_{|\boldsymbol{\omega}|=1} \frac{\delta''(\boldsymbol{\omega} \cdot \boldsymbol{e}_r)}{\boldsymbol{\omega} \cdot \boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} \,\mathrm{d}\boldsymbol{\omega} , \tag{65}$$

where $r = |\boldsymbol{x}|$, $\boldsymbol{e}_r = \boldsymbol{x}/r$, *holding for a symmetric and invertible acoustic tensor* $A_{ij}(\boldsymbol{\omega})$.

The integral in the potential function Φ, equation (65), can be rewritten as

$$\int_{|\boldsymbol{\omega}|=1} Z(\boldsymbol{\omega})\, \delta''(\boldsymbol{\omega} \cdot \boldsymbol{e}_r) \,\mathrm{d}\boldsymbol{\omega} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} Z(\theta, \phi)\, \delta''(\cos\phi) \,\mathrm{d}\phi , \tag{66}$$

where

$$Z(\boldsymbol{\omega}) = \frac{1}{\boldsymbol{\omega} \cdot \boldsymbol{A}^{-1}(\boldsymbol{\omega})\boldsymbol{\omega}} , \tag{67}$$

with the change in variable $y = \cos\phi$ and using the property of the derivative of the delta function we arrive at the expression for the potential Φ

$$\Phi(\boldsymbol{x}) = \frac{1}{8\pi^2 r^3} \int_0^{2\pi} \left.\frac{\partial^2 Z(\theta, \phi)}{\partial \phi^2}\right|_{\phi=\pi/2} \mathrm{d}\theta . \tag{68}$$

5 Conical localization of deformation

The aim of this section is to use the Green's function (19) as a perturbing agent to explore the conditions of a material prestressed near the boundary of ellipticity loss, corresponding to the formation of shear bands. To this purpose, a specific constitutive law for the material has to be introduced. We will refer to the J_2-deformation theory of plasticity.

5.1 J_2-deformation theory of plasticity

For the sake of simplicity, we consider here an elastic nonlinear material deformed and stressed in an axisymmetric way about the x_3–axis. In this case, following Bigoni & Gei [31],

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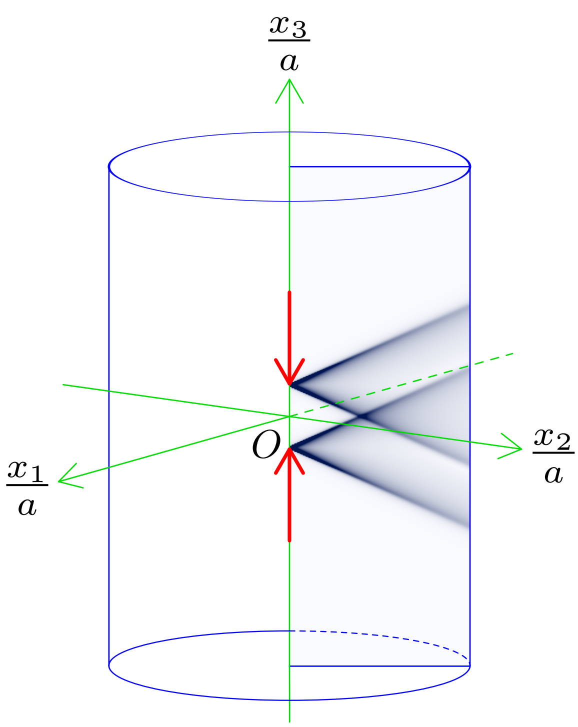


(a) *Rotation of the map.*

(b) *Corresponding solid field representation.*

Figure 3: Level sets of the modulus of the incremental displacement field (multiplied by Ka) generated by a force dipole in an incompressible material subjected to a high uniaxial compressive stress, close to the elliptic boundary. The dipole lies on the x_3–axis, parallel to the prestress direction. Note that the incremental displacement field is focused along four *conical* shear surfaces.

a general incremental constitutive equation can be written in terms of Jaumann increment of Cauchy stress $\overset{\nabla}{\boldsymbol{T}}$ and Eulerian strain increment $\boldsymbol{D}$ as

$$\overset{\nabla}{\boldsymbol{T}} = \overset{\nabla}{p}\boldsymbol{I} + \mathbb{E}[\boldsymbol{D}]\,, \tag{69}$$

where $\mathbb{E}$ is a fourth-order tensor possessing all symmetries, which can be represented in the form

$$\mathbb{E}_{ijkl} = \frac{\Gamma_1}{2}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) + \Gamma_2 G_{ij}G_{kl} + \Gamma_3\big(G_{ik}\delta_{jl} + \delta_{ik}G_{jl}\big) + \Gamma_4\delta_{ij}G_{kl}\,, \tag{70}$$

in which $G_{ij} = \delta_{i3}\delta_{j3}$ is the dyad corresponding to the symmetry axis and the parameters Γ_i ($i = 1, \ldots, 4$), function of the state variables, are subjected to the constraint

$$\Gamma_2 + 2\Gamma_3 + 3\Gamma_4 = 0\,. \tag{71}$$

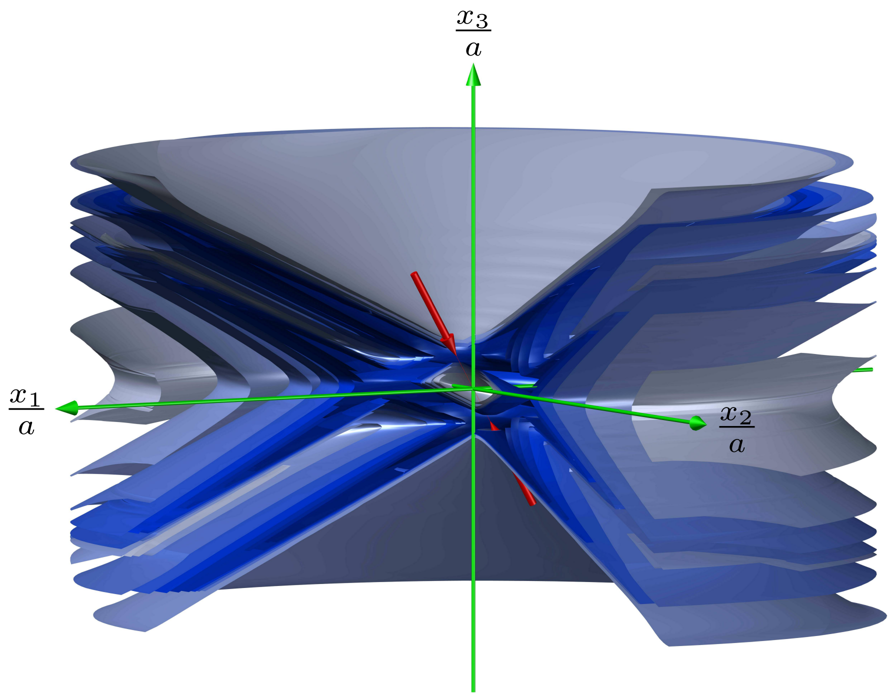
In a relative Lagrangean description equation (69) can be transformed into a relation involving the increment of nominal stress as follows

$$\dot{\boldsymbol{t}} = \dot{p}\,\boldsymbol{I} + \mathbb{E}[\boldsymbol{D}] - \boldsymbol{T}\boldsymbol{W} - \boldsymbol{D}\boldsymbol{T}\,, \tag{72}$$

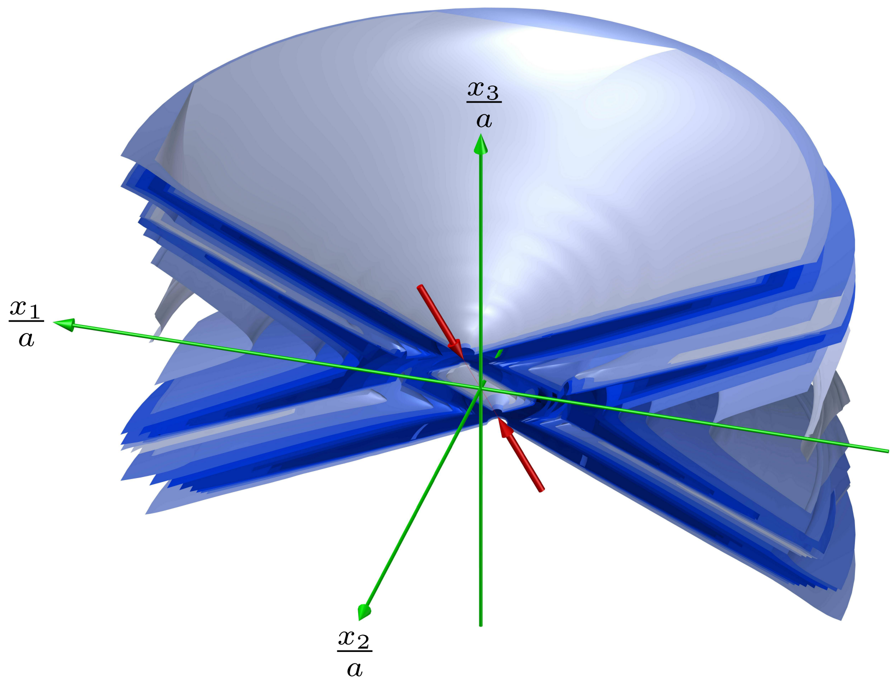
which compared to equation (22) yields the definition of $\mathbb{K}$ in terms of $\mathbb{E}$

$$\mathbb{K}_{ijkl} = \mathbb{E}_{ijkl} + \frac{1}{2}\big(T_{ik}\delta_{jl} - T_{il}\delta_{jk}\big) - \frac{1}{2}\big(T_{jl}\delta_{ik} + T_{jk}\delta_{il}\big)\,. \tag{73}$$

It is expedient now [31] to re-write parameters Γ_i in terms of the three incremental moduli μ_1,

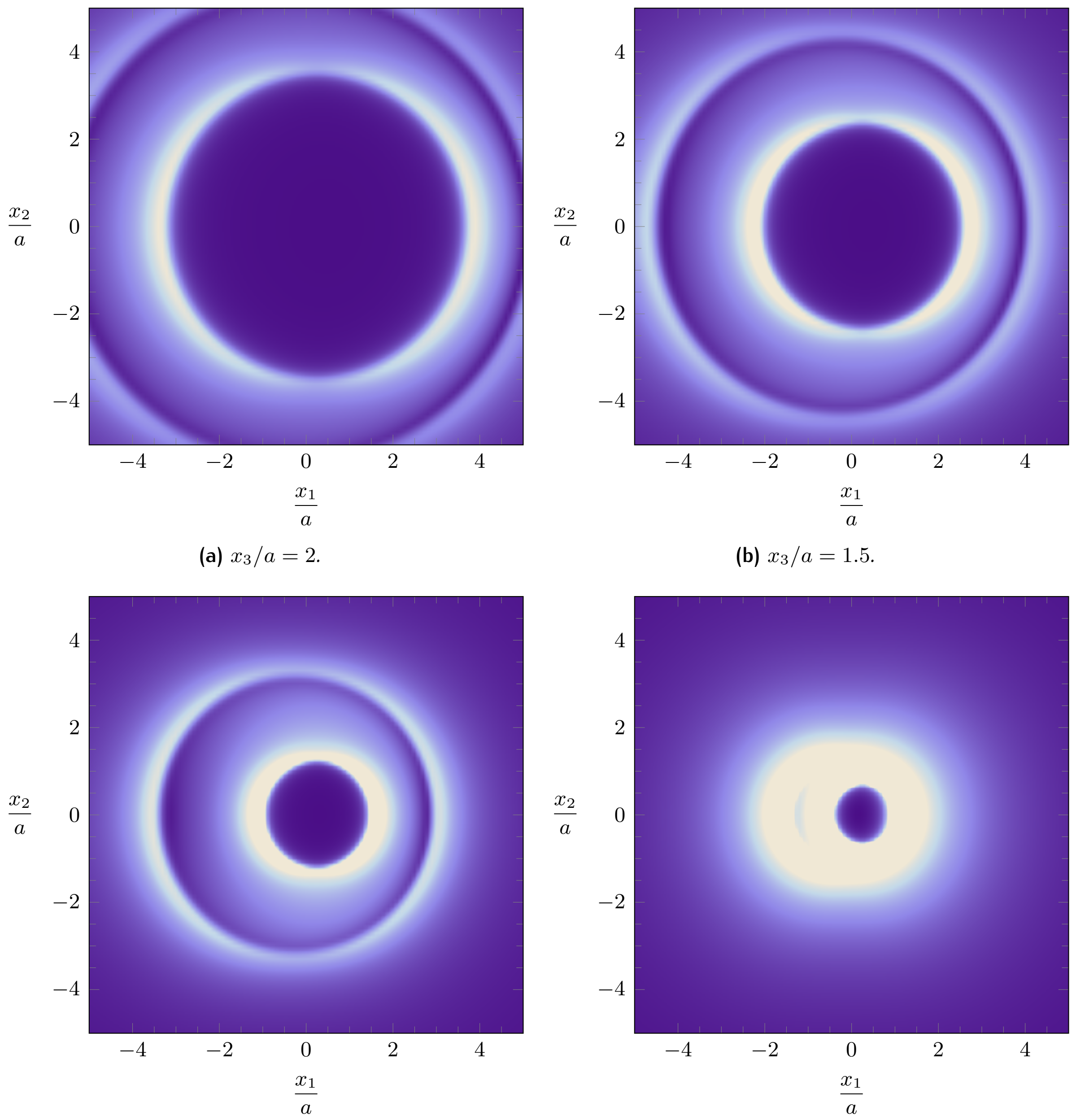


(a) *The field is 'cut' with two orthogonal planes, one of which contains the force dipole and the x_3–axis.*



(b) *The field is 'cut' with a plane containing the force dipole and the x_3–axis.*

Figure 4: Two different views of the level sets of the modulus of the incremental displacement (multiplied by Ka) field generated by a force dipole in an incompressible material subjected to a high uniaxial compressive stress, close to the elliptic boundary. The dipole lies on the x_1–x_3 plane and is inclined at $30°$ with respect to the x_3–axis, which is parallel to the prestress direction. Note that the incremental displacement field is focused along four *conical* shear surfaces.



(a) $x_3/a = 2$. **(b)** $x_3/a = 1.5$. **(c)** $x_3/a = 1$. **(d)** $x_3/a = 0.1$.

Figure 5: Section-cuts of the representation of the level sets of the modulus of the incremental displacement field generated by a force dipole shown in Figure 4. Cuts have been taken orthogonally to the x_3–axis.

μ_2 and μ_3 as

$$\Gamma_1 = 4\mu_2 - 2\mu_1\,, \tag{74a}$$
$$\Gamma_2 = 2\mu_1 + 2\mu_2 - 4\mu_3\,, \tag{74b}$$
$$\Gamma_3 = 2\mu_1 - 4\mu_2 + 2\mu_3\,, \tag{74c}$$
$$\Gamma_4 = 2\mu_2 - 2\mu_1\,, \tag{74d}$$

so that the constraint (71) is automatically satisfied. Referring for simplicity to a state of uniaxial Cauchy stress σ aligned parallel to the symmetry axis and introducing a cylindrical reference system with the z–axis coincident with the x_3–axis, tensor $\mathbb{K}$ can be shown to possess the following non-null components

$$\begin{aligned}
&\mathbb{K}_{rrrr} = \mathbb{K}_{\theta\theta\theta\theta} = 2\mu_2\,,\\
&\mathbb{K}_{rr\theta\theta} = \mathbb{K}_{\theta\theta rr} = 2(\mu_1 - \mu_2)\,,\\
&\mathbb{K}_{zzzz} = 2\mu_1 - \sigma\,,\\
&\mathbb{K}_{zrzr} = \mathbb{K}_{z\theta z\theta} = \mu_3 + \frac{\sigma}{2}\,,\\
&\mathbb{K}_{rzzr} = \mathbb{K}_{zrrz} = \mathbb{K}_{rzrz} = \mathbb{K}_{z\theta\theta z} = \mathbb{K}_{\theta zz\theta} = \mathbb{K}_{\theta z\theta z} = \mu_3 - \frac{\sigma}{2}\,,\\
&\mathbb{K}_{r\theta r\theta} = \mathbb{K}_{r\theta\theta r} = \mathbb{K}_{\theta rr\theta} = \mathbb{K}_{\theta r\theta r} = 2\mu_2 - \mu_1\,.
\end{aligned} \tag{75}$$

For the J_2-deformation of plasticity, defined by a hardening exponent $N \in (0,1]$ and a constitutive parameter $K/3$ representing an initial shear modulus (Bigoni [29]), the coefficients μ_i can be expressed as functions of the longitudinal stretch λ_3 in the form

$$\mu_1 = \frac{KN}{3}\varepsilon_{\mathrm{e}}^{N-1}\,, \tag{76a}$$
$$\mu_2 = \frac{K}{6}(N+1)\varepsilon_{\mathrm{e}}^{N-1}\,, \tag{76b}$$
$$\mu_3 = \frac{K}{2}\varepsilon_{\mathrm{e}}^{N-1}\frac{\lambda_3^3+1}{\lambda_3^3-1}\ln\lambda_3\,, \tag{76c}$$

where the effective strain ε_{e} is defined as $\varepsilon_{\mathrm{e}} = |\ln\lambda_3|$. Moreover, the axial Cauchy stress σ (representing a state of prestress when a perturbation in terms of a concentrated force is applied) can be written as

$$\sigma = K\varepsilon_{\mathrm{e}}^{N-1}\ln\lambda_3\,, \tag{77}$$

so that the state of strain and the uniaxial stress are controlled by the axial stretch λ_3.

The two (one corresponding to tension and another to compression) critical stretches for ellipticity loss are the two solutions of the following nonlinear equation

$$1 - \frac{6\varepsilon}{3N+1}\coth\left(\frac{3\varepsilon}{2}\right) + \left(\frac{3\ln\varepsilon}{3N+1}\right)^2 = 0\,, \qquad \text{provided} \quad |\varepsilon| > N + \frac{1}{3}\,, \tag{78}$$

where $\varepsilon = \ln\lambda_3$, so that for $N = 0.4$ the critical logarithmic strains for failure of ellipticity are $\varepsilon = \pm 1.0891$, corresponding to the two critical stretches 2.9716 and $0.336\,52$. Once the critical logarithmic strain for ellipticity loss is know from equation (78), coefficients (76) and the prestress (77) can be calculated, so that the shear band inclination ϕ can be obtained by solving the following equation

$$\left(1-\frac{\sigma}{2\mu_3}\right)\tan^4\phi+\left(\frac{\mu_1}{\mu_3}+\frac{\mu_2}{\mu_3}-1\right)\tan^2\phi+\left(1+\frac{\sigma}{2\mu_3}\right)=0\,. \tag{79}$$

5.2 The perturbed displacement fields

We are now in a position to analyse the effect of a force dipole (two equal and opposite forces at a distance a) as an agent perturbing an infinite elastic incompressible media, prestrained with a stretch $\lambda_3 = 0.337$ and obeying the J_2-deformation theory of plasticity with a value of the hardening parameter $N = 0.4$. For this hardening exponent the critical stretch for failure of ellipticity in compression is $0.336\,52$, a value very close to that employed to generate the maps of incremental displacements (that have been obtained through superposition and subsequent implementation of equation (19)).

We consider first the situation in which the dipole is aligned with the x_3–axis, as sketched in Figure 3a. Due to the axisymmetric conditions, it is possible to plot the modulus of the incremental displacements (made dimensionless through multiplication by Ka) only in the plane Ox_2x_3 (Figure 3a), so that a solid representation can be obtained through rotation about the x_3–axis (Figure 3b). Note that the modulus of the displacement field is focused along four cones, representing *localized conical zones of intense shear deformation*. At ellipticity loss, equation (79) provides an inclination for the shear bands equal to $66.16°$ with respect to the x_3–axis, which clearly agrees with the inclinations of the conical surfaces shown in Figure 3.

The perturbing force dipole has been assumed inclined at an angle of $30°$ to produce the incremental displacement maps shown in Figure 4. Even if the symmetry is now broken, conical shear surfaces are formed, again with an inclination almost coincident with that predicted at failure of ellipticity.

The conical shear surfaces can also be appreciated from Figure 5, where three section-cuts of Figure 4 are reported, taken orthogonal to the x_3–axis at $x_3/a = 0.1$, 1, 1.5, and 2 (Figures 5d, 5c, 5b, and 5a, respectively). We can note that higher shear deformations are localized in correspondence with the applied forces, so that two sorts of 'menisci' are visible. Moreover, the cones of intense deformations are not coaxial.

We may conclude that the formation of conical zones of intense shearing strain is a typical feature of failure of ductile materials as induced by a small defect or perturbation (in this case a force dipole). Therefore, the presented results may provide an explanation for the cup-cone failure mechanism typically observed in ductile bars under tension.

6 Conclusions

The infinite-body Green's function set (incremental displacements and incremental mean stress) and the related boundary integral equations have been obtained for the incremental response of a nonlinear elastic solid prestrained homogeneously. These results, essential to build boundary element solutions for nonlinear elasticity, have been employed to show how a material deformed near the boundary of ellipticity loss behaves when perturbed with a force dipole. This perturbation reveals the formation of cones (instead of the usual 'bands') of localized shear deformation, which may explain the formation of shear surfaces during failure of ductile materials.

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