

# Anisotropic effective higher-order response of heterogeneous Cauchy elastic materials

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## Abstract

The homogenization results obtained by Bacca et al. (Mindlin second-gradient elastic properties from dilute two-phase Cauchy-elastic composites. Part I: Closed form expression for the effective higher-order constitutive tensor. *Int. J. Sol. Struct.*, DOI: 10.1016/j.ijsolstr.2013.08.014), to identify the effective second-gradient elastic materials from heterogeneous Cauchy elastic solids, are extended here to the case of phases having non-isotropic tensors of inertia. It is shown that the nonlocal constitutive tensor for the homogenized material depends on both the inertia properties of the RVE and the difference between the effective and the matrix local elastic tensors. Results show that: (i.) orthotropic nonlocal effects follow from homogenization of a dilute distribution of aligned elliptical holes and, in the limit case, of cracks; (ii.) even under the dilute assumption and isotropic local effective behaviour, homogenization may lead to effective nonlocal orthotropic properties.

*Keywords:* Second-order homogenization; Higher-order elasticity; Effective nonlocal continuum; Characteristic length-scale; Cracked materials, Dilute suspension.

## 1 Introduction

Microstructures introduce length-scales and nonlocal effects in the mechanical modeling of solids, requiring the use of higher-order theories, in particular when large strain gradients are involved [as in the case when material instabilities (Dal Corso and Willis, 2011; Piccolroaz et al. 2006) or cracks occur (Mishuris et al., 2012)].

Nonlocality, often introduced phenomenologically (Cosserat and Cosserat, 1909; Koiter, 1964; Mindlin, 1964; Aifantis, 1978), can be analyzed from the fundamental and scarcely explored<sup>1</sup> point of view in which it is linked to the microstructure of a Cauchy elastic heterogeneous material via homogenization theory. This approach was followed by Bigoni and Drugan

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<sup>1</sup>The attempt of relating the microstructure to higher-order effect goes back to Dean and Urgate (1968) and Banks and Sokolowski (1968) and has been pursued, among others, by (Berglund, 1982; Pideri and Seppecher, 1997; Forest, 1998; Wang and Stronge, 1999; Ostoja-Starzewski et al., 1999; Bouyge et al., 2001; Auffray et al. 2010; Forest and Trinh, 2011; Dal Corso and Deseri, 2013).

(2007) and extended by Bacca et al. (2013a) who obtained a closed-form solution for the non-local effective response through a second-order identification procedure, based on the dilute approximation.<sup>2</sup>

Although the dilute assumption is often associated to *randomness* of the inclusions' distribution, the effective properties derived under that assumption still hold for periodic suspensions. Therefore, the overall local response is independent of the spatial distribution of inclusions in the dilute limit, in other words, in the asymptotic representation of effective properties truncated at first-order in the volume fraction.<sup>3</sup> As a consequence, the homogenization technique presented by Bigoni and Drugan (2007) and by Bacca et al. (2013a) formally holds, in the dilute limit, for periodic distributions of inclusions. The aim of the present article is: (i.) to explicitly consider the case of periodicity, thus introducing the possibility that the RVE may have a non-spherical ellipsoid of inertia, so that generic periodic distributions can be considered, and (ii.) to allow the possibility of inclusions with an arbitrary tensor of inertia, so that for instance the case of elliptical holes or cracks can be investigated, Fig. 1 left.

"instance the case of elliptical"

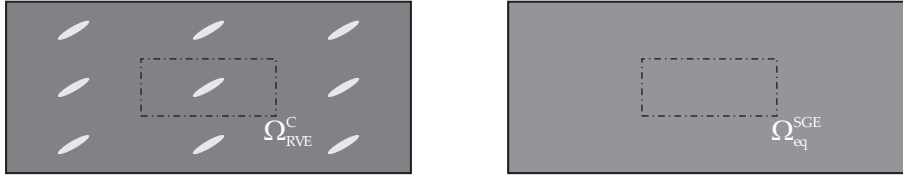


Figure 1: Heterogeneous Cauchy material (left) made up of an inclusion phase (taken elliptical for the sake of illustration) with constitutive tensor  $\mathbf{C}^{(2)}$  embedded within a matrix with constitutive tensor  $\mathbf{C}^{(1)}$  and homogeneous equivalent SGE material (right) with effective tensors  $\mathbf{C}^{eq}$  and  $\mathbf{A}^{eq}$ .

By matching the elastic energy of a Cauchy elastic heterogeneous material (Fig. 1, left) with that of a homogeneous higher-order elastic solid (Fig. 1, right), both subject to the same displacement conditions on the boundary, it is shown that a sixth-order constitutive tensor can be obtained, defining a second-gradient elastic (SGE) material (Mindlin and Eshel, 1968). The results show how the shape and the constitutive symmetry class of the phases influence the higher-order response and are exploited to investigate two particular cases of special interest.

One case is that of a dilute distribution of aligned elliptical voids (and the corresponding limit case of cracks) in a Cauchy elastic isotropic material, which yields an orthotropic nonlocal effective behaviour.

The other case is that of a Cauchy elastic heterogeneous material with the RVE having a non-spherical ellipsoid of inertia (as for instance a rectangular lattice of small circular elastic inclusions in an isotropic elastic matrix, Fig. 2, right). In this case the effective nonlocal behaviour is found to be orthotropic, even though the local behaviour remains isotropic, consistently with the diluteness. This finding, consistent with early results (Auffray et al., 2010, where anisotropic high-order effects are numerically obtained for materials with an isotropic local response), implies that the proposed homogenization procedure provides different nonlocal effective responses for topologically equivalent unit cells, but having different tensors of inertia. This apparent paradox is related to the contrasting effects between nonlocality and diluteness

<sup>2</sup>The effective response under dilute approximation is obtained by matching the energy stored within the RVE and the equivalent material when effective boundary displacement conditions are imposed.

<sup>3</sup>For instance, a dilute suspension of spherical voids in an isotropic Cauchy elastic matrix results always equivalent to an isotropic effective Cauchy material independently of the geometrical distribution of the voids.

on the evaluation of effective higher-order properties. In fact, our results show that, even at small volume fractions and differently from the evaluation of the local effective properties, the nonlocal effective response depends on the boundary conditions imposed on the RVE, so that it can be only estimated and not exactly determined.



Figure 2: Two composites  $\mathcal{M}$  and  $\mathcal{N}$  containing a dilute suspension of isotropic circular inclusions in an isotropic matrix (with the same volume ratio  $f \ll 1$ ). While standard homogenization leads to the same effective isotropic constitutive fourth-order tensor for both distributions,  $\mathbf{C}^{eq}(\mathcal{M}) = \mathbf{C}^{eq}(\mathcal{N})$ , second-order homogenization leads to different effective nonlocal properties,  $\mathbf{A}^{eq}(\mathcal{M}) \neq \mathbf{A}^{eq}(\mathcal{N})$ . In particular, while  $\mathbf{A}^{eq}(\mathcal{M})$  is isotropic,  $\mathbf{A}^{eq}(\mathcal{N})$  is orthotropic.

## 2 Nonlocal effective response

Through application of the second-order homogenization procedure, the evaluation of the effective nonlocal response of a Second Gradient Elastic solid (Mindlin and Eshel, 1968) obtained by Bacca et al. (2013a), and valid for a RVE having inclusion and matrix with spherical ellipsoids of inertia, is extended to the case of phases with generic ellipsoid of inertia.

The homogenization is performed on a heterogeneous Cauchy RVE (volume  $\Omega_{RVE}^C$ ; Fig. 1, left), where a *dilute* distribution of inclusions (phase (2), local constitutive tensor  $\mathbf{C}^{(2)}$ , volume  $\Omega_2^C = f\Omega_{RVE}^C$ , volume fraction  $f \ll 1$ ) is embedded within a matrix (phase (1), local constitutive tensor  $\mathbf{C}^{(1)}$  volume  $\Omega_1^C = (1-f)\Omega_{RVE}^C$ ). The homogenization leads to a homogeneous SGE material (with effective local constitutive tensor  $\mathbf{C}^{eq}$  and nonlocal constitutive tensor  $\mathbf{A}^{eq}$ , volume  $\Omega_{eq}^{SGE} = \Omega_{RVE}^C$ ; Fig. 1, right). The equivalent response is obtained through the annihilation of the strain energy mismatch between the heterogeneous and equivalent materials, which is

$$\mathcal{W}_{RVE}^C(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}) - \mathcal{W}_{eq}^{SGE}(\mathbf{C}^{eq}, \mathbf{A}^{eq}) = 0, \quad (1)$$

when the following second-order (linear and quadratic) displacement field  $\bar{\mathbf{u}}$  is imposed on both the boundaries of the RVE and the SGE

$$\mathbf{u} = \bar{\mathbf{u}}, \quad \text{on } \partial\Omega_{RVE}^C, \quad \text{and} \quad \begin{cases} \mathbf{u} = \bar{\mathbf{u}}, \\ D\mathbf{u} = D\bar{\mathbf{u}}, \end{cases} \quad \text{on } \partial\Omega_{eq}^{SGE}, \quad (2)$$

with

$$\bar{u}_i = \underbrace{\alpha_{ij}x_j}_{\bar{u}_i^\alpha} + \underbrace{\beta_{ijk}x_jx_k}_{\bar{u}_i^\beta}, \quad (3)$$

where  $\alpha_{ij}$  and  $\beta_{ijk}$  are constant coefficients, the latter satisfying the symmetry  $\beta_{ijk} = \beta_{ikj}$ . The strain energies for the heterogeneous Cauchy material and the equivalent SGE material are

given by integration over the domain of the respective strain energy density, namely

$$\begin{aligned}\mathcal{W}_{RVE}^C &= \frac{1}{2} \int_{\Omega_1^C} \mathbf{C}_{ijhk}^{(1)} u_{i,j} u_{h,k} + \frac{1}{2} \int_{\Omega_2^C} \mathbf{C}_{ijhk}^{(2)} u_{i,j} u_{h,k}, \\ \mathcal{W}_{eq}^{SGE} &= \frac{1}{2} \int_{\Omega_{SGE}^{eq}} \left[ \mathbf{C}_{ijhk}^{eq} u_{i,j} u_{h,k} + \mathbf{A}_{ijklmn}^{eq} u_{k,ij} u_{n,lm} \right].\end{aligned}\quad (4)$$

To explicitly evaluate the effective nonlocal response, some of the geometrical assumptions introduced by Bacca et al. (2013a) are now modified. In particular, coincidence of the centroids of matrix and inclusion, both centered at the origin of the  $x_i$ -axes (i.e. geometrical property GP1 in Bacca et al. 2013a) is still assumed, namely, the static moment vectors of the inclusion, of the matrix and of the RVE are null

$$\mathbf{S}(\Omega_1^C) = \mathbf{S}(\Omega_2^C) = \mathbf{S}(\Omega_{RVE}^C) = \mathbf{0}, \quad \mathbf{S}(V) = \int_V \mathbf{x}, \quad (5)$$

but no restriction is now assumed about the shape and the directions of the ellipsoids of inertia for matrix and the inclusion (i.e. the geometrical property GP2 of Bacca et al. 2013a is removed).

Introducing the normalized inertia tensor  $\mathbf{B}$  for a generic solid occupying the region  $V$ , defined as the second-order Euler tensor of inertia  $\mathbf{E}$  divided by the RVE volume (or area in plane-strain)  $\Omega_{RVE}^C$ ,

$$\mathbf{B}(V) = \frac{\mathbf{E}(V)}{\Omega_{RVE}^C}, \quad \mathbf{E}(V) = \int_V \mathbf{x} \otimes \mathbf{x}, \quad (6)$$

the normalized tensors of inertia for the matrix  $\mathbf{B}^{(1)} = \mathbf{B}(\Omega_1^C)$ , the inclusion  $\mathbf{B}^{(2)} = \mathbf{B}(\Omega_2^C)$  and the RVE  $\mathbf{B}^{RVE} = \mathbf{B}(\Omega_{RVE}^C)$  are given by

$$\begin{aligned}\mathbf{B}^{(1)} &= (1-f) \sum_{k=1}^N \left[ \rho_k^{(1)} \right]^2 \mathbf{e}_{[k]}^{(1)} \otimes \mathbf{e}_{[k]}^{(1)}, \\ \mathbf{B}^{(2)} &= f \sum_{k=1}^N \left[ \rho_k^{(2)} \right]^2 \mathbf{e}_{[k]}^{(2)} \otimes \mathbf{e}_{[k]}^{(2)}, \\ \mathbf{B}^{RVE} &= \sum_{k=1}^N \left[ \rho_k^{RVE} \right]^2 \mathbf{e}_{[k]}^{RVE} \otimes \mathbf{e}_{[k]}^{RVE},\end{aligned}\quad (7)$$

where  $\{\mathbf{e}_{[k]}^{(1)}, \mathbf{e}_{[k]}^{(2)}, \mathbf{e}_{[k]}^{RVE}\}$  are the  $k$ -th unit vectors identifying the principal system for the matrix, the inclusion, and the RVE, and  $\{\rho_k^{(1)}, \rho_k^{(2)}, \rho_k^{RVE}\}$  are the respective radii of gyration, and  $N$  is equal to 3 (or 2 for plane strain case).

Note that from eqns (6) and (7) it follows

$$\mathbf{B}^{RVE} = \mathbf{B}^{(1)} + \mathbf{B}^{(2)}, \quad (8)$$

so that the normalized tensors of inertia for the RVE can be isotropic,  $\mathbf{B}^{RVE} = (\rho^{RVE})^2 \mathbf{I}$ , even when the tensors of inertia of the matrix and inclusion are not isotropic. In other words, since  $\mathbf{B}^{(1)}$ ,  $\mathbf{B}^{(2)}$  and  $\mathbf{B}^{RVE}$  are geometrical quantities,  $\mathbf{B}^{RVE}$  corresponds to the normalized inertia of the region enclosed by the the external contour of the RVE.

Considering the normalized inertia tensors (7), it is assumed that for the inclusion all the radii of gyration  $\rho_k^{(2)}$  vanish in the limit of null inclusion volume fraction  $f$  (a modification of the geometrical property GP3 in Bacca et al. 2013a), namely,

$$\lim_{f \rightarrow 0} \rho_k^{(2)}(f) = 0, \quad \forall k = 1, \dots, N. \quad (9)$$

Similarly to the computations in (Bacca et al., 2013a), the energy mismatch condition (1) is now used to obtain a nonlocal material equivalent to the heterogeneous Cauchy RVE. Since Lemmas 1 and 2 in (Bacca et al., 2013a) are not affected by the geometrical assumption GP2 (which is now removed), the mutual energy is still null at first-order in  $f$ . Considering that  $\mathbf{C}^{eq}$  is obtained through a first-order homogenization, the energy mismatch (1) is given only by the contribution related to the sole quadratic displacement boundary condition,

$$\mathcal{W}_{RVE}^C(\bar{\mathbf{u}}) - \mathcal{W}_{eq}^{SGE}(\bar{\mathbf{u}}, D\bar{\mathbf{u}}) = \mathcal{W}_{RVE}^C(\bar{\mathbf{u}}^{\beta\circ*}) - \mathcal{W}_{eq}^{SGE}(\bar{\mathbf{u}}^{\beta\circ*}, D\bar{\mathbf{u}}^{\beta\circ*}) + o(f), \quad (10)$$

where the symbol  $\diamond$  denotes the fact that the quadratic displacement field is restricted to produce a self-equilibrated stress in an auxiliary homogeneous medium with local tensor  $\mathbf{C}^*$ , taken as a first-order perturbation in  $f$  to the equivalent local constitutive tensor  $\mathbf{C}^{eq}$  (see Bacca et al., 2013a, for details).<sup>4</sup>

Lemmas 3 and 4 in (Bacca et al., 2013a) can be extended by considering the definition of normalized Euler tensors of inertia given by eqns (7), and from which the strain energies instrumental to compute the energy mismatch can be obtained as

$$\mathcal{W}_{RVE}^C(\bar{\mathbf{u}}^{\beta\circ*}) = 2\Omega_{RVE}^C B_{lm}^{RVE} \mathbf{C}_{ijhk}^{(1)} \beta_{ijl}^{\circ*} \beta_{hkm}^{\circ*} + o(f), \quad (11)$$

and

$$\mathcal{W}_{eq}^{SGE}(\bar{\mathbf{u}}^{\beta\circ*}, D\bar{\mathbf{u}}^{\beta\circ*}) = 2\Omega_{RVE}^C \left( B_{lm}^{RVE} \mathbf{C}_{ijhk}^{eq} + \mathbf{A}_{jlikmh}^{eq} \right) \beta_{ijl}^{\circ*} \beta_{hkm}^{\circ*} + o(f). \quad (12)$$

By using eqns (11) and (12) to impose the annihilation of the energy mismatch (1) for a generic purely quadratic boundary condition (and leading to a self-equilibrated stress within the arbitrary medium characterized by local tensor  $\mathbf{C}^*$ ), the nonlocal sixth-order tensor  $\mathbf{A}^{eq}$  of the equivalent SGE material is evaluated (at first-order in  $f$ ) as

$$\mathbf{A}_{ijhlmn}^{eq} = -\frac{f}{4} \left( \tilde{\mathbf{C}}_{ihln} B_{jm}^{RVE} + \tilde{\mathbf{C}}_{ihmn} B_{jl}^{RVE} + \tilde{\mathbf{C}}_{jhln} B_{im}^{RVE} + \tilde{\mathbf{C}}_{jhmn} B_{il}^{RVE} \right), \quad (13)$$

where  $\tilde{\mathbf{C}}$  is the first-order discrepancy tensor, defined as

$$\mathbf{C}^{eq} = \mathbf{C}^{(1)} + f\tilde{\mathbf{C}}, \quad (14)$$

and assumed to be known from standard homogenization.

Solution (13) represents the evaluation –in the dilute case– of the nonlocal behaviour of the equivalent SGE material under the geometrical assumptions (5) and (9). Regardless of the ellipsoid of inertia of the inclusion phase, this solution reduces to that given in (Bacca et al.,

<sup>4</sup>The energy match employed to identify the higher-order response is performed considering prescribed quadratic displacements at the boundary of the RVE and of the equivalent SGE. The fact that the imposed quadratic displacements generate a self-equilibrated stress state in an auxiliary homogeneous material (with local constitutive tensor  $\mathbf{C}^*$ ) is fundamental to provide a perfect match between the equations and the unknown non-local parameters. Indeed, the equilibrium constraint is eventually relaxed by the arbitrariness in the choice of the auxiliary material.

2013a) when a RVE enclosing a region with a spherical ellipsoid of inertia,  $B_{ij}^{RVE} = (\rho^{RVE})^2 \delta_{ij}$ , is considered<sup>5</sup>

$$\mathbf{A}_{ijhlmn}^{eq} = -f \frac{(\rho^{RVE})^2}{4} \left( \tilde{\mathbf{C}}_{ihln} \delta_{jm} + \tilde{\mathbf{C}}_{ihmn} \delta_{jl} + \tilde{\mathbf{C}}_{jhln} \delta_{im} + \tilde{\mathbf{C}}_{jhm n} \delta_{il} \right). \quad (15)$$

It is worth to remark that, while the normalized inertia tensor  $\mathbf{B}^{RVE}$  is dependent only on the shape and size of the external boundary of the RVE, the first-order discrepancy tensor  $\tilde{\mathbf{C}}$  is dependent on the constitutive tensors of both phases and on the shape of the inclusion phase.

Similarly to the case of phases with spherical ellipsoid of inertia (Bacca et al., 2013b), it can be noted that:

- the equivalent SGE material is positive definite if and only if  $\tilde{\mathbf{C}}$  is negative definite;
- the constitutive higher-order tensor  $\mathbf{A}^{eq}$  is linear in  $f$  for dilute concentrations;

but, differently, the higher-order material symmetries of the equivalent SGE solid involves *not only* the material symmetries of the first-order discrepancy tensor  $\tilde{\mathbf{C}}$ . Indeed, when a RVE encloses a region with a non-spherical ellipsoid of inertia (see Appendix A for details),

*the symmetry class of the higher-order response for the effective material coincides with the symmetry class of both the RVE's normalized inertia tensor  $\mathbf{B}^{RVE}$  and the first-order discrepancy tensor  $\tilde{\mathbf{C}}$ .*

Therefore, an isotropic higher-order behaviour (in the dilute case) is obtained only when both the first-order discrepancy tensor  $\tilde{\mathbf{C}}$  and the normalized inertia tensor  $\mathbf{B}^{RVE}$  are isotropic.

Note that, in cases when topologically equivalent unit cells having different normalized inertia tensor  $\mathbf{B}^{RVE}$  can be defined for a periodic distribution, result (13) yields different evaluations for  $\mathbf{A}^{eq}$  for the same dilute composite. As discussed in the Introduction, this is related to the contrast between nonlocality and diluteness, so that the eqn (13) represents a mere estimate.

### 3 Application cases

With reference to composites with phases having isotropic Cauchy behaviour

$$\mathbf{C}_{ijhk}^{(1)} = \lambda_1 \delta_{ij} \delta_{hk} + \mu_1 (\delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh}), \quad \mathbf{C}_{ijhk}^{(2)} = \lambda_2 \delta_{ij} \delta_{hk} + \mu_2 (\delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh}), \quad (16)$$

the anisotropic nonlocal effective response is analyzed by considering applications of the homogenization result (13) to the following specific geometries:

- RVE enclosing a region with a spherical ellipsoid of inertia;
- inclusion with non-spherical ellipsoid of inertia.

The above examples show that, similarly to the usual first-order homogenization, specific assumptions on the geometry of the phases lead to anisotropic nonlocal effective response and that, more interestingly, in the case of RVE enclosing a region with non-spherical ellipsoid inertia (and under the dilute assumption), anisotropic nonlocal effective response can arise even in composites having isotropic local effective behaviour.

<sup>5</sup>Note that in (Bacca et al., 2013a) the spherical radius of gyration of the RVE is indicated with  $\rho$ , while within this article is denoted by  $\rho^{RVE}$ .

### 3.1 RVE enclosing a region with non-spherical ellipsoid inertia

Restricting attention to inclusions which geometry is such that the first-order discrepancy tensor, eqn (14), is isotropic

$$\tilde{\mathbf{C}}_{ijhk} = \tilde{\lambda}\delta_{ij}\delta_{hk} + \tilde{\mu}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \quad (17)$$

and considering the principal directions of the RVE ellipsoid of inertia<sup>6</sup> as the reference system, the components of the normalized tensor of inertia  $\mathbf{B}^{RVE}$ , eqn (7)<sub>3</sub>, can be written as

$$B_{ij}^{RVE} = \sum_{k=1}^N (\rho_k^{RVE})^2 \delta_{ik}\delta_{jk}, \quad (18)$$

and the solution (13) leads to the following effective nonlocal orthotropic tensor<sup>7</sup>

$$\mathbf{A}_{ijhlmn}^{eq} = \sum_{k=1}^N \left\{ \begin{aligned} & \frac{a_2^{[k]}}{2} (\delta_{ih}\delta_{kj} + \delta_{jh}\delta_{ki}) (\delta_{ln}\delta_{km} + \delta_{mn}\delta_{kl}) \\ & + \frac{a_4^{[k]}}{2} \delta_{hn} [\delta_{kj} (\delta_{im}\delta_{kl} + \delta_{il}\delta_{km}) + \delta_{ki} (\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})] \\ & + \frac{a_5^{[k]}}{2} [\delta_{in}\delta_{kj} (\delta_{hl}\delta_{km} + \delta_{hm}\delta_{kl}) + \delta_{jn}\delta_{ki} (\delta_{hm}\delta_{kl} + \delta_{hl}\delta_{km})] \end{aligned} \right\}, \quad (19)$$

where the  $3N$  nonlocal parameters are given by

$$a_2^{[k]} = -f \frac{(\rho_k^{RVE})^2}{2} \tilde{\lambda}, \quad a_4^{[k]} = a_5^{[k]} = -f \frac{(\rho_k^{RVE})^2}{2} \tilde{\mu}, \quad k = 1, \dots, N. \quad (20)$$

Note that the effective nonlocal tensor  $\mathbf{A}^{eq}$ , eqn (19), is an orthotropic sixth-order tensor and that it reduces to an isotropic sixth-order tensor when the ellipsoid of inertia of the RVE becomes a sphere (or a circle in plane strain),  $\rho_k^{RVE} = \rho^{RVE}$  for  $k = 1, \dots, N$ . Furthermore, the homogenization procedure leads to a positive definite SGE material when the discrepancy tensor  $\tilde{\mathbf{C}}$  is negative definite, corresponding to

$$\tilde{K} < 0, \quad \tilde{\mu} < 0, \quad (21)$$

where  $\tilde{K}$  is the bulk modulus, equal to  $\tilde{\lambda} + 2\tilde{\mu}/3$  when  $N = 3$  and  $\tilde{\lambda} + \tilde{\mu}$  when  $N = 2$ .

It is worth to mention that the  $3(N - 1)$  nonlocal parameters  $\{a_2^{[k]}; a_4^{[k]}; a_5^{[k]}\}$  ( $k = 2, N$ ) can be expressed as functions of the two nonlocal parameters  $\{a_2^{[1]}; a_4^{[1]}\}$  as

$$a_2^{[k]} = \left( \frac{\rho_k^{RVE}}{\rho_1^{RVE}} \right)^2 a_2^{[1]}, \quad a_4^{[k]} = a_5^{[k]} = \left( \frac{\rho_k^{RVE}}{\rho_1^{RVE}} \right)^2 a_4^{[1]}, \quad k = 2, N. \quad (22)$$

Equation (22) reveals that (for inclusions less stiff than the matrix and) if  $\rho_k^{RVE}/\rho_1^{RVE} > 1$  ( $< 1$ ), namely, when the distance between the inclusions is larger (smaller) in the direction  $k$

<sup>6</sup>The nomenclature ‘RVE ellipsoid of inertia’ means the inertia of the region enclosed within the external contour of the RVE, according to eqn (8).

<sup>7</sup>The nonlocal parameters appearing in definition of the orthotropic tensor, eqn (19), have been denoted by  $\{a_2^{[k]}; a_4^{[k]}; a_5^{[k]}\}$  to use the same nomenclature as in (Bacca et al., 2013b).



than in direction 1, then the nonlocal behaviour is stiffer (less stiff) in the former direction than in the latter (Fig. 2, right).

The explicit evaluation of the non-local parameters is provided below for two simple cases, by considering the discrepancy parameters  $\tilde{\lambda}$  and  $\tilde{\mu}$  reported in (Bacca et al., 2013b), since the first-order homogenization result is not affected by the external shape of the RVE in the dilute case.

**Circular elastic inclusion within a rectangular RVE.** In the case of a circular elastic inclusion (in plane strain condition) of radius  $r$  embedded in a rectangular RVE with edges  $h_1$  and  $h_2$  (parallel to the directions  $x_k$ ,  $k = 1, 2$ ), the three nonlocal parameters  $\{a_2^{[1]}; a_4^{[1]}; a_5^{[1]}\}$  are obtained through eqn (20) as

$$\begin{aligned} a_2^{[1]} &= \frac{\pi r^2 h_1}{24 h_2} \left[ \frac{(K_1 - K_2)(K_1 + \mu_1)}{K_2 + \mu_1} - \frac{2\mu_1(\mu_1 - \mu_2)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)} \right], \\ a_4^{[1]} &= a_5^{[1]} = \frac{\pi r^2 h_1}{12 h_2} \frac{\mu_1(\mu_1 - \mu_2)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}, \end{aligned} \quad (23)$$

from which, by using relation (22), the remaining three parameters  $\{a_2^{[2]}; a_4^{[2]}; a_5^{[2]}\}$  follow in the form

$$a_2^{[2]} = \left(\frac{h_2}{h_1}\right)^2 a_2^{[1]}, \quad a_4^{[2]} = a_5^{[2]} = \left(\frac{h_2}{h_1}\right)^2 a_4^{[1]}. \quad (24)$$

Note that a positive definite equivalent SGE material is obtained only when the inclusion phase is ‘softer’ than the matrix in terms of *both* shear and bulk moduli, namely,

$$\mu_2 < \mu_1, \quad K_2 < K_1. \quad (25)$$

**Spherical inclusion within a parallelepipedic RVE.** In the case of a spherical elastic inclusion of radius  $r$  embedded in a parallelepiped with sides  $h_1$ ,  $h_2$  and  $h_3$  (parallel to the directions  $x_k$ ,  $k = 1, 2, 3$ ), the three nonlocal parameters  $\{a_2^{[1]}; a_4^{[1]}; a_5^{[1]}\}$  are obtained through eqn (20) as

$$\begin{aligned} a_2^{[1]} &= \frac{\pi r^3}{18} \frac{h_1}{h_2 h_3} \left[ \frac{(3K_1 + 4\mu_1)(K_1 - K_2)}{3K_2 + 4\mu_1} - \frac{2}{3} \frac{5\mu_1(\mu_1 - \mu_2)(3K_1 + 4\mu_1)}{\mu_1(3K_1 + 4\mu_2) + 2(3K_1 + 4\mu_1)(\mu_2 + \mu_1)} \right], \\ a_4^{[1]} &= a_5^{[1]} = \frac{\pi r^3}{18} \frac{h_1}{h_2 h_3} \frac{5\mu_1(\mu_1 - \mu_2)(3K_1 + 4\mu_1)}{\mu_1(3K_1 + 4\mu_2) + 2(3K_1 + 4\mu_1)(\mu_2 + \mu_1)}, \end{aligned} \quad (26)$$

from which, by using relation (22), the remaining six parameters  $\{a_2^{[2]}; a_4^{[2]}; a_5^{[2]}; a_2^{[3]}; a_4^{[3]}; a_5^{[3]}\}$  follow in the form

$$\begin{aligned} a_2^{[2]} &= \left(\frac{h_2}{h_1}\right)^2 a_2^{[1]}, \quad a_4^{[2]} = a_5^{[2]} = \left(\frac{h_2}{h_1}\right)^2 a_4^{[1]}, \\ a_2^{[3]} &= \left(\frac{h_3}{h_1}\right)^2 a_2^{[1]}, \quad a_4^{[3]} = a_5^{[3]} = \left(\frac{h_3}{h_1}\right)^2 a_4^{[1]}. \end{aligned} \quad (27)$$

Similarly to the case of circular elastic inclusions, the equivalent SGE material is positive definite only when condition (25) is satisfied.



### 3.2 Inclusion with non-spherical ellipsoid inertia

It has been shown that solution (15), obtained by Bacca et al. (2013a), remains valid regardless of the ellipsoid of inertia of the inclusions, whenever the RVE encloses a region with a spherical ellipsoid of inertia. Solution (15) is now exploited to achieve the nonlocal effective behaviour of a composite with aligned elliptical holes and, in the limit of vanishing ellipse axes ratio, of cracks.

**Elliptical hole within an isotropic matrix.** The presence of equally-oriented inclusions with non-spherical ellipsoid of inertia within a composite introduces an anisotropy in the effective behaviour. In particular, when a system of dilute aligned elliptic holes (with semi-axis  $b_k$  parallel to direction  $x_k$ ,  $k = 1, 2$ ) is considered within an isotropic matrix, the effective behaviour is orthotropic with orthotropy axes coincident with those of the elliptical hole, so that the first-order discrepancy tensor  $\tilde{\mathbf{C}}$  described in the reference system of orthotropy is (Tsukrov and Kachanov, 2000)

$$\tilde{\mathbf{C}}_{ijkh} = \tilde{\lambda} \delta_{ij} \delta_{hk} + \tilde{\mu} (\delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh}) + \tilde{\xi} (\delta_{i1} \delta_{j2} + \delta_{i2} \delta_{j1}) (\delta_{h1} \delta_{k2} + \delta_{h2} \delta_{k1}) + \tilde{\omega} \delta_{i1} \delta_{j1} \delta_{h1} \delta_{k1}, \quad (28)$$

with

$$\begin{aligned} \tilde{\lambda} &= -(\lambda_1 + 2\mu_1) \frac{\lambda_1 (\lambda_1 + 2\mu_1) (1 + \Lambda^2) - 2\Lambda\mu_1^2}{2\Lambda\mu_1(\lambda_1 + \mu_1)}, \\ \tilde{\mu} &= -(1 + \Lambda) (\lambda_1 + 2\mu_1) \frac{\lambda_1(1 - \Lambda) + 2\mu_1}{2\Lambda(\lambda_1 + \mu_1)}, \\ \tilde{\xi} &= (1 - \Lambda^2) \frac{\lambda_1 + 2\mu_1}{2\Lambda}, \\ \tilde{\omega} &= (1 - \Lambda^2) \frac{\lambda_1 + 2\mu_1}{\Lambda}, \end{aligned} \quad (29)$$

where the parameter  $\Lambda$  is the ratio between the ellipse's semi-axes,  $\Lambda = b_2/b_1$ .

Considering that the RVE encloses a region with a spherical ellipsoid of inertia,  $B_{ij}^{RVE} = (\rho^{RVE})^2 \delta_{ij}$ , the effective nonlocal tensor  $\mathbf{A}^{eq}$  can be obtained, exploiting solution (13), as the following positive-definite orthotropic sixth-order tensor<sup>8</sup>

$$\begin{aligned} \mathbf{A}_{ijhlmn}^{eq} &= \frac{a_2}{2} [\delta_{ih} (\delta_{jl} \delta_{mn} + \delta_{jm} \delta_{ln}) + \delta_{jh} (\delta_{il} \delta_{mn} + \delta_{im} \delta_{ln})] \\ &+ a_4 (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \delta_{hn} + \frac{a_5}{2} [\delta_{in} (\delta_{jl} \delta_{hm} + \delta_{jm} \delta_{hl}) + \delta_{jn} (\delta_{il} \delta_{hm} + \delta_{im} \delta_{hl})] \\ &+ \frac{a_6}{2} \{ (\delta_{i1} \delta_{h2} + \delta_{i2} \delta_{h1}) [(\delta_{l1} \delta_{n2} + \delta_{l2} \delta_{n1}) \delta_{jm} + (\delta_{m1} \delta_{n2} + \delta_{m2} \delta_{n1}) \delta_{jl}] \\ &+ (\delta_{j1} \delta_{h2} + \delta_{j2} \delta_{h1}) [(\delta_{l1} \delta_{n2} + \delta_{l2} \delta_{n1}) \delta_{im} + (\delta_{m1} \delta_{n2} + \delta_{m2} \delta_{n1}) \delta_{il}] \} \\ &+ \frac{a_9}{2} [\delta_{i1} (\delta_{l1} \delta_{jm} + \delta_{m1} \delta_{jl}) + \delta_{j1} (\delta_{l1} \delta_{im} + \delta_{m1} \delta_{il})] \delta_{h1} \delta_{n1}, \end{aligned} \quad (30)$$

with the following nonlocal parameters

$$a_2 = -f \frac{(\rho^{RVE})^2}{2} \tilde{\lambda}, \quad a_4 = a_5 = -f \frac{(\rho^{RVE})^2}{2} \tilde{\mu}, \quad a_6 = -f \frac{(\rho^{RVE})^2}{2} \tilde{\xi}, \quad a_9 = -f \frac{(\rho^{RVE})^2}{2} \tilde{\omega}. \quad (31)$$

<sup>8</sup>The five nonlocal parameters appearing in definition of the orthotropic tensor, eqn (30), have been denoted by  $\{a_2; a_4; a_5; a_6; a_9\}$  to use the same nomenclature as in (Bacca et al., 2013b).

In the case that the RVE has a square boundary, and by considering the first-order discrepancy quantities given by eqn (29), the nonlocal parameters simplify to

$$\begin{aligned}
a_2 &= \frac{\pi b_1^2}{48} \frac{\lambda_1 (\lambda_1 + 2\mu_1) (1 + \Lambda^2) - 2\Lambda\mu_1^2}{\mu_1(\lambda_1 + \mu_1)} (\lambda_1 + 2\mu_1), \\
a_4 = a_5 &= \frac{\pi b_1^2}{48} \frac{\lambda_1(1 - \Lambda) + 2\mu_1}{(\lambda_1 + \mu_1)} (1 + \Lambda) (\lambda_1 + 2\mu_1), \\
a_6 &= -\frac{\pi b_1^2}{48} (1 - \Lambda^2) (\lambda_1 + 2\mu_1), \\
a_9 &= -\frac{\pi b_1^2}{24} (1 - \Lambda^2) (\lambda_1 + 2\mu_1).
\end{aligned} \tag{32}$$

The nonlocal effective constants  $\{a_2; a_4 = a_5; a_6 = a_9/2\}$  given by eqn (32) are reported in Fig. 3 as functions of the ellipse semi-axes ratio  $b_2/b_1$  and for different values of the Poisson's ratio of the matrix  $\nu_1 = \lambda_1/(2(\lambda_1 + \mu_1))$ . Note that both the constants  $a_6$  and  $a_9$ , defining the orthotropic nonlocal effective behaviour, approach zero in the limit  $b_2/b_1 \rightarrow 1$ , so that the nonlocal orthotropic behaviour turns out to be isotropic when the elliptical voids become circular holes.

**Effective nonlocal parameters in the limit case of aligned cracks.** Although the sixth-order tensor (30) has been obtained excluding the possibility that inclusions have non-null radius of inertia at null volume ratio (namely, eqn (9)), the nonlocal parameters of the effective tensor  $\mathbf{A}^{eq}$  for a dilute distribution of aligned ellipses, eqn (32), can be used to obtain the limit values for a dilute distribution of aligned cracks ( $\Lambda = b_2/b_1 \rightarrow 0$ ) with length  $2b_1$  within a square RVE as

$$\begin{aligned}
a_2 &= \frac{\pi b_1^2}{48} \frac{\lambda_1(\lambda_1 + 2\mu_1)^2}{\mu_1(\lambda_1 + \mu_1)}, & a_4 = a_5 &= \frac{\pi b_1^2}{48} \frac{(\lambda_1 + 2\mu_1)^2}{\lambda_1 + \mu_1}, \\
a_6 &= -\frac{\pi b_1^2}{48} (\lambda_1 + 2\mu_1), & a_9 &= -\frac{\pi b_1^2}{24} (\lambda_1 + 2\mu_1).
\end{aligned} \tag{33}$$

## 4 Conclusions

Identification of effective nonlocal constants has been provided through energetic equivalence between a second-gradient homogeneous elastic material and a heterogeneous Cauchy elastic material containing a *dilute* distribution of inclusions of arbitrary shape in an arbitrary RVE (under the condition that inclusion and RVE have a coincident center of mass). Results show that the shape of the RVE influences the effective nonlocal behaviour even under the dilute approximation, in contrast with the usual homogenization scheme valid for effective local properties.

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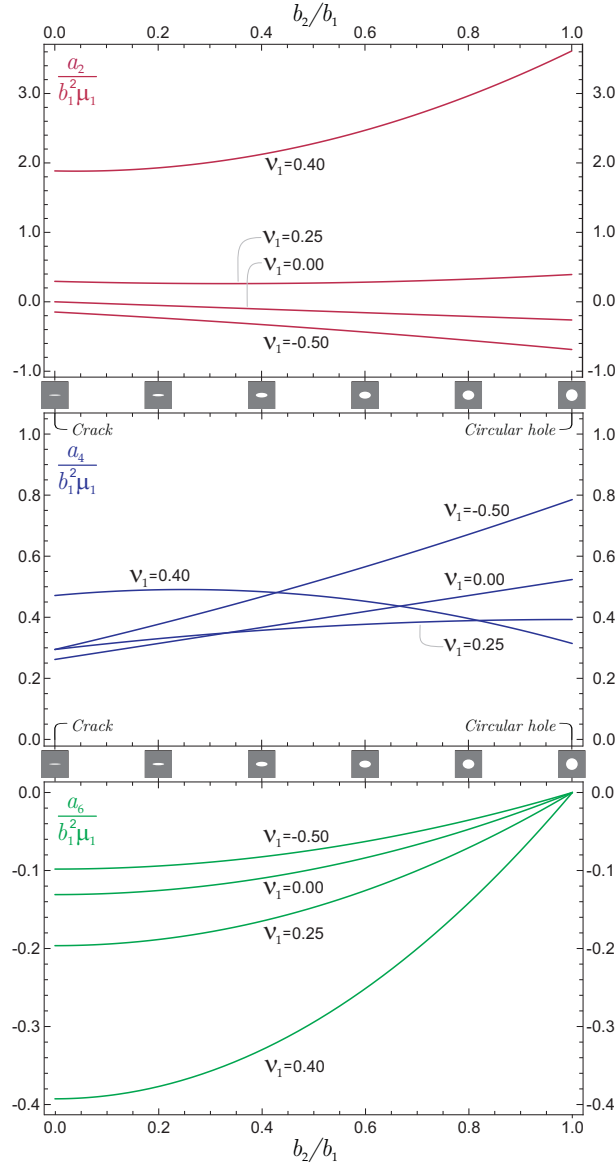


Figure 3: Higher-order constitutive parameters  $a_2$ ,  $a_4 = a_5$  and  $a_6 = a_9/2$  of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of aligned elliptical voids (with major semi-axis  $b_1$ ), as functions of the semi-axes ratio  $b_2/b_1$ , and for different values of the Poisson's ratio of the matrix  $\nu_1 = \{-0.5; 0; 0.25; 0.4\}$ . The constants are made dimensionless through division by parameter  $b_1^2\mu_1$ . Note that both constants  $a_6$  and  $a_9$  approach zero in the limit  $b_2/b_1 \rightarrow 1$ , so that the nonlocal orthotropic behaviour turns out to become isotropic when the elliptical voids become circular holes.

## A Coincidence of symmetry class of $\mathbf{A}^{eq}$ with the intersection of the symmetry classes of $\tilde{\mathbf{C}}$ and $\mathbf{B}^{RVE}$

Considering the sixth-order tensor  $\mathbf{D}^{eq}$  defined as

$$\mathbf{D}_{ijhlmn}^{eq} = -f\tilde{\mathbf{C}}_{ihln}B_{jm}^{RVE}, \quad (\text{A.1})$$

and having the following symmetries

$$\mathbf{D}_{ijhlmn}^{eq} = \mathbf{D}_{lmnijh}^{eq} = \mathbf{D}_{hjilmn}^{eq} = \mathbf{D}_{ijhnmnl}^{eq}, \quad (\text{A.2})$$

the sixth-order tensor  $\mathbf{A}^{eq}$ , eqn (13), is obtained through application of the symmetrization operator  $\mathcal{S}$  on  $\mathbf{D}^{eq}$  as follows

$$\mathbf{A}^{eq} = \mathcal{S}(\mathbf{D}^{eq}), \quad \mathbf{A}_{ijhlmn}^{eq} = \frac{1}{4} \left( \mathbf{D}_{ijhlmn}^{eq} + \mathbf{D}_{ijhmln}^{eq} + \mathbf{D}_{jihlmn}^{eq} + \mathbf{D}_{jihmln}^{eq} \right), \quad (\text{A.3})$$

so that  $\mathbf{A}^{eq}$  has the following symmetries

$$\mathbf{A}_{ijhlmn}^{eq} = \mathbf{A}_{lmnijh}^{eq} = \mathbf{A}_{jihlmn}^{eq} = \mathbf{A}_{ijhnmnl}^{eq}. \quad (\text{A.4})$$

The relation (A.3) can be inverted through the inverse symmetrization operator  $\mathcal{S}^{-1}$  defined as

$$\begin{aligned} \mathbf{D}^{eq} = \mathcal{S}^{-1}(\mathbf{A}^{eq}), \quad \mathbf{D}_{ijhlmn}^{eq} = & \mathbf{A}_{ijhlmn}^{eq} + \mathbf{A}_{jihmnl}^{eq} + \mathbf{A}_{hijnlm}^{eq} - \mathbf{A}_{ijhnlm}^{eq} - \mathbf{A}_{hijlmn}^{eq} \\ & + \mathbf{A}_{ijhnmnl}^{eq} + \mathbf{A}_{jihlmln}^{eq} - \mathbf{A}_{jihnlm}^{eq} - \mathbf{A}_{hijmnl}^{eq}. \end{aligned} \quad (\text{A.5})$$

A material symmetry (with respect to an orthogonal transformation) for a tensor  $\mathbf{M}$  corresponds to the condition

$$\mathcal{Q}(\mathbf{M}) = \mathbf{M}, \quad (\text{A.6})$$

where the orthogonal transformation  $\mathcal{Q}$  (defined with respect to an orthogonal tensor  $\mathbf{Q}$ ) applied to the RVE's normalized inertia tensor  $\mathbf{B}^{RVE}$ , to the first-order discrepancy tensor  $\tilde{\mathbf{C}}$ , and to the sixth-order tensors  $\mathbf{A}^{eq}$  and  $\mathbf{D}^{eq}$  is given respectively by

$$\begin{aligned} [\mathcal{Q}(\mathbf{B}^{RVE})]_{ij} &= Q_{ip}Q_{jq}\mathbf{B}_{pq}^{RVE}, & [\mathcal{Q}(\tilde{\mathbf{C}})]_{ijhl} &= Q_{ip}Q_{jq}Q_{hr}Q_{ls}\tilde{\mathbf{C}}_{pqrs}, \\ [\mathcal{Q}(\mathbf{A}^{eq})]_{ijhlmn} &= Q_{ip}Q_{jq}Q_{hr}Q_{ls}Q_{mt}Q_{nu}\mathbf{A}_{pqrst}^{eq}, & [\mathcal{Q}(\mathbf{D}^{eq})]_{ijhlmn} &= Q_{ip}Q_{jq}Q_{hr}Q_{ls}Q_{mt}Q_{nu}\mathbf{D}_{pqrst}^{eq}. \end{aligned} \quad (\text{A.7})$$

Since the symmetrization and the inverse symmetrization operators,  $\mathcal{S}$  and  $\mathcal{S}^{-1}$ , are commutative with the orthogonal operator  $\mathcal{Q}$ ,

$$\mathcal{Q}(\mathcal{S}(\mathbf{D}^{eq})) = \mathcal{S}(\mathcal{Q}(\mathbf{D}^{eq})), \quad \mathcal{Q}(\mathcal{S}^{-1}(\mathbf{A}^{eq})) = \mathcal{S}^{-1}(\mathcal{Q}(\mathbf{A}^{eq})), \quad (\text{A.8})$$

the symmetry class of  $\mathbf{A}^{eq}$  is coincident to that of  $\mathbf{D}^{eq}$ , namely

$$\mathcal{Q}(\mathbf{A}^{eq}) = \mathbf{A}^{eq} \quad \Leftrightarrow \quad \mathcal{Q}(\mathbf{D}^{eq}) = \mathbf{D}^{eq}, \quad (\text{A.9})$$

which, considering that the sixth-order tensor  $\mathbf{D}^{eq}$  is given by eqn (A.1), coincide with the intersection of symmetry classes of  $\tilde{\mathbf{C}}$  and  $\mathbf{B}^{RVE}$ , namely

$$\mathcal{Q}(\mathbf{A}^{eq}) = \mathbf{A}^{eq}, \quad \Leftrightarrow \quad \mathcal{Q}(\tilde{\mathbf{C}}) = \tilde{\mathbf{C}} \quad \text{and} \quad \mathcal{Q}(\mathbf{B}^{RVE}) = \mathbf{B}^{RVE}. \quad (\text{A.10})$$

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