# Effects of temperature and thermo-mechanical couplings on material instabilities and strain localization of inelastic materials

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## Abstract

A general framework for rate-independent, small-strain, thermoinelastic material behaviour is presented, which includes thermo-plasticity and -damage as particular cases. In this context, strain localization and the development of material instabilities are investigated to highlight the roles of thermal effects and thermomechanical couplings. During a loading process, it is shown that two conditions play the essential roles and correspond to the singularity of the isothermal and the adiabatic acoustic tensors. Under quasi-static conditions, strain localization (in a classical sense) may occur when either of these two conditions is met. It involves a jump in temperature rate in the latter case, whereas temperature rate remains continuous in the former, but a discontinuity in the spatial derivatives of the heat flux must occur. This is consistent with the condition of stationarity of acceleration waves, which are shown to be homothermal and propagate with a velocity related to the eigenvalues of the isothermal acoustic tensor. A linear perturbation analysis further clarifies the above findings. In particular, for a quasi-static path of an infinite medium, failure of positive definiteness of either of the acoustic tensors corresponds to bifurcations in wave-like modes of arbitrary wave-length and infinite rate of growth. Under dynamic conditions, unbounded growth of perturbations is associated only to the short wavelength regime and corresponds to divergence growth or flutter phenomena relative to the isothermal acoustic tensor.
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## 1. Introduction

Thermal effects have important consequences on behaviour of materials and are fully accessible to measurements. For instance, experimental results relative to polymers demonstrate that fracture initiation and growth may be influenced by temperature variations ahead of the crack tip (Rittel, 1999, 2000). Thermal loading strongly affects the plastic deformation of soils and rocks and consequently strain localization in these materials (Hueckel and Baldi, 1990; Hueckel and Pellegrini, 1992; Hueckel et al., 1994). In structural steel, Marchand and Duffy (1988) have observed the important effect of heating on nucleation and growth of shear bands. These experimental evidences suggest that thermal phenomena can play an important role in promoting or inhibiting global and local instabilities in materials, but, surprisingly, coupled thermal effects are usually ignored in inelastic modelling of materials. As a matter of fact, only relatively few contributions are available in the literature (Mandel, 1969; Raniecki, 1972, 1976, 1979; Raniecki and Sawczuk, 1975; Mróz and Raniecki, 1976; Raniecki and Bruhns, 1981; Raniecki and Samanta, 1989; Simo and Miehe, 1992; Armero and Simo, 1993; Miehe, 1996); moreover, localization of plastic deformation, which is a thoroughly analyzed and well-understood topic for isothermal deformations, was apparently only addressed by Benallal (1992a, b) in the context of coupled, inviscid thermoplasticity, while several contributions are restricted to adiabatic conditions (see, among others, Batra and Zhu, 1991; Duszek et al., 1992; Duszek-Perzyna and Perzyna, 1993; Perzyna, 1994; Steinmann et al., 1999). However, the main result of the present paper evidences that adiabatic analyses may be not sufficient to understand the development of material instabilities and strain localization phenomena.

It may be important to add to the above discussion that thermoelasticity is comparatively much more developed than thermoplasticity. In particular, different notions of stability were analyzed in that context (Truesdell and Noll, 1965, Sections 71, 89, 90; Ericksen, 1966; Dafermos, 1968) and, more in detail, findings presented by Abeyaratne and Knowles (1999) are closely related to results obtained in the present paper.

Another point that is worth to consider is that the problems of *quasi-static* deformation of fluid-saturated, inelastic porous media and of thermoplastic solids share certain similarities. In particular, the diffusion of the fluid through the solid skeleton plays a role corresponding to heat conduction, so that some of the results obtained by Loret and Harireche (1991) and Benallal and Comi (2002; 2003a, b) can be translated to the thermo-mechanical context. The major difference between the two problems lies in the important fact that for thermo-plasticity, the adiabatic constitutive tangent moduli are always non-symmetric, even for associative flow rule. It will be shown later that a consequence of this is that a hierarchy between failure of positive definiteness of the isothermal and adiabatic acoustic tensors does not exist, even in the context of the associative flow law.

Different thermal effects and thermo-mechanical couplings affect the mechanical behaviour of solids. In particular, while it is known that all mechanical properties are usually temperature dependent and constitute concurrent causes of thermal softening, other less recognized phenomena such as thermal expansion, heat conduction, mechanical dissipation and phase transformations need to be considered. However, the analysis

that will be presented, restricted to rate-independent material beahviour [1] , reveals that the roles of these phenomena may be more or less important.

The present article is organized as follows. A broad constitutive framework for thermo-inelastic material behaviour is introduced in the second section, where all thermal effects and their couplings are described in details. In particular, the presented constitutive framework—limited for simplicity to small strains and time-independent behaviour—includes non-associative elastoplasticity and damage models and the rate-constitutive equations are specified both under isothermal and adiabatic conditions, defining the relevant tangent moduli, that will become essential later. Within the above constitutive framework and after a review of jump conditions (third section), the strain localization under quasi-static conditions is addressed in the fourth section, using the classical shear band analysis restricted to the loading branch of the constitutive operator (Hill, 1962; Rice, 1976), though now extended to include thermal effects and thermomechanical couplings. This extension is performed by employing, additionally to the usual discontinuity conditions on mechanical variables, the jump conditions associated to the thermal variables, essentially, the temperature rate and the heat flux. Localization is shown to first occur when either of the thresholds corresponding to singularity of the isothermal or adiabatic acoustic tensor is reached. The mechanical interpretation of such a condition involves the formation of a discontinuity of the heat flux divergence across a band in the former case or of the temperature rate in the latter case. The hierarchy of the two thresholds is also analyzed. Plastic acceleration waves are briefly considered in the fifth section, to show that propagation velocity is related to the eigenvalues of the isothermal acoustic tensor, while the adiabatic acoustic tensor does not play a role. A consequence of this is that, while strain localization corresponds to stationarity of acceleration waves in the isothermal case, when thermal effects are considered, localization may occur in adiabatic conditions, with still propagating acceleration waves. Results presented in the fourth and fifth sections are complemented by a linear perturbation analysis developed in the sixth section. In particular, at a certain stage of a homogeneous deformation of an infinite body, an infinitely small perturbation is applied and the resulting perturbed motion analysed. Assuming a linearized response around the homogeneous solution (see Bigoni and Petryk, 2002, for a discussion on the limits of this assumption), the conditions for unbounded growth of perturbation are derived. Under dynamic conditions, unbounded growth is associated only to the short wavelength regime and to divergence growth or flutter phenomena governed by the *isothermal* acoustic tensor. When inertia is neglected, the perturbation analysis corresponds to a bifurcation analysis in a sense much similar to the internal instabilities defined by Biot (1965). It is therefore shown that for a quasi-static path—which in general is neither isothermal nor adiabatic—internal instability corresponds either to the singularity of the isothermal acoustic tensor (localization in the incrementally isothermal conditions) or to the singularity of the adiabatic acoustic tensor (localization in the

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[1] Rate sensitivity may become a key issue in the modelling of metals, particularly where temperature activates dislocation motion. It may be worth mentioning that there is a large literature devoted to strain localization on thermo-sensitive, viscoplastic materials (see among others Anand et al., 1987; Bai, 1982; Molinari, 1985); this paper remains however confined to inviscid constitutive laws.

incrementally adiabatic conditions) and is associated to the full range of wavelengths. Results for a particular set of elastic–plastic constitutive equations with associative flow rule conclude the paper (seventh section). The constitutive framework assumed in the last section was tailored to model the behaviour of the Inconel 718 alloy employed in turbine discs in jet engines. In particular, the complete thermo-mechanical behaviour of the material in service conditions has been identified through monotonic and cyclic tests for a wide range of temperature (20–800°C) by Benallal and Ben Cheikh (1987). The model is employed to show that a hierarchy between the two localization thresholds does not exist, and to quantify the 'gap' in a loading process between them.

## 2. Thermoinelastic constitutive equations

A broad class of rate-independent, coupled thermo-irreversible constitutive equations is considered, including thermoplasticity and thermodamage. For simplicity, the presentation is restricted to small strain. The main assumptions from which the theory is built are listed below.

- The Helmholtz free energy per unit mass

$$\Psi = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}, T), \quad \Psi = e - Ts, \tag{1}$$

  where $\boldsymbol{\varepsilon}$ is the (small) strain tensor, $T$ the absolute temperature and $\boldsymbol{\alpha}$ is a generic collection of internal variables of various tensorial nature (scalars, vectors or second-order tensors) describing different physical mechanisms governing inelastic deformation. Moreover, $e$ and $s$ are the specific internal energy and the specific entropy, respectively.
- The reversibility domain, defining the range in which inelastic processes are excluded, is defined through the yield (or damage) function

$$f(\mathbf{A}, \boldsymbol{\alpha}, T) \leqslant 0, \tag{2}$$

  where $\mathbf{A}$ are the thermodynamical forces associated to the internal variables $\boldsymbol{\alpha}$. Inelastic deformations are therefore possible only if $f = 0$, and during plastic flow the evolution of the internal variables must satisfy Prager's consistency $\dot{f} = 0$.
- The evolution of internal variables given by

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda}\mathbf{P}, \quad \mathbf{P} = \frac{\partial F}{\partial \mathbf{A}}, \tag{3}$$

  where the potential function $F = F(\mathbf{A}, \boldsymbol{\alpha}, T)$ is a function of the state variables and $\dot{\lambda}$ satisfies the Kuhn–Tucker conditions

$$\dot{\lambda} \geqslant 0, \quad f \leqslant 0, \quad \dot{\lambda}f = 0. \tag{4}$$

  We note that the intrinsic dissipation, defined as

$$\mathscr{D} = \dot{\boldsymbol{\alpha}} \cdot \mathbf{A} = \dot{\lambda}\mathbf{P} \cdot \mathbf{A} \geqslant 0, \tag{5}$$

is assumed not negative, a condition implying restrictions on the possible pairs of tensors $\mathbf{P}$ and $\mathbf{A}$.

Now, the rate equations can be obtained as follows. The stress tensor $\boldsymbol{\sigma}$, the thermodynamical forces $\mathbf{A}$ associated to the internal variables $\boldsymbol{\alpha}$ and the entropy $s$ are calculated employing the state laws:

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{A} = -\rho \frac{\partial \Psi}{\partial \boldsymbol{\alpha}}, \quad s = -\frac{\partial \Psi}{\partial T}, \tag{6}$$

where $\rho$ is the mass density. Time differentiation of (6) together with (3) yields

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}_{\mathrm{i}}[\dot{\boldsymbol{\varepsilon}}] - \dot{\lambda} \mathbf{M}_{\mathrm{i}} + \dot{T} \mathbf{B},$$

$$\dot{\mathbf{A}} = \frac{\partial \mathbf{A}}{\partial \boldsymbol{\varepsilon}}[\dot{\boldsymbol{\varepsilon}}] + \dot{\lambda} \frac{\partial \mathbf{A}}{\partial \boldsymbol{\alpha}}[\mathbf{P}] + \frac{\partial \mathbf{A}}{\partial T} \dot{T},$$

$$\dot{s} = -\frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon} \partial T} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \frac{\partial^2 \Psi}{\partial \boldsymbol{\alpha} \partial T} \cdot \frac{\partial F}{\partial \mathbf{A}} - \frac{\partial^2 \Psi}{\partial T^2} \dot{T}, \tag{7}$$

where

$$\mathbb{E}_{\mathrm{i}} = \rho \frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon}^2} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{M}_{\mathrm{i}} = -\rho \left( \frac{\partial^2 \Psi}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\varepsilon}} \right)^{\mathrm{T}} \left[ \frac{\partial F}{\partial \mathbf{A}} \right] = -\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\alpha}}[\mathbf{P}], \tag{8}$$

are the *isothermal elastic incremental fourth-order tensor* and the *isothermal flow mode tensor*. Moreover, the second-order symmetric tensor

$$\mathbf{B} = \rho \frac{\partial^2 \Psi}{\partial T \partial \boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\sigma}}{\partial T}, \tag{9}$$

is the stress-temperature tensor.

## 2.1. Incremental constitutive laws with isothermal quantities

When positive, the plastic multiplier $\dot{\lambda}$ may be calculated from Prager's consistency:

$$\dot{f} = \mathbf{Q} \cdot \dot{\mathbf{A}} + \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \mathbf{P} + \frac{\partial f}{\partial T} \dot{T} = 0, \tag{10}$$

where

$$\mathbf{Q} = \frac{\partial f}{\partial \mathbf{A}}, \tag{11}$$

is the yield function gradient with respect to the thermodynamical forces.

Using Eq. (7)$_2$ in the consistency condition (10), we get

$$\dot{\lambda} = \frac{\langle \mathbf{N}_{\mathrm{i}} \cdot \dot{\boldsymbol{\varepsilon}} - \beta \dot{T} \rangle}{H_{\mathrm{i}}}, \tag{12}$$

where the operator $\langle \rangle$ is defined for every scalar $\alpha$ as $\langle \alpha \rangle = (\alpha + |\alpha|)/2$, moreover

$$\mathbf{N}_{\mathrm{i}} = -\rho \frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\alpha}}[\mathbf{Q}], \quad \beta = \rho \frac{\partial^2 \Psi}{\partial T \partial \boldsymbol{\alpha}} \cdot \mathbf{Q} - \frac{\partial f}{\partial T}, \tag{13}$$

and the isothermal plastic modulus

$$H_i = \rho \mathbf{Q} \cdot \frac{\partial^2 \Psi}{\partial \boldsymbol{\alpha}^2}[\mathbf{P}] - \frac{\partial f}{\partial \boldsymbol{\alpha}} \cdot \mathbf{P}, \tag{14}$$

is assumed strictly positive. Finally, one can substitute Eq. (12) into Eq. (7) to get the rate constitutive laws

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}_i[\dot{\boldsymbol{\varepsilon}}] - \frac{1}{H_i}\langle \mathbf{N}_i \cdot \dot{\boldsymbol{\varepsilon}} - \beta\dot{T}\rangle \mathbf{M}_i + \dot{T}\,\mathbf{B}, \tag{15}$$

expressed in terms of isothermal quantities. In particular, in isothermal conditions, $\dot{T}=0$, the tangent moduli associated to the loading branch of constitutive equation (15) is

$$\mathbb{D}_i = \mathbb{E}_i - \frac{1}{H_i}\mathbf{M}_i \otimes \mathbf{N}_i. \tag{16}$$

Note that tensor $\mathbb{D}_i$ possesses the major symmetry in the special case of normality, defined as

$$\mathbf{N}_i = \mathbf{M}_i. \tag{17}$$

The constitutive equations (15) are expressed in the direct form, relating the rate of stress to the strain rate. These can be inverted, as reported in Appendix A.

## 2.2. Incremental constitutive laws with adiabatic quantities

Local conservation of energy or the First Law of Thermodynamics is (see e.g. Truesdell and Toupin, 1960; Germain, 1973; Lemaitre and Chaboche, 1985)

$$\rho\dot{e} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \operatorname{div}\mathbf{q} + r, \tag{18}$$

where $r$ is the heat supply per unit volume generated by internal sources, $\mathbf{q}$ is the heat flux, taken to obey a generalized Fourier law of heat conduction

$$\mathbf{q} = -\mathbf{K}\nabla T, \tag{19}$$

in which tensor $\mathbf{K} = \mathbf{K}(\mathbf{A}, \boldsymbol{\alpha}, T)$ is assumed positive definite. Note that local adiabatic conditions correspond to $r = \operatorname{div}\mathbf{q} = 0$.

Using Eqs. (1)$_2$ and (6)$_3$, we obtain two alternative, equivalent forms of the local conservation of energy (18)

$$c\rho\dot{T} = T\mathbf{B} \cdot \dot{\boldsymbol{\varepsilon}} + \dot{\lambda}T\chi - \operatorname{div}\mathbf{q} + r,$$
$$\rho\dot{s}T = \dot{\lambda}(\mathbf{A} \cdot \mathbf{P}) - \operatorname{div}\mathbf{q} + r, \tag{20}$$

where

$$c = -T\frac{\partial^2 \Psi}{\partial T^2}, \tag{21}$$

is the specific heat at constant strain and internal variables—assumed positive for simplicity—and internal variables and

$$\chi = \left(\rho\frac{\partial^2 \Psi}{\partial \boldsymbol{\alpha}\partial T} + \frac{\mathbf{A}}{T}\right) \cdot \mathbf{P}. \tag{22}$$

It is important to remark that the Clausius–Duhem inequality (Second Law of Thermodynamics, see e.g. Truesdell and Noll, 1965, Sect. 79)

$$\rho\dot{s} \geqslant -\frac{\text{div}\,\mathbf{q}}{T} + \frac{\mathbf{q}\cdot\nabla T}{T^2} + \frac{r}{T}, \tag{23}$$

using the balance of energy $(20)_2$, can be written in the form

$$\mathscr{D} - \frac{\mathbf{q}\cdot\nabla T}{T} \geqslant 0. \tag{24}$$

From expression (24) it follows that the assumptions of non-negative inelastic dissipation (5) and of Fourier law of heat conduction, Eq. (19), imply *a-priori* satisfaction of the Clausius–Duhem inequality (24), which will be therefore not further considered.

Eq. $(20)_1$ shows that the variation of temperature at a given point of the solid is due to the external heat supply $r$, the heat conduction $\text{div}\,\mathbf{q}$, the mechanical dissipation $\dot{\lambda}\mathbf{A}\cdot\mathbf{P}$, the isentropic thermo-elastic term $T\mathbf{B}\cdot\dot{\boldsymbol{\varepsilon}}$, and finally to the thermo-inelastic coupling contribution

$$\dot{\lambda}T\rho\,\frac{\partial^2\Psi}{\partial\boldsymbol{\alpha}\partial T}\cdot\mathbf{P}.$$

The thermo-elastic contribution includes in particular thermal expansion but also thermal softening due to the decrease of elastic modulus with increasing temperature. Thermal softening is also related to the thermo-plastic term, due to the decrease of the yield stress and hardening phenomena with temperature. However, the thermo-plastic term may also include various important effects, as for instance, latent energy in phase transformations.

We obtain from Eqs. (12) and (20)

$$\dot{\lambda} = \frac{\langle\mathbf{N}_\text{a}\cdot\dot{\boldsymbol{\varepsilon}} - \beta\gamma/(\rho c)\rangle}{H_\text{a}}, \tag{25}$$

where

$$\mathbf{N}_\text{a} = \mathbf{N}_\text{i} - \frac{T\beta}{\rho c}\,\mathbf{B}, \tag{26}$$

$$\gamma = r - \text{div}\,\mathbf{q}, \tag{27}$$

and

$$H_\text{a} = H_\text{i} + \frac{T\beta\chi}{\rho c}, \tag{28}$$

is the adiabatic plastic modulus that is assumed strictly positive. As will become clear later, positiveness of $H_\text{a}$ excludes locking of the adiabatic stress strain curve. For simplicity, in the following we will not consider the heat supply, $r = 0$, so that $\gamma$ will be reduced to the only contribution of the heat flux.

Using Eq. (25) in Eq. $(7)_1$ and keeping into account Eq. (20) we may rewrite the rate equations (15) in a form involving adiabatic quantities, namely

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}_\text{a}[\dot{\boldsymbol{\varepsilon}}] - \frac{1}{H_\text{a}}\left\langle\mathbf{N}_\text{a}\cdot\dot{\boldsymbol{\varepsilon}} - \frac{\beta\gamma}{\rho c}\right\rangle\mathbf{M}_\text{a} + \frac{\gamma}{\rho c}\,\mathbf{B}, \tag{29}$$

where

$$\mathbb{E}_a = \mathbb{E}_i + \frac{T}{\rho c} \mathbf{B} \otimes \mathbf{B}, \tag{30}$$

is the elastic adiabatic incremental fourth-order tensor and

$$\mathbf{M}_a = \mathbf{M}_i - \frac{T\chi}{\rho c}\mathbf{B}. \tag{31}$$

The inversion of constitutive equations (29) is reported in Appendix B.

It may be worth noting that in locally adiabatic conditions, $\gamma = 0$, the tangent moduli associated to the loading branch of (29) is

$$\mathbb{D}_a = \mathbb{E}_a - \frac{1}{H_a} \mathbf{M}_a \otimes \mathbf{N}_a, \tag{32}$$

which corresponds to (16) when the index 'i' is replaced by 'a'. Note that, differently from $\mathbb{D}_i$, tensor $\mathbb{D}_a$ does not possess the major symmetry even in the special case of normality, condition (17).

As a clarification of the presented constitutive framework, we report in Appendix C an example of specialization of the presented constitutive equations to damage, whereas a specialization to elastoplasticity will be given in Section 7.

## 3. Jump conditions

As a necessary premise to the strain localization and acceleration wave analyses that will follow, we briefly review the discontinuity conditions across moving and stationary discontinuity surfaces, with an emphasis on certain details that will become important later.

### 3.1. Jump conditions for propagating discontinuity surfaces

Let us denote by $\Sigma$ a discontinuous surface with unit normal $\mathbf{n}$ moving with normal speed $C$ with respect to the material. If $\mathbf{v}$ is the velocity of the material points and $\mathbf{w}$ is the speed of the surface $\Sigma$ with respect to the particles instantaneously situated upon it, we may write

$$\mathbf{w} \cdot \mathbf{n} = -C. \tag{33}$$

The jump conditions across a discontinuity surface can be found in (Truesdell and Toupin, 1960; Chadwick and Powdrill, 1965; Chadwick, 1976). These are listed below under small strain hypothesis.

Assuming continuity of the velocity $\mathbf{v}$ field, the Hadamard's compatibility relations imply that across $\Sigma$ the following relation holds

$$[\![\nabla \mathbf{v}]\!] = \mathbf{g} \otimes \mathbf{n}, \quad \mathbf{g} = -\frac{1}{C}[\![\dot{\mathbf{v}}]\!], \tag{34}$$

where $\dot{\mathbf{v}}$ is the acceleration. Assuming continuity of the stress $\boldsymbol{\sigma}$, the compatibility relations also yield

$$[\![\operatorname{div}\boldsymbol{\sigma}]\!] = -\frac{1}{C}[\![\dot{\boldsymbol{\sigma}}]\!]\mathbf{n}. \tag{35}$$

Mass conservation requires

$$\llbracket \rho C \rrbracket = 0, \tag{36}$$

while balance of momentum leads to

$$- C\rho \llbracket \mathbf{v} \rrbracket = \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket = \mathbf{0}. \tag{37}$$

Starting from the equations of motion

$$\operatorname{div} \boldsymbol{\sigma} = \rho \dot{\mathbf{v}}, \tag{38}$$

and using Eq. (35), balance of momentum also implies

$$\llbracket \dot{\boldsymbol{\sigma}} \rrbracket \mathbf{n} = -\rho C \llbracket \dot{\mathbf{v}} \rrbracket. \tag{39}$$

Finally, conservation of energy gives

$$- C\rho \llbracket e + \tfrac{1}{2} \mathbf{v}^2 \rrbracket = \llbracket \mathbf{v} \cdot \boldsymbol{\sigma} \mathbf{n} + \mathbf{q} \cdot \mathbf{n} \rrbracket. \tag{40}$$

Assuming continuity of internal energy $e$, Eq. (40) becomes

$$\llbracket \mathbf{q} \cdot \mathbf{n} \rrbracket = 0 \tag{41}$$

implying the continuity of the heat flux. When the temperature field is continuous across $\Sigma$, from Hadamard jump conditions, one obtains

$$\llbracket \nabla T \rrbracket = \mu \mathbf{n}, \quad \llbracket \dot{T} \rrbracket = -C\mu, \tag{42}$$

where $\mu$ is a scalar. Therefore, using Fourier's law in Eq. (41) and then Eq. (42)$_1$, the positive definiteness of the conduction tensor $\mathbf{K}$ provides

$$\llbracket \nabla T \rrbracket = 0, \tag{43}$$

and, consequently, Eqs. (42) yield

$$\llbracket \dot{T} \rrbracket = 0. \tag{44}$$

## 3.2. Stationary discontinuities

The conditions to be satisfied at a stationary discontinuity surface may be obtained from Eqs. (39) and (40) in the limit in which the velocity of propagation is equal to zero, $C = 0$. Under the small strain hypothesis, and *assuming continuity of the velocity* $\mathbf{v}$, the balance laws reduce to the kinematical conditions of compatibility on the velocity gradient

$$\llbracket \nabla \mathbf{v} \rrbracket = \mathbf{g} \otimes \mathbf{n}, \quad \mathbf{g} = \llbracket \nabla \mathbf{v} \rrbracket \mathbf{n}, \tag{45}$$

and to the requirement of continuity of the fluxes and their rates

$$\llbracket \boldsymbol{\sigma} \rrbracket \mathbf{n} = \mathbf{0}, \quad \llbracket \dot{\boldsymbol{\sigma}} \rrbracket \mathbf{n} = \mathbf{0}, \tag{46}$$

$$\llbracket \mathbf{q} \rrbracket \cdot \mathbf{n} = 0, \quad \llbracket \dot{\mathbf{q}} \rrbracket \cdot \mathbf{n} = 0. \tag{47}$$

Spatial continuity of the temperature gives again condition (43). However, it is important to note that, differently from the propagation case $C \neq 0$, now Eqs. (43) and (47)$_2$ *do not imply* that $\dot{T}$ is continuous. In other words, Eq. (44) does not necessarily hold for stationary discontinuities and cannot be obtained just setting $C = 0$ in Eq. (42)$_2$.

## 4. Strain localization

Strain localization in the sense of Hill (1962), Rudnicki and Rice (1975), Rice (1976) is a static phenomenon in essence and corresponds to the emergence of a strain rate field discontinuous across a planar band. It turns out in the isothermal case that strain localization corresponds to vanishing speed of the acceleration wave having the propagation direction orthogonal to the discontinuity band (Hill, 1962; Rice, 1976). Strain localization analysis is performed below for a fully coupled thermal setting, with reference to the loading branch of constitutive operator (so that the possibility of discontinuities occurring with elastic unloading are not considered). We begin defining the two following acoustic tensors.

The isothermal elastoplastic acoustic tensor:

$$\mathbf{A}_i^{ep}(\mathbf{n}) = \mathbf{A}_i^e(\mathbf{n}) - \frac{1}{H_i}\,\mathbf{M}_i\mathbf{n} \otimes \mathbf{N}_i\mathbf{n}, \tag{48}$$

and the adiabatic elastoplastic acoustic tensor:

$$\mathbf{A}_a^{ep}(\mathbf{n}) = \mathbf{A}_a^e(\mathbf{n}) - \frac{1}{H_a}\,\mathbf{M}_a\mathbf{n} \otimes \mathbf{N}_a\mathbf{n}, \tag{49}$$

where

$$\mathbf{A}_i^e(\mathbf{n})\mathbf{g} = \mathbb{E}_i[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}, \quad \mathbf{A}_a^e(\mathbf{n})\mathbf{g} = \mathbb{E}_a[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}, \tag{50}$$

are, respectively, the isothermal and adiabatic elastic acoustic tensors satisfying

$$\mathbf{A}_a^e(\mathbf{n}) = \mathbf{A}_i^e(\mathbf{n}) + \frac{T}{\rho c}\,\mathbf{Bn} \otimes \mathbf{Bn}, \tag{51}$$

with the following inverse formula

$$[\mathbf{A}_a^e(\mathbf{n})]^{-1} = [\mathbf{A}_i^e(\mathbf{n})]^{-1} - \frac{T}{\rho c}\,\frac{[\mathbf{A}_i^e(\mathbf{n})]^{-1}\mathbf{Bn} \otimes [\mathbf{A}_i^e(\mathbf{n})]^{-1}\mathbf{Bn}}{1 + (T/\rho c)\mathbf{Bn} \cdot [\mathbf{A}_i^e(\mathbf{n})]^{-1}\mathbf{Bn}}. \tag{52}$$

The form of the inverse (52) clarifies that invertibility of the isothermal elastic acoustic tensor implies invertibility of the adiabatic elastic acoustic tensor. The converse do not need to be true, but we assume for simplicity both elastic acoustic tensors to be positive definite in the following. It is important to remark that in the special case of associative flow rule, the isothermal elastoplastic acoustic tensor becomes symmetric, whereas the adiabatic remains unsymmetric. We note that the inverse of the elastoplastic acoustic tensors can also be calculated, resulting in

$$[\mathbf{A}_x^{ep}(\mathbf{n})]^{-1} = [\mathbf{A}_x^e(\mathbf{n})]^{-1} + \frac{[\mathbf{A}_x^e(\mathbf{n})]^{-1}\mathbf{M}_x\mathbf{n} \otimes [\mathbf{A}_x^e(\mathbf{n})]^{-1}\mathbf{N}_x\mathbf{n}}{H_x - \mathbf{M}_x\mathbf{n} \cdot [\mathbf{A}_x^e(\mathbf{n})]^{-1}\mathbf{N}_x\mathbf{n}}, \tag{53}$$

where $x$ stands for 'i' or 'a' denoting the isothermal or adiabatic quantities and the formula has been obtained under the hypothesis that the denominator of the fraction be different from zero.

In the usual strain localization analysis, the current state is assumed *given* and homogeneous, so that Eqs. $(46)_1$ and $(47)_1$ are trivially satisfied. In the present context, a state corresponds to all usual mechanical variables, plus the thermal ones, including

temperature and heat flux, so that we assume that these quantities are continuous functions of the place and we look for the possibility of non-trivial jumps in rates of stress, strain, temperature, and—in a sense to be pinpointed later—of the divergence of the heat flux, variable $\gamma$. Application of rate conditions (45)–(47) with the constitutive law (15) and the local conservation of energy (20) yields in terms of isothermal quantities

$$\mathbf{A}_{i}^{ep}(\mathbf{n})\mathbf{g} + \left( \frac{\beta}{H_i} \mathbf{M}_i\mathbf{n} + \mathbf{Bn} \right) [\![\dot{T}]\!] = \mathbf{0},$$

$$[\![\gamma]\!] = \left( c\rho + \frac{\beta\chi T}{H_i} \right) [\![\dot{T}]\!] - T \left( \mathbf{Bn} + \frac{\chi}{H_i} \mathbf{N}_i\mathbf{n} \right) \cdot \mathbf{g}, \tag{54}$$

or, equivalently, in terms of adiabatic quantities [i.e. using Eq. (29) instead of Eq. (15)]

$$\mathbf{A}_{a}^{ep}(\mathbf{n})\mathbf{g} + \frac{1}{\rho c} \left( \frac{\beta}{H_a} \mathbf{M}_a\mathbf{n} + \mathbf{Bn} \right) [\![\gamma]\!] = \mathbf{0},$$

$$c\rho[\![\dot{T}]\!] = T \left( \mathbf{Bn} + \frac{\chi}{H_a} \mathbf{N}_a\mathbf{n} \right) \cdot \mathbf{g} + \left( 1 - \frac{\beta}{\rho c H_a} \right) [\![\gamma]\!]. \tag{55}$$

Though the two systems of Eqs. (54) and (55) are fully equivalent, form (54) is useful in case when $[\![\dot{T}]\!]$ is given, whereas the other form (55) may be more convenient when $[\![\gamma]\!]$ is prescribed.

If both $\mathbf{A}_{a}^{ep}$ and $\mathbf{A}_{i}^{ep}$ are non-singular, Eqs. (54)$_1$ and (55)$_1$ can be solved yielding

$$\mathbf{g} = -[\![\dot{T}]\!][\mathbf{A}_{i}^{ep}(\mathbf{n})]^{-1} \left( \frac{\beta}{H_i} \mathbf{M}_i\mathbf{n} + \mathbf{Bn} \right),$$

$$\mathbf{g} = -\frac{[\![\gamma]\!]}{\rho c}[\mathbf{A}_{a}^{ep}(\mathbf{n})]^{-1} \left( \frac{\beta}{H_a} \mathbf{M}_a\mathbf{n} + \mathbf{Bn} \right). \tag{56}$$

Let us discuss now conditions (54) and (55). If we assume that the divergence of the heat flux is a continuous function of place, so that $[\![\gamma]\!] = 0$, the problem setting corresponds to that assumed for uniqueness analysis by Mróz and Raniecki (1976), Raniecki (1979) and Raniecki and Bruhns (1981). Using Eq. (55) we conclude that $\dot{T}$, $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$ remain continuous as long as $\mathbf{A}_{a}^{ep}$ is not singular. When $\det \mathbf{A}_{a}^{ep} = 0$, non-trivial jumps in the rate quantities become possible, with still a null jump in $\gamma$, a feature which motivates the denomination 'localization in incrementally adiabatic conditions'. Note that these jumps are in a sense undetermined since the quantities depend on the modulus of $\mathbf{g}$, which is an eigenvector of the adiabatic acoustic tensor. More in general, if $[\![\gamma]\!] \neq 0$ is a prescribed quantity, we see from Eq. (56)$_2$ that until the adiabatic acoustic tensor is invertible, the solution in terms of $\mathbf{g}$ and $[\![\dot{T}]\!]$ is unique, but this uniqueness is lost at the adiabatic strain localization.

Let us generalize now the previous situation, assuming that all fields defining the state are given, with the exception of $[\![\gamma]\!] \neq 0$, which is now (together with $[\![\dot{T}]\!]$) an *unprescribed* quantity and is allowed to jump across a narrow zone bonded by two planes. Until the two acoustic tensors are not singular, there is a one-to-one correspondence between the two variables, whereas infinite solutions occur when one of the acoustic tensors becomes singular. In particular, when the isothermal acoustic tensor is

singular, system (54) yields $[\![\dot{T}]\!] = 0$ for appropriate $\mathbf{g}$ and $[\![\gamma]\!] \neq 0$ becomes arbitrary. The fact that the jump in the rate of temperature vanishes motivates the denomination 'localization in incrementally isothermal conditions'.

A difficulty with the above point regarding the interpretation of singularity of the isothermal acoustic tensor is that it is not easy to envisage a situation in which the gradient of the heat flux is not assigned. However, one can think a evolution problem and imagine a bifurcation mode where the state changes instantaneously, and this is the request for having a localization with a null jump in temperature rate. However, the difficulty can also be partially avoided by considering a second-order problem, in the way illustrated by Bigoni (1996) for smooth bifurcations, thus assuming both $[\![\ddot{T}]\!]$ and $[\![\dot{\gamma}]\!]$ as free parameters. We do not want to embark here in this analysis, since we believe that the interpretation of singularity of the isothermal acoustic tensor involves a change of the usual setting assumed for strain localization, so that the recurse to higher-order theories do not really modifies the scenario.

As a conclusion, both the thresholds of strain localization in incrementally isothermal and adiabatic conditions should be regarded as relevant to localization. These correspond to the singularity of the isothermal and adiabatic acoustic tensors

$$\det \mathbf{A}_x^{\mathrm{ep}}(\mathbf{n}) = 0, \tag{57}$$

where $x$ stands for 'i' or 'a'. Obviously, strain localization remains possible to occur also in between the two thresholds corresponding to singularity of the two acoustic tensors. Such a localization should occur with peculiar features that may be worth of analyzing but fall beyond the scope of the present paper.

### 4.1. Hierarchy between isothermal and adiabatic strain localization

Conditions (57) can be written in terms of a critical value of plastic modulus. For a fixed band normal $\mathbf{n}$, this corresponds to the vanishing of the denominator in (53), so that the critical condition for localization occurs when the isothermal plastic modulus $H_i$ satisfies

$$H_i = H_i^{\mathrm{cr}}(\mathbf{n}) = \max \left\{ \mathbf{M}_i \mathbf{n} \cdot [\mathbf{A}_i^{\mathrm{e}}]^{-1} \mathbf{N}_i \mathbf{n}, \ \mathbf{M}_a \mathbf{n} \cdot [\mathbf{A}_a^{\mathrm{e}}]^{-1} \mathbf{N}_a \mathbf{n} - \frac{T\beta\chi}{\rho c} \right\}, \tag{58}$$

in which both conditions of isothermal and adiabatic strain localization have been considered. If we define the isothermal hardening modulus $h_i$ as

$$h_i = H_i - \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{M}_i], \tag{59}$$

where $\mathbb{C}_i$ is the inverse of $\mathbb{E}_i$ in the sense detailed in Appendix A, condition (58) can be expressed in the form

$$h_i = h_i^{\mathrm{cr}}(\mathbf{n}) = \max \left\{ \mathbf{M}_i \mathbf{n} \cdot [\mathbf{A}_i^{\mathrm{e}}]^{-1} \mathbf{N}_i \mathbf{n}, \ \mathbf{M}_a \mathbf{n} \cdot [\mathbf{A}_a^{\mathrm{e}}]^{-1} \mathbf{N}_a \mathbf{n} - \frac{T\beta\chi}{\rho c} \right\}$$
$$- \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{M}_i]. \tag{60}$$

Conditions (58) and (60) can be easily rewritten in an equivalent way, but in terms of adiabatic moduli $H_a$ and $h_a$ (for the definition of the latter see Appendix B)

$$H_a = H_a^{cr}(\mathbf{n}) = H_i^{cr}(\mathbf{n}) + \frac{T\beta\chi}{\rho c}, \quad h_a = h_a^{cr}(\mathbf{n}) = H_i^{cr}(\mathbf{n}) + \frac{T\beta\chi}{\rho c} - \mathbf{N}_a \cdot \mathbb{C}_a[\mathbf{M}_a]. \tag{61}$$

In practice, the first occurrence during a loading process is of importance and this is obtained via the usual constrained maximization of one of the conditions (58)–(61) as function of the unit vector $\mathbf{n}$. Such a maximization has been explicitly solved for isotropic elasticity and isotropic $\mathbf{B}$ by a number of Authors (Bigoni and Hueckel, 1991; Benallal and Comi, 1996, among others), but for anisotropic elasticity (or generic $\mathbf{B}$) results are still limited to special cases (Bigoni and Loret 1999, Bigoni et al., 2000).

Under the additional assumption of isotropic elasticity it is possible to prove (following Rice, 1977) that the critical isothermal hardening modulus is never positive for associative flow rule. Now the question arises if there exists some hierarchy between the two localization thresholds corresponding to isothermal and adiabatic conditions.

A distance between failure of positive definiteness of the two isothermal and adiabatic acoustic tensors at a fixed band normal $\mathbf{n}$ can be quantified in terms of the following parameter

$$d(\mathbf{n}) = -\mathbf{M}_i\mathbf{n} \cdot [\mathbf{A}_i^e]^{-1}\mathbf{N}_i\mathbf{n} + \mathbf{M}_a\mathbf{n} \cdot [\mathbf{A}_a^e]^{-1}\mathbf{N}_a\mathbf{n} - \frac{T\beta\chi}{\rho c}, \tag{62}$$

a quantity that when negative (positive) indicates that the determinant of the isothermal (adiabatic) acoustic tensor vanishes when the adiabatic (isothermal) is still positive definite, for the given $\mathbf{n}$. Now, assuming $\mathbf{A}_i^e$ positive definite, simple algebra yields

$$d(\mathbf{n}) = -\kappa(\mathbf{n})(\mathbf{N}_i\mathbf{n} \cdot [\mathbf{A}_i^e(\mathbf{n})]^{-1}\mathbf{Bn} + \beta)(\mathbf{M}_i\mathbf{n} \cdot [\mathbf{A}_i^e(\mathbf{n})]^{-1}\mathbf{Bn} + \chi), \tag{63}$$

where

$$\kappa(\mathbf{n}) = \frac{1}{\rho c/T + \mathbf{Bn} \cdot [\mathbf{A}_i^e(\mathbf{n})]^{-1}\mathbf{Bn}}, \tag{64}$$

is strictly positive. Due to the fact that $\chi$ and $\beta$ are different quantities, $d(\mathbf{n})$ is not *a-priori* negative, even in the particular case of associative flow rule, $\mathbf{M}_i = \mathbf{N}_i$. In particular, $\chi$ and $\beta$ are non-negative scalars which depend on the specific constitutive assumptions, so that, if $\mathbf{n}$ is considered as a free parameter, it can be easily shown that $d(\mathbf{n})$ can be made of any sign. However, $\mathbf{n}$ is not free, but is determined as a solution of a constrained maximization problem in strain localization. Anyway, specific examples postponed to Section 7 show that $d(\mathbf{n})$ may take any sign, even for associative flow rule. This is an important difference to porous plastic, fluid-saturated media, where strain localization in drained conditions always occurs before strain localization in undrained conditions for associative flow rule (Benallal and Comi, 2002).

## 5. Acceleration waves

An acceleration wave is a surface moving in a material, across which the second-order derivatives of a *smooth* displacement field (i.e. the acceleration field) and related

quantities suffer jump discontinuities. Propagation of such waves including thermal effects is a thoroughly analyzed problem in elasticity (see Chadwick and Currie, 1972; Chen, 1968, 1973 and references quoted therein), but only scarcely considered in thermoplasticity. Following Mandel (1962, 1969) and Raniecki (1976), we assume continuity of the thermodynamical state and of the specific energy. In the class of inelastic material considered here, four types of acceleration waves are possible, namely, elastic waves (the state of the material in front and behind the wave is elastic), plastic waves (the state of the material in front and behind the wave is plastic), loading waves (the state of the material in front the wave is elastic, but is plastic behind), and unloading waves (the state of the material in front the wave is plastic, but is elastic behind).

The presentation is limited here to plastic waves and the necessary conditions for propagation of acceleration waves follow from the Hadamard restrictions on possible jumps in the material. In particular, following Mandel (1962, 1969), Hill (1962) and Raniecki (1976), we proceed noting that Eq. (44) implies that *acceleration waves are homothermal*. Moreover, considering Eq. (15) and using Eqs. (43), (34) and (37)$_2$ into the equations of motion the propagation condition is obtained

$$(\mathbf{A}_i^{ep}(\mathbf{n}) - \rho C^2\, \mathbf{I})[\![\dot{\mathbf{v}}]\!] = \mathbf{0}, \tag{65}$$

where $\mathbf{A}_i^{ep}(\mathbf{n})$ is the elastic–plastic isothermal acoustic tensor, Eq. (48).

We may conclude that vanishing speed of an acceleration wave occurs when the isothermal elastoplastic acoustic tensor $\mathbf{A}_i^{ep}(\mathbf{n})$ becomes singular for some propagation direction $\mathbf{n}$.

It is important to remark that in the analysis of acceleration waves, the quantity $\gamma$, Eq. (26)$_2$, must suffer a discontinuity across the acceleration front. This may be calculated using Eq. (20) as

$$[\![\gamma]\!] = -T\mathbf{B} \cdot [\![\dot{\boldsymbol{\varepsilon}}]\!] - \chi T [\![\dot{\lambda}]\!], \tag{66}$$

which implies that in the analysis of acceleration waves the div $\mathbf{q}$ is discontinuous, a condition that can be satisfied even when the temperature gradient is continuous.

When temperature effects are not present, stationarity of acceleration waves coincides with strain localization (Hill, 1962; Rice, 1977), understood as the appearance of a discontinuous strain rate field superimposed to a homogeneous field, whereas in the present context vanishing speed of acceleration waves coincides with localization in isothermal conditions, which also involves a jump in $\gamma$, see Section 4. There is an analogous of the present situation in porous, fluid saturated elastoplastic materials. In that context vanishing of the speed of an acceleration wave corresponds to singularity of the acoustic tensor in incrementally drained conditions (Loret and Harireche, 1991; Benallal and Comi, 2002).

## 6. Perturbations in thermomechanical problems

Certain mechanical interpretations have been provided in the previous sections to the condition of singularity of the isothermal and adiabatic acoustic tensors. We complement the investigation providing here an interpretation in terms of stability of an

infinite medium with respect to perturbations in the form of harmonic waves, in the short and long wavelength limits. In essence, we follow the approach of Simões et al. (1999), developed in the context of porous, fluid-saturated media, but we consider a full linearization (as in Benallal and Comi, 2002) of equations instead of working directly with rates. The investigation is restricted to the loading branch of the constitutive operator, so that a full elastoplastic analysis is not performed (see Bigoni and Petryk, 2002, for a thorough discussion on the limits of this approach).

An infinite, homogeneously deformed medium is considered for which the linearized governing equations are obtained by superposing to the reference solution a perturbation motion and linearizing this around the reference solution. The equations are listed in the following.

- Equations of motion

$$\text{div } \boldsymbol{\sigma}' = \rho \ddot{\mathbf{u}}', \tag{67}$$

  where the superscript $(\ )'$ is used to denote a perturbation solution and $\mathbf{u}$ is the displacement vector.
- Conservation of energy

$$T(\boldsymbol{\varepsilon}' \cdot \mathbb{A}_{\boldsymbol{\varepsilon}}[\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}, \dot{T}] + \boldsymbol{\alpha}' \cdot \mathbb{A}_{\boldsymbol{\alpha}}[\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}, \dot{T}] + T'\mathbb{A}_T[\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}, \dot{T}])$$

$$- T'\left(\mathbf{B} \cdot \dot{\boldsymbol{\varepsilon}} - \frac{\partial \mathbf{A}}{\partial T} \cdot \dot{\boldsymbol{\alpha}} - \frac{c}{T}\right) - \mathbf{A}' \cdot \dot{\boldsymbol{\alpha}}$$

$$+ \rho c \dot{T}' = T\mathbf{B} \cdot \dot{\boldsymbol{\varepsilon}}' - T\left(\frac{\partial \mathbf{A}}{\partial T} - \frac{\mathbf{A}}{T}\right) \cdot \dot{\boldsymbol{\alpha}}' - \text{div } \mathbf{q}', \tag{68}$$

  where

$$\mathbb{A}_{\mathbf{X}}[\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}, \dot{T}] = -\left(\frac{\partial \mathbf{B}}{\partial \mathbf{X}}\right)^{\mathrm{T}}[\dot{\boldsymbol{\varepsilon}}] + \left(\frac{\partial^2 \mathbf{A}}{\partial T \partial \mathbf{X}}\right)^{\mathrm{T}}[\dot{\boldsymbol{\alpha}}] - \frac{1}{\rho}\frac{\partial^3 \psi}{\partial \mathbf{X} \partial T^2}\dot{T}, \tag{69}$$

  in which $\mathbf{X}$ stands for $\boldsymbol{\varepsilon}$, $\boldsymbol{\alpha}$ or $T$ and

$$\mathbf{A}' = \frac{\partial \mathbf{A}}{\partial \boldsymbol{\varepsilon}}[\boldsymbol{\varepsilon}'] + \frac{\partial \mathbf{A}}{\partial \boldsymbol{\alpha}}[\boldsymbol{\alpha}'] + \frac{\partial \mathbf{A}}{\partial T}T'. \tag{70}$$

- Evolution law of internal variables

$$\dot{\boldsymbol{\alpha}}' = \mathbf{P}\dot{\lambda}' + \dot{\lambda}\mathbf{P}', \tag{71}$$

  where

$$\mathbf{P}' = \frac{\partial F}{\partial \mathbf{A}} \cdot \mathbf{A}' + \frac{\partial F}{\partial \boldsymbol{\alpha}} \cdot \boldsymbol{\alpha}' + \frac{\partial F}{\partial T}T'. \tag{72}$$

- Constitutive equations

$$\boldsymbol{\sigma}' = \mathbb{E}_{\mathrm{i}}[\boldsymbol{\varepsilon}'] + \rho \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\alpha}}[\boldsymbol{\alpha}'] + T'\mathbf{B}. \tag{73}$$

- Prager's consistency

$$\mathbf{N}_{\mathrm{i}} \cdot \boldsymbol{\varepsilon}' - \beta T' + \left[\left(\frac{\partial \mathbf{A}}{\partial \boldsymbol{\alpha}}\right)^{\mathrm{T}}[\mathbf{Q}] + \frac{\partial f}{\partial \boldsymbol{\alpha}}\right] \cdot \boldsymbol{\alpha}' = 0. \tag{74}$$

The perturbation field is now sought in the harmonic wave form

$$\{\mathbf{u}', T', \boldsymbol{\alpha}'\} = \{\hat{\mathbf{u}}, \hat{T}, \hat{\boldsymbol{\alpha}}\} e^{ik(\mathbf{n} \cdot \mathbf{x} + vt)}, \tag{75}$$

where $\mathbf{x}$ is the position, $\mathbf{n}$ the wave normal, $k$ is the wave number, $v$ the wave speed, $t$ denotes the time and the constant amplitudes $\hat{\mathbf{u}}$, $\hat{T}$, $\hat{\boldsymbol{\alpha}}$ are usually complex. We note that the velocity of propagation $v$ and the wave a number $k$ are in general complex quantities.

The form (75) of the perturbation enables us to observe that

$$\{\dot{\mathbf{u}}', \dot{T}', \dot{\boldsymbol{\alpha}}'\} = ikv\{\mathbf{u}', T', \boldsymbol{\alpha}'\}, \tag{76}$$

a condition which, used together with Eq. (75) into Eqs. (71) and (74) and substituted in Eq. (68), gives

$$\rho c \hat{T} = \chi T \frac{ik\hat{\mathbf{u}} \cdot \mathbf{N}_i \mathbf{n} - \beta \hat{T}}{H_i} + ikT\hat{\mathbf{u}} \cdot \mathbf{Bn} + \frac{ik}{v} \mathbf{n} \cdot \mathbf{Kn} \hat{T} + \mathscr{L}_1 \left( \frac{\hat{\mathbf{u}}}{v}, \frac{\hat{\boldsymbol{\alpha}}}{kv}, \frac{\hat{T}}{kv} \right), \tag{77}$$

where $\mathscr{L}_1$ denotes a linear function of its arguments, which does not need to be specified for further evaluation. Use of Eqs. (75), (76) in Eq. (73) and finally in Eq. (67) yields the equations of motion in the form

$$-k^2 \mathbf{A}_i^{ep}(\mathbf{n})\hat{\mathbf{u}} + ik \frac{\beta \hat{T}}{H_i} \mathbf{M}_i \mathbf{n} + ik\hat{T} \mathbf{Bn} = -k^2 v^2 \rho \hat{\mathbf{u}} + \mathscr{L}_2 \left( \frac{k\hat{\mathbf{u}}}{v}, \frac{\hat{\boldsymbol{\alpha}}}{v}, \frac{\hat{T}}{v} \right), \tag{78}$$

where, again, $\mathscr{L}_2$ is a linear function of its arguments that need not to be further specified. In the short wavelength limit, $k \to \infty$ at fixed $v$, Eq. (77) becomes

$$\hat{T} = -\frac{vT}{\mathbf{n} \cdot \mathbf{Kn}} \left[ \frac{\chi}{H_i} \mathbf{N}_i \mathbf{n} + \mathbf{Bn} \right] \cdot \hat{\mathbf{u}}, \tag{79}$$

and the equations of motion (78) give

$$(\mathbf{A}_i^{ep}(\mathbf{n}) - \rho v^2 \mathbf{I})\hat{\mathbf{u}} = \mathbf{0}. \tag{80}$$

Eq. (80) shows that stability (defined as real and positive squares of the wave speeds) corresponds to positive and real eigenvalues of the elastoplastic isothermal acoustic tensor, so that two possibilities of instability may arise: a *divergence instability* when an eigenvalue becomes negative, and a *flutter instability*, when two eigenvalues become complex conjugate. Flutter instability, was never considered for thermal problems, but is not addressed here for conciseness (the interested reader is reminded to Bigoni, 2000; Bigoni and Loret, 1999; Loret et al., 2000, for extensive reviews). However, it is important to note that flutter instability is governed by the isothermal acoustic tensor, which becomes symmetric in the case of associative flow rule. Therefore, *flutter instability is always excluded when normality holds.*

The short wavelength limit investigated here represents a material instability, in the sense that it may become localized in a vanishing small volume of the solid. Moreover, the rate of growth of these perturbations given by $kv$ is unbounded.

Conditions on the isothermal acoustic tensor is the result of the perturbative, asymptotic harmonic analysis. It is interesting now to analyse the quasi-static problem. Neglecting inertia, conditions are found on both the isothermal and the adiabatic elastoplastic acoustic tensor. In particular, only the latter merits to be considered (since

the former can be obtained simply setting $\rho = 0$ in Eq. (80)). When inertia is neglected, the equations of motion reduce to

$$\text{div}\, \boldsymbol{\sigma}' = \mathbf{0}, \tag{81}$$

and the parameter $v$ in representation (75) looses the meaning of velocity, but becomes a parameter governing growth of internal variables and heat flux. In this sense, it can be used as a perturbation parameter, so that the limit $v \to \infty$ can be performed at fixed $k$. Taking the limit, Eq. (77) becomes

$$\hat{T} = \frac{ikT}{\rho c} \left[ \mathbf{B}\mathbf{n} + \frac{\chi}{H_a} \mathbf{N}_a \mathbf{n} \right] \cdot \hat{\mathbf{u}}, \tag{82}$$

in which Eqs. (26) and (28) have been employed. A substitution of Eq. (82) and use of Eqs. (26), (28) and (31) yield

$$\mathbf{A}_a^{ep}(\mathbf{n})\, \hat{\mathbf{u}} = \mathbf{0}, \tag{83}$$

which admits only the trivial solution except when the adiabatic elastoplastic acoustic tensor becomes singular. When singularity occurs, non-trivial solutions in the form (75) are possible of arbitrary wavelength and infinite rate of growth. This may correspond to a form of bifurcation of an infinite body, much like the internal instability of Biot (1965), a situation fully consistent with the formation of discontinuity bands for the deformation and temperature rates, already presented.

## 7. An application to thermoplasticity of metals

A time independent version of the elastic-viscoplastic model developed by Benallal and Ben Cheikh (1987) is presented below to describe the behaviour under thermo-mechanical loadings of a Nickel based superalloy (Inconel 718), employed in turbine disks of jet engines. A strain localization analysis is performed, under the simplifying assumption of negligible variation of elastic moduli with temperature.

### 7.1. Constitutive model

The usual summation rule of elastic and plastic deformation is adopted

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p. \tag{84}$$

The sets of internal variables $\boldsymbol{\alpha}$ and associated thermodynamical forces are:

$$\boldsymbol{\alpha} = \{\boldsymbol{\varepsilon}_p, \boldsymbol{\kappa}, p\}, \quad \mathbf{A} = \{\boldsymbol{\sigma}, \mathbf{X}, R\}, \tag{85}$$

where $\mathbf{X}$ is the back stress and $\boldsymbol{\kappa}$ is its associated internal variable, $R$ is the thermo-dynamical force (representative of the change of the radius of the yield function due to hardening, as will become clear later) associated to $p$, representing the accumulated plastic strain

$$p = \int |\dot{\boldsymbol{\varepsilon}}^p|\, dt. \tag{86}$$

The free energy is now a function of the elastic strain $\varepsilon_e = \varepsilon - \varepsilon_p$ (note that $\operatorname{tr}\varepsilon = \operatorname{tr}\varepsilon_e$), of the cumulated plastic strain $p$ through function $g$, and of $\boldsymbol{\kappa}$

$$\Psi = \frac{3K_i - 2G_i}{6\rho}(\operatorname{tr}\varepsilon)^2 + \frac{G_i}{\rho}(\varepsilon - \varepsilon_p)\cdot(\varepsilon - \varepsilon_p) - \frac{3\alpha}{\rho}K_i(T - T_0)\operatorname{tr}\varepsilon$$

$$- cT\log\frac{T}{T_0} + c(T - T_0) - (T - T_0)s_0 + b\boldsymbol{\kappa}\cdot\boldsymbol{\kappa} + \frac{1}{\rho}g(p, T) + \Psi_0, \quad (87)$$

where $K_i$ and $G_i$ are the isothermal elastic bulk and shear moduli, respectively; $\alpha$ is the coefficient of thermal expansion and $T_0$ is the absolute temperature at which, for null deformation, the material is unstressed. Moreover, $s_0$ and $\Psi_0$ are the values of entropy and free energy when $T = T_0$ and $\varepsilon = 0$. From (87) we get

$$\mathbf{B} = -3\alpha K_i\mathbf{I}, \quad R = -\frac{\partial g}{\partial p}. \quad (88)$$

Moreover, we restrict attention to an elastic–plastic model with isotropic and non-linear kinematic hardening describing cyclic behaviour of metals. For the material considered here, only the yield stress and the isotropic hardening were found temperature dependent. The yield function takes therefore the form:

$$f = |\mathbf{S} - \mathbf{X}| - k(T) + R, \quad J_2 = \tfrac{1}{2}\mathbf{S}\cdot\mathbf{S}, \quad (89)$$

where $\mathbf{S} = \boldsymbol{\sigma} - \mathbf{I}\operatorname{tr}\boldsymbol{\sigma}/3$ represents the deviatoric stress and $k$ is the radius of the initial yield function, equal to $\sqrt{2/3}$ times the initial yield stress in uniaxial tension. Parameter $R$ measures the isotropic hardening, representing the variation of the radius of the yield surface with plastic deformation. The back stress $\mathbf{X}$ traces the movement of its center. The plastic potential $F$ is given by

$$F = |\mathbf{S} - \mathbf{X}| + \frac{\mathbf{X}\cdot\mathbf{X}}{2a} + R, \quad (90)$$

where $a$ is a material parameter. The yield function and plastic potential gradients result therefore in the forms

$$\mathbf{Q} = \left\{\frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|}, -\frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|}, 1\right\}, \quad \mathbf{P} = \left\{\frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|}, -\frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|} + \frac{\mathbf{X}}{a}, 1\right\}, \quad (91)$$

so that from Eq. (3), the evolution laws of the internal variables become

$$\{\dot{\varepsilon}^p, \dot{\boldsymbol{\kappa}}, \dot{p}\} = \dot{\lambda}\left\{\frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|}, -\frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|} + \frac{\mathbf{X}}{a}, 1\right\}. \quad (92)$$

We note in passing here that even if the yield function $f$ is different from the plastic potential $F$, the model is associative in the sense that $\mathbf{M}_i = \mathbf{N}_i$. The rate constitutive equations in terms of isothermal quantities are characterized by

$$\mathbb{E}_i = \left(K_i - \frac{2}{3}G_i\right)\mathbf{I}\otimes\mathbf{I} + 2G_i\mathbb{S}, \quad (93)$$

where $\mathbb{S}$ is the symmetrizing fourth-order tensor (see Appendix A) and

$$\mathbf{M}_i = \mathbf{N}_i = 2G_i \frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|}, \quad \mathbf{B} = -3\alpha K_i \mathbf{I}, \quad \beta = \left( \frac{\partial^2 g}{\partial T \partial p} + \frac{\partial k}{\partial T} \right),$$

$$H_i = 2G_i + \frac{\partial^2 g}{\partial p^2}. \tag{94}$$

In terms of adiabatic quantities the constitutive equations are characterized by

$$\mathbb{E}_a = \left( K_i - \frac{2}{3} G_i + 9 \frac{T \alpha^2 K_i^2}{\rho c} \right) \mathbf{I} \otimes \mathbf{I} + 2G_i \mathbb{S}, \tag{95}$$

and

$$\mathbf{M}_a = \mathbf{N}_i + \frac{\chi T}{\rho c} 3\alpha K_i \mathbf{I}, \quad \mathbf{N}_a = \mathbf{N}_i + \frac{\beta T}{\rho c} 3\alpha K_i \mathbf{I}, \quad H_a = H_i + \frac{T}{\rho c} \beta \chi, \tag{96}$$

where

$$\chi = \frac{|\mathbf{S} - \mathbf{X}| + R}{T} + \frac{\partial^2 g}{\partial T \partial p} = \frac{k(T)}{T} + \frac{\partial^2 g}{\partial T \partial p}. \tag{97}$$

The model parameters have been identified by Benallal and Ben Cheikh (1982), using monotonic and cyclic tests carried out for a wide range of temperature (from 20°C to 800°C), so that the full temperature dependence of the material behaviour was available to us, specified by the followings laws (where coefficients are listed in Table 1).

- Isothermal elastic modulus and Poisson's ratio

$$E_i(T) = e_1(1 - e^{e_2 T}) + e_3, \quad v_i = 0.3. \tag{98}$$

- Isotropic hardening

$$g(p, T) = [a_1(1 - e^{-a_2 T}) + a_3] \left( p + \frac{e^{-\gamma p}}{\gamma} \right). \tag{99}$$

- Yield stress

$$k(T) = k_1(1 - e^{k_2 T}) + k_3. \tag{100}$$

- Coefficient of thermal expansion

$$\alpha(T) = \alpha_1 T + \alpha_2. \tag{101}$$

- Kinematic hardening, as mentioned previously, was found to be temperature independent.

The coefficients not listed in Table 1 are the following

$$\gamma = 60, \quad a = 340 \text{ MPa}, \quad b = 170\,000 \text{ MPa}, \quad \rho = 8220 \text{ Kg m}^{-3},$$

$$c = 435 \text{ J Kg}^{-1\circ}\text{C}^{-1}.$$

It may be worth noting that the adopted model, together with the parameter identification fits experimental results in a quite accurate way. In particular, a comparison (taken from Benallal and Ben Cheikh, 1987) between experimental results (in the center) and model simulation (on the right), reported in Fig. 1 for the thermomechanical loading program sketched in the left of the figure, is pretty satisfactory.

Table 1
Material parameters entering laws (98)–(100)

| | | |
|---|---|---|
| $e_1 = 30\,000$ MPa | $e_2 = 1.45 \times 10^{-3}\,^\circ\mathrm{C}^{-1}$ | $e_3 = 206\,000$ MPa |
| $a_1 = -180$ MPa | $a_2 = 3 \times 10^{-3}\,^\circ\mathrm{C}^{-1}$ | $a_3 = -40$ MPa |
| $k_1 = 50$ MPa | $k_2 = 3.4 \times 10^{-3}\,^\circ\mathrm{C}^{-1}$ | $k_3 = 920$ MPa |
| $\alpha_1 = 4.475 \times 10^{-9}\,^\circ\mathrm{C}^{-2}$ | $\alpha_2 = 1.3 \times 10^{-5}\,^\circ\mathrm{C}^{-1}$ | — |
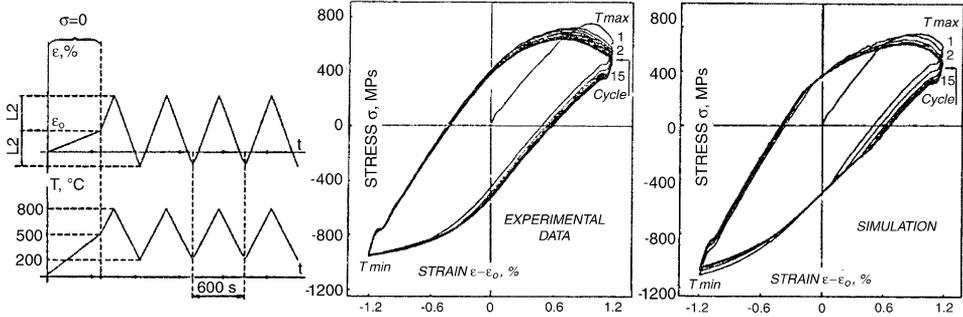


Fig. 1. Comparison between model prediction (on the right) and experimental results (center) for a particular thermomechanical loading program (sketch on the left).

### 7.2. Strain localization analysis

Due to the fact that both the elastic tensor $\mathbb{E}_i$ and the stress-temperature tensor **B** are isotropic, the localization analysis can be performed employing the explicit solution for maximization of the hardening modulus as a function of the direction **n** derived by Bigoni and Hueckel (1991) and by Benallal and Comi (1996), the latter authors employing a geometrical method. In particular, the two tensors **S** and **X** are coaxial, deviatoric and symmetric, so that we may operate in their principal reference system. In this system, the normal to shear band lies in the principal plane orthogonal to the axis corresponding to the intermediate principal value of **S** and is therefore singled out by the angle $\theta$ taken from axis 1, in the plane, say, 1–3, Fig. 2. The maximization corresponding to Eq. (60) can be split into the two problems

$$(h_i^{\mathrm{cr}})^{\mathrm{isoth.}} = \max_{\mathbf{n},\,|\mathbf{n}|=1} \{ \mathbf{M}_i\mathbf{n} \cdot [\mathbf{A}_i^{\mathrm{e}}]^{-1}\mathbf{N}_i\mathbf{n} \} - \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{M}_i], \tag{102}$$

representing the critical isothermal hardening modulus for the singularity of the isothermal acoustic tensor and

$$(h_i^{\mathrm{cr}})^{\mathrm{adiab.}} = \max_{\mathbf{n},\,|\mathbf{n}|=1} \{ \mathbf{M}_a\mathbf{n} \cdot [\mathbf{A}_a^{\mathrm{e}}]^{-1}\mathbf{N}_a\mathbf{n} \} - \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{M}_i] - \frac{T\beta\chi}{\rho c}, \tag{103}$$

representing the critical isothermal hardening modulus for the singularity of the adiabatic acoustic tensor. The constrained maximization problems (102) and (103) can be
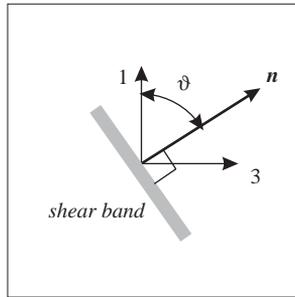
Fig. 2. Geometry and reference system for shear band analysis.

solved following Bigoni and Hueckel (1991), with reference to the yield function and plastic potential gradients defined in stress space, which in our case are

$$\mathbf{Q}_i = \mathbf{P}_i = \mathbb{C}_i \mathbf{N}_i = \frac{\mathbf{S} - \mathbf{X}}{|\mathbf{S} - \mathbf{X}|},$$

$$\mathbf{Q}_a = \mathbb{C}_a \mathbf{N}_a = \mathbf{Q}_i + \beta q \mathbf{I}, \quad \mathbf{P}_a = \mathbb{C}_a \mathbf{M}_a = \mathbf{Q}_i + \chi q \mathbf{I}, \tag{104}$$

where $\mathbb{C}_i$ and $\mathbb{C}_a$ are the inverse of elastic tensors (see Appendices A and B) and

$$q = \frac{\alpha T}{\rho c (1 + 9\alpha^2 T / (\rho c) K_i)}. \tag{105}$$

For the range of constitutive parameters considered here, the maximization of the hardening moduli (102) and (103) yields

$$\frac{(h_i^{cr})^{\text{isoth.}}}{G_i} = -2(1 + v_i) Q_{i2}^2,$$

$$\frac{(h_i^{cr})^{\text{adiab.}}}{G_i} = \frac{1 + v_a}{2} \left[ q^2 (\chi - \beta)^2 \frac{1 + v_a}{1 - v_a} - 4q(\chi - \beta)(Q_{i2} + \beta q) - 4(Q_{i2} + \beta q)^2 \right]$$

$$- \frac{T\beta\chi}{\rho c G_i} (1 - 9\alpha q K_i), \tag{106}$$

where $v_i$ and $v_a$ are the adiabatic and isothermal Poisson's ratios, the index 2 denotes the intermediate principal value of $\mathbf{S} - \mathbf{X}$. The corresponding critical band inclinations are given by

$$(\cos^2 \theta)^{\text{isoth.}} = \frac{Q_{i1} + v_i Q_{i2}}{Q_{i1} - Q_{i3}},$$

$$(\cos^2 \theta)^{\text{adiab.}} = \frac{Q_{i1} + v_a Q_{i2} + (1 + v_a)\beta q}{Q_{i1} - Q_{i3}} + \frac{(\chi - \beta)(1 + v_a)}{2(Q_{i1} - Q_{i3})}. \tag{107}$$

Being tensor $\mathbf{Q}_i$ deviatoric and unit norm, it may be shown that

$$\left. \begin{array}{c} Q_{i1} \\ Q_{i3} \end{array} \right\} = \frac{1 - Q_{i2} \pm \sqrt{2 - 3Q_{i2}^2}}{2}, \quad Q_{i2} \in \left[ -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right], \tag{108}$$
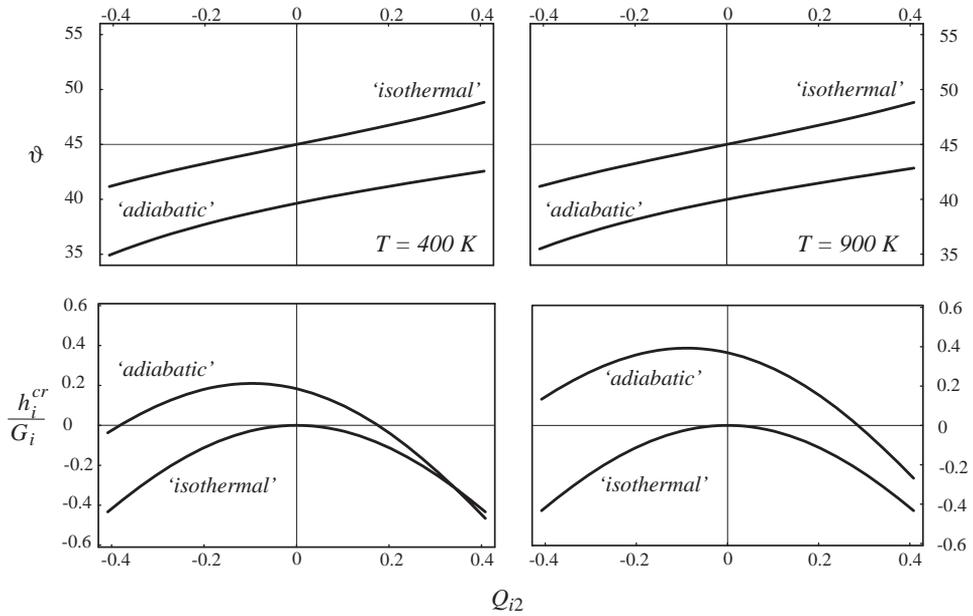
Fig. 3. Critical hardening modulus and band inclination (in degrees) for the incrementally isothermal and adiabatic localization as function of the loading parameter $Q_{i2}$ for two different values of temperature.

so that $Q_{i1} - Q_{i3}$ does not vanish. Therefore, the critical hardening modulus and band inclination can be expressed as functions of the parameter $Q_{i2}$ only.

The critical isothermal hardening moduli for localization in isothermal and adiabatic conditions (normalized through division by $G_i$, Eq. (106)) and the band normal inclination, Eq. (107), are reported as functions of the intermediate principal value $Q_{i2}$ in Fig. 3, where angle $\theta$ is expressed in degrees. It can be concluded from the figure that, except in a small range of values $Q_{i2}$ for $T = 400$ K, instability in the incrementally adiabatic conditions occurs before than instability in isothermal conditions. Moreover, while the critical hardening modulus for the latter solid is never positive (as is clearly indicated by Eq. (106)$_1$), *the critical hardening modulus for localization in the incrementally adiabatic conditions can be positive* (even for associative plasticity, as in the current example).

The critical hardening modulus for incrementally adiabatic localization, as a function of $Q_{i2}$ reaches a maximum when

$$Q_{i2} = -\frac{q}{2}(\chi + \beta), \tag{109}$$

and the corresponding critical hardening modulus can be calculated from (106)$_2$ to be

$$\frac{(h_i^{cr})^{adiab.}}{G_i} = \frac{1 + v_a}{1 - v_a} q^2(\chi - \beta)^2 - \frac{T\beta\chi}{\rho c G_i}(1 - 9\alpha q K_i). \tag{110}$$

It may be important to note that the inclination of the band for localization in the incrementally adiabatic conditions is always inferior than the inclination corresponding to localization in the isothermal solid. The difference between the two band inclinations is approximately 5°–6°, a quantity which can be experimentally detected.

Finally, we mention that the part of the graph relative to $T = 400$ K, where localization in incrementally adiabatic conditions occurs after localization in incrementally isothermal conditions, is important at least from conceptual point of view, because it shows that *a hierarchy between the localization criteria does not exist in general*. To develop this point in more detail, we note that assuming $v_a = v_i = v$ we get from Eqs. (106)

$$\frac{(h_i^{cr})^{adiab.} - (h_i^{cr})^{isoth.}}{G_i} = \frac{1+v}{2}\left[ q^2(\chi - \beta)^2 \frac{1+v}{1-v} - 4q(\chi - \beta)(Q_{i2} + \beta q) \right.$$

$$\left. - 4\beta q(2Q_{i2} + \beta q) \right] - \frac{T\beta\chi}{\rho c G_i}(1 - 9\alpha q K_i). \tag{111}$$

The condition for which the two critical hardening moduli become equal can be obtained imposing the vanishing of (111) and solving for $Q_{i2}$

$$Q_{i2} = \frac{1}{4q(\chi + \beta)}\left[ q^2(\chi - \beta)^2 \frac{1+v}{1-v} - 4q^2\beta(\chi - \beta) \right.$$

$$\left. - 4\beta^2 q^2 - \frac{4T\beta\chi}{\rho c E_i}(1 - 9\alpha q K_i) \right]. \tag{112}$$

The right-hand side of condition (112) is plotted as a function of the temperature in Fig. 4. The intersections of the resulting curve with the lines corresponding to $Q_{i2} = \pm 1/\sqrt{6}$ single out (under the approximation $v_i = v_a$) the range of temperature for which an interval exists of parameter $Q_{i2}$ in which localization in the incrementally adiabatic conditions occurs after localization in isothermal conditions. It is easily seen from the figure that this range of parameters corresponds to temperatures inferior to 625 K.
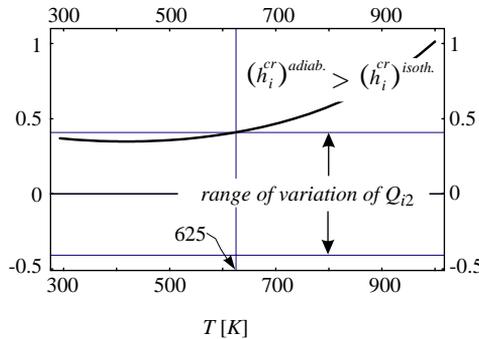


Fig. 4. The right-hand side of condition (112) as a function of temperature. Localization in incrementally adiabatic conditions occurs before than localization in incrementally isothermal conditions for temperatures superior to 625 K.

## 8. Discussion and conclusions

Under isothermal conditions, strain localization, stationarity of acceleration waves, and unbounded growth of perturbations coincide with the singularity of the acoustic tensor, corresponding to the loss of ellipticity of the governing field equations. In addition, all wavelengths of perturbations grow unboundedly at the same time. It has been shown in this paper that consideration of thermal effects and thermo-mechanical couplings definitively change this situation. In particular, the presented results show that for coupled problems, while propagation (and stationarity) of acceleration waves is governed by isothermal strain rate, (static) strain localization and unbounded growth of perturbations are both marked by the singularity of either the isothermal or adiabatic acoustic tensors. Under quasi-static conditions, localization and unbounded growth of perturbations are related to the conditions of localization under isothermal or adiabatic conditions, whereas under dynamic conditions, unbounded growth is related either to the stationarity of acceleration waves or to the appearance of the flutter phenomenon and both are set by the isothermal properties. This is an important point since flutter instability results excluded for associative flow law.

The analyses performed in the present paper share analogies with those relative to deformation of porous, fluid-saturated inelastic materials, where pore pressure plays a role similar to heat conduction. Isothermal and adiabatic conditions are replaced in that context by drained and undrained conditions. However, differently from the porous plastic problem, the adiabatic constitutive tangent operator is not symmetric for thermoplasticity, even in the relevant case of associative flow rule. An important consequence of this is that a hierarchy between instability in incrementally isothermal and adiabatic conditions does not exist, so that the latter may be critical for stability.

The analysis was limited in the present study to a homogeneous and infinite solid. A current analysis shows that many of the results contained in this paper remain valid when one considers heterogeneous situations and includes boundary conditions. These boundary conditions however bring new features.

It may be also important to mention, in closure, that the obtained results may be of interest in the numerical treatment of coupled problems, with emphasis on the consequences of material instabilities on numerical analyses. When perturbations may grow unboundedly, the initial boundary-value problem becomes ill-posed in the sense specified by Schaeffer (1992) and Benallal (1992a, b) and one should expect mesh and *time step* dependency of the results, as already pointed out in Benallal and Comi (1997, 2003a) for porous media, and our findings support the results by Armero and Park (2003a, b) in their analysis of localized thermoplastic problems.

## Appendix A. Inversion of isothermal constitutive laws (15)

Direct isothermal constitutive equations (15) can be inverted to relate strain to stress rates under the hypothesis that the restriction of the elastic, isothermal tensor $\mathbb{E}_i$ to the space of all symmetric tensors is invertible:

$$\mathbb{E}_i \mathbb{C}_i = \mathbb{C}_i \mathbb{E}_i = \mathbb{S}, \tag{A.1}$$

where $\mathbb{C}_i$ is the elastic, isothermal compliance tensor and $\mathbb{S}$ is the fourth-order symmetrizing tensor, transforming every second-order tensor $\mathbf{H}$ in its symmetric part, $\mathbb{S}[\mathbf{H}] = (\mathbf{H} + \mathbf{H}^T)/2$.

Now we proceed obtaining the strain rate from Eq. (7)

$$\dot{\boldsymbol{\varepsilon}} = \mathbb{C}_i[\dot{\boldsymbol{\sigma}}] + \dot{\lambda}\mathbb{C}_i[\mathbf{M}_i] - \dot{T}\mathbb{C}_i[\mathbf{B}], \tag{A.2}$$

and, going back to Prager's consistency (10), using Eq. (7) with $\dot{\boldsymbol{\varepsilon}}$ given by Eq. (A.2), we obtain the plastic multiplier

$$\dot{\lambda} = \frac{1}{h_i}\langle \mathbf{N}_i \cdot \mathbb{C}_i[\dot{\boldsymbol{\sigma}}] - (\beta + \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{B}])\dot{T}\rangle, \tag{A.3}$$

where

$$h_i = H_i - \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{M}_i], \tag{A.4}$$

is the *the isothermal hardening modulus*, assumed strictly positive in the above inversion of constitutive laws (15).

Finally, we obtain from Eq. (A.2)

$$\dot{\boldsymbol{\varepsilon}} = \mathbb{C}_i[\dot{\boldsymbol{\sigma}}] + \frac{1}{h_i}\langle \mathbf{N}_i \cdot \mathbb{C}_i[\dot{\boldsymbol{\sigma}}] - (\beta + \mathbf{N}_i \cdot \mathbb{C}_i[\mathbf{B}])\dot{T}\rangle \mathbb{C}_i[\mathbf{M}_i] - \dot{T}\mathbb{C}_i[\mathbf{B}], \tag{A.5}$$

which is the inverted form of Eq. (15).

It may be important to observe that the isothermal hardening modulus $h_i$ has been assumed to be strictly positive in inverting the rate constitutive laws (15). Consequently, constitutive equations (A.5) may only describe positive isothermal hardening. However, direct constitutive equations (15) are more general than the inverted form (A.5) and may describe also strain softening and ideally plastic behaviour. These correspond to negative and null values of $h_i$, which is related to $H_i$ through Eq. (A.4).

## Appendix B. Inversion of adiabatic constitutive laws (29)

Direct adiabatic constitutive equations (29) can be inverted using Eq. (A.2) into Eq. (20) to obtain the temperature rate as a function of the stress rate and plastic multiplier:

$$\dot{T} = \frac{T\mathbf{B} \cdot \mathbb{C}_i[\dot{\boldsymbol{\sigma}}] + \dot{\lambda}T(\mathbf{B} \cdot \mathbb{C}_i[\mathbf{M}_i] + \chi) + \gamma}{\rho c + T\mathbf{B} \cdot \mathbb{C}_i[\mathbf{B}]}. \tag{B.1}$$

A substitution of Eq. (B.1) into Eq. (A.3) gives

$$\dot{\lambda} = \frac{1}{h_a}\left\langle \mathbf{N}_a \cdot \mathbb{C}_a[\dot{\boldsymbol{\sigma}}] - \frac{\gamma}{\rho c}(\mathbf{N}_a \cdot \mathbb{C}_a[\mathbf{B}] + \beta)\right\rangle, \tag{B.2}$$

where

$$h_a = H_a - \mathbf{N}_a \cdot \mathbb{C}_a[\mathbf{M}_a], \tag{B.3}$$

is the *adiabatic hardening modulus*, assumed strictly positive, and $\mathbb{C}_a$ is the adiabatic, elastic compliance tensor

$$\mathbb{C}_a = \mathbb{C}_i - \frac{T}{\rho c + T\mathbf{B} \cdot \mathbb{C}_i[\mathbf{B}]} \mathbb{C}_i[\mathbf{B}] \otimes \mathbb{C}_i[\mathbf{B}], \qquad \mathbb{C}_a \mathbb{E}_a = \mathbb{E}_a \mathbb{C}_a = \mathbb{S}. \tag{B.4}$$

A substitution of Eqs. (B.1) and (B.2) into Eq. (A.2) provides, finally, the rate inverse equations with adiabatic quantities

$$\dot{\boldsymbol{\varepsilon}} = \mathbb{C}_a[\dot{\boldsymbol{\sigma}}] + \frac{1}{h_a} \left\langle \mathbf{N}_a \cdot \mathbb{C}_a[\dot{\boldsymbol{\sigma}}] - \frac{\gamma}{\rho c}(\mathbf{N}_a \cdot \mathbb{C}_a[\mathbf{B}] + \beta) \right\rangle \mathbb{C}_a[\mathbf{M}_a] - \frac{\gamma}{\rho c} \mathbb{C}_a[\mathbf{B}]. \tag{B.5}$$

## Appendix C. A simple damage model

Damage models fit in the presented constitutive framework. An example is provided here of calculation of the relevant quantities for a simple isotropic damage model. In particular, the sets of internal variables and associated thermodynamical forces are assumed to be:

$$\boldsymbol{\alpha} = \{D\}, \qquad \mathbf{A} = \{Y\}, \tag{C.1}$$

and the specific free energy is now a function of the damage parameter $D \in [0, 1]$

$$\Psi = (1 - D) \left\{ \frac{3K_i - 2G_i}{6\rho} (\mathrm{tr}\, \boldsymbol{\varepsilon})^2 + \frac{G_i}{\rho} \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} - \frac{3\alpha}{\rho} K_i (T - T_0) \mathrm{tr}\, \boldsymbol{\varepsilon} \right\}$$
$$- cT \log(T/T_0) + c(T - T_0) - (T - T_0)s_0 + \Psi_0. \tag{C.2}$$

Moreover, we assume the following loading function

$$f(Y, D, T) = Y - Y_0(T) - MD, \tag{C.3}$$

where $M$ is a material parameter, $Y_0$ is the activation threshold for damage (dependent on the temperature). The loading function gradient is simply equal to unity, $\mathbf{Q} = 1$. From Eqs. (3) and (6), assuming $\mathbf{P} = \mathbf{Q}$ (implying normality), we get

$$\boldsymbol{\sigma} = (1 - D) \left\{ \frac{3K_i - 2G_i}{3} (\mathrm{tr}\, \boldsymbol{\varepsilon})\mathbf{I} + 2G_i\boldsymbol{\varepsilon} - 3\alpha K_i(T - T_0)\mathbf{I} \right\},$$

$$Y = \frac{3K_i - 2G_i}{6} (\mathrm{tr}\, \boldsymbol{\varepsilon})^2 + G_i\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} - 3\alpha K_i(T - T_0) \mathrm{tr}\, \boldsymbol{\varepsilon}, \tag{C.4}$$

and the evolution law of internal variable

$$\dot{D} = \dot{\lambda}. \tag{C.5}$$

The stress-temperature tensor results to be

$$\mathbf{B} = -3\alpha(1 - D)K_i\mathbf{I}, \tag{C.6}$$

and the constitutive equations, in terms of isothermal quantities, are defined by

$$\mathbb{E}_i = (1-D)\{(K_i - \tfrac{2}{3}G_i)\mathbf{I} \otimes \mathbf{I} + 2G_i\mathbb{S}\}, \tag{C.7}$$

and

$$\mathbf{M}_i = \mathbf{N}_i = \frac{1}{1-D}\boldsymbol{\sigma}, \quad \beta = 3\alpha K_i(\mathrm{tr}\,\boldsymbol{\varepsilon}) + Y_0', \quad H_i = M, \tag{C.8}$$

with $Y_0' = \partial Y_0/\partial T$. In terms of adiabatic quantities the constitutive equations are defined by

$$\mathbb{E}_a = \mathbb{E}_i^{\mathrm{iso}} + 9(1-D)^2\,\frac{T\alpha^2 K_i^2}{\rho c}\mathbf{I} \otimes \mathbf{I}, \tag{C.9}$$

and

$$\chi = \frac{Y}{T} + 3\alpha K_i\,\mathrm{tr}\,\boldsymbol{\varepsilon}. \tag{C.10}$$

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