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UNIQUENESS AND LOCALIZATION—II. COUPLED ELASTOPLASTICITY

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Abstract—Uniqueness of the incremental response is investigated for an elastoplastic coupling model with elastic stiffness degradation. The criteria of second order work positiveness and strain localization are used in the form derived by Bigoni and Hueckel (1991, *Int. J. Solids Structures* **28**, 197–213). A numerical example shows that in a triaxial test the loss of second order work positiveness occurs at the positive hardening modulus, whereas the condition of strain localization is never fulfilled. Thus shear bands or splits are predicted not to form, despite strong elasticity degradation and non-associativity.

1. INTRODUCTION

The behavior of pressure sensitive dilatant materials such as rock, concrete or soil may be characterized by a visible degradation of elastic stiffness during plastic deformation (Fig. 1). Degradation is a phenomenological counterpart of generation and/or growth of defects and voids in the material. The decrease in elastic stiffness may thus be a convenient indirect measure of material damage. It is believed that in an elastoplastic continuum the above-mentioned damage leads eventually to strain localization or other forms of loss of uniqueness of the incremental response even in the presence of small strains and rotations.

To represent the elasticity degradation, several models were developed. Dougill proposed a model (1976) in which the material is elastic "fracturing", i.e. all deformations are recovered after total unloading, but the unloading–reloading curves show a strong stiffness degradation. In the model proposed by Hueckel (1975, 1976), a coupling was introduced between elastic and plastic deformations. The behavior of the idealized material results in a combination of both the plastic strain and the elastic degradation. The model was analyzed in terms of stability and uniqueness by Maier and Hueckel (1977, 1979). Variational principles for elastoplastic coupling were established by Capurso (1979) using quadratic optimization. A thermodynamic strain space formulation for coupled laws was given by Dafalias (1977). Recently the coupled elastoplasticity has been used to model the behavior of concrete under complex loading conditions (Chen and Han, 1988; Han and Chen, 1986).

The aim of this paper is to analyze the possible influence of plastically induced changes in elasticity moduli on uniqueness and on strain localization in small strain elastoplasticity theory employing the criteria in the form developed by Bigoni and Hueckel (1991). Uniqueness of the rate response is studied for a material with varying elastic stiffness, as described by the coupled elastoplastic model.

For any material element, subjected to progressive damage, the determination of the validity range of the constitutive law is a crucial and often complicated question. Loss of uniqueness of the rate response of such an element terminates the validity of the stress–strain relationship. Localization is an extreme form of loss of uniqueness, at which a discontinuous strain rate mode becomes admissible. The coupled elastoplasticity leads to a particular form of non-associative constitutive law in which the deviation from normality is a function of elasticity degradation (Hueckel, 1975, 1976; Maier and Hueckel, 1979; Hueckel and Maier, 1977). The non-associativity of the flow rule is known to have an unstabilizing effect on the material behavior (Rudnicki and Rice, 1975; Rice, 1976). Thus, the question arises how much the coupling-induced non-associativity, and therefore degradation of elasticity, affects the uniqueness of the response. As a sufficient condition for

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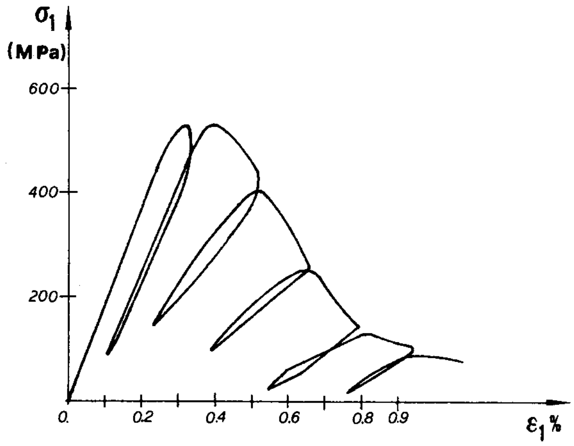

Fig. 1. Uniaxial compression of a rock-like material.

uniqueness (Raniecki, 1979) the local criterion of second order work positiveness is used (Maier and Hueckel, 1979). The particular case of loss of uniqueness in the form of strain localization into planar bands (Rudnicki and Rice, 1975; Rice, 1976) is analyzed here employing an explicit uncoupled criterion obtained by Bigoni and Hueckel (1991). In a numerical example, experiments are simulated in triaxial compression at constant isotropic stress and the relative threshold values of the second order work and the localization are calculated and discussed.

2. BASIC EQUATIONS FOR COUPLED ELASTOPLASTICITY

The main characteristic of the coupled elastoplastic law is that it admits change in the elastic properties during the irreversible loading process. The elastic properties are thus described by the current elastic compliance tensor **C**, which is assumed to be a function of plastic deformation $\boldsymbol{\varepsilon}^p$:

$$\mathbf{C} = \mathbf{C}(\boldsymbol{\varepsilon}^p) \tag{1}$$

where the fourth order tensor **C** is assumed to be symmetric and positive definite.

The elastic strain is given by the usual stress–strain relationship:

$$\boldsymbol{\varepsilon}^e = \mathbf{C} : \boldsymbol{\sigma} \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress tensor. It may be seen that during plastic yielding, the change in the elastic compliance tensor $\dot{\mathbf{C}}$ generates an additional elastic strain rate, which is not proportional to the stress rate, but rather to the stress itself, i.e.:

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{C} : \dot{\boldsymbol{\sigma}} + \dot{\mathbf{C}} : \boldsymbol{\sigma}, \quad \text{for} \quad f = 0 \quad \text{and} \quad \dot{f} = 0 \tag{3}$$

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{C} : \dot{\boldsymbol{\sigma}}, \quad \text{for} \quad f < 0 \quad \text{or} \quad f = 0, \quad \dot{f} < 0 \tag{4}$$

where $f(\boldsymbol{\sigma}, k(\boldsymbol{\varepsilon}^p))$ is the yield function, in which k is a scalar parameter governing the plastic hardening. The latter term in (3) is irreversible, because the change in the elastic compliance tensor is irreversible.

Clearly, change in elastic modulus is not the only source of irreversible deformation. In fact, yielding of the material produces an irreversible strain which may be unlimited and the rate of which is not proportional to stress as above but rather to the stress rate. This strain rate will be referred to as plastic and it is assumed to be associated with the yield function, via a normality rule:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\Lambda}\mathbf{Q} \tag{5}$$

where $\dot{\Lambda}$ is the plastic multiplier and $\mathbf{Q}$ is the yield surface gradient. In consequence of the small strain assumption, the additivity of the elastic and plastic strain rates holds and thus (3) and (5) yield:

$$\dot{\varepsilon}_{ij} = C_{ijkh}\dot{\sigma}_{kh} + \frac{\partial C_{ijkh}}{\partial \varepsilon^p_{st}}\dot{\varepsilon}^p_{st}\sigma_{hk} + \dot{\varepsilon}^p_{ij}. \tag{6}$$

Making use of the flow rule (5), the following stress–strain relationship is finally obtained:

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{C}:\dot{\boldsymbol{\sigma}} + \dot{\Lambda}(\mathbf{B}+\mathbf{Q}) \tag{7}$$

where the symmetric second order tensor $\mathbf{B}$ is defined as:

$$B_{ij} = \frac{\partial C_{ijkh}}{\partial \varepsilon^p_{st}}\sigma_{hk}Q_{st}. \tag{8}$$

Equation (7) is subject to the conditions:

$$\dot{\Lambda}\dot{f} = 0, \quad \dot{f} \leqslant 0, \quad \dot{\Lambda} \geqslant 0. \tag{9}$$

Here:

$$\dot{f} = \mathbf{Q}:\dot{\boldsymbol{\sigma}} - H\dot{\Lambda} \tag{10}$$

where the plastic hardening modulus H is defined as:

$$H = -\frac{\partial f}{\partial \boldsymbol{\varepsilon}^p}:\mathbf{Q}. \tag{11}$$

From eqn (7) it follows that the coupled elastoplasticity adopted here results in a particular type of non-associative law, in which the variable degree of non-associativity, measured by tensor $\mathbf{B}$, is a function of the change in the elastic properties. In fact, the mode of the irreversible strain rate in eqn (7) is determined by the tensor sum of the two second order symmetric tensors $\mathbf{Q}$ and $\mathbf{B}$.

3. CRITERIA OF SECOND ORDER WORK AND LOCALIZATION

In what follows the criteria for a generic non-associative plastic flow rule will be specialized to the coupled elastoplasticity. At a given stress state, with a known accumulated plastic strain $\boldsymbol{\varepsilon}^p$, the check against the criterion of the zero second order work is performed by comparing the actual plastic hardening modulus H with the critical hardening modulus now expressed as:

$$H^u_{cr} = \tfrac{1}{2}[\sqrt{(\mathbf{Q}:\mathbf{E}:\mathbf{Q})(\mathbf{Q}:\mathbf{E}:\mathbf{Q}+2\mathbf{B}:\mathbf{E}:\mathbf{Q}+\mathbf{B}:\mathbf{E}:\mathbf{B})} - \mathbf{Q}:\mathbf{E}:\mathbf{Q} - \mathbf{Q}:\mathbf{E}:\mathbf{B}] \tag{12}$$

where $\mathbf{E} = \mathbf{C}^{-1}$.

Uniqueness is ensured if $H > H^u_{cr}$. The deformation rate at which second order work reduces to zero is:

$$\dot{\boldsymbol{\varepsilon}}^0 = \alpha[\sqrt{\mathbf{Q}:\mathbf{E}:\mathbf{Q}+2\mathbf{Q}:\mathbf{E}:\mathbf{B}+\mathbf{B}:\mathbf{E}:\mathbf{B}}\,\mathbf{Q} + \sqrt{\mathbf{Q}:\mathbf{E}:\mathbf{Q}}(\mathbf{Q}+\mathbf{B})], \quad \forall \alpha \in \mathbb{R}-\{0\}. \tag{13}$$

The elastic tensor **E** and the tensor **B**+**Q** depend explicitly on plastic strain, as opposed to the usual plasticity formulation. Therefore, at the same stress point the critical hardening modulus may be different for different plastic strain histories.

Localization is a form of non-unique rate response of material in which a strain rate discontinuity forms across a planar band (Rudnicki and Rice, 1975; Rice, 1976). Explicit expressions for the critical hardening modulus for localization and for the components of vector **n** normal to the discontinuity band are derived elsewhere (Bigoni and Hueckel, 1991). Excluding the cases of infinite bands, the normal to the band is necessarily defined by one of the following sets of components (cases i, ii and iii).

(i) None of the components of versor **n** is zero:

$$n_1^2 = \frac{1}{\Delta}\{2d(P_2-P_3)(Q_2-Q_3)-b[(P_1-P_3)(Q_2-Q_3)+(P_2-P_3)(Q_1-Q_3)]\}$$

$$n_2^2 = \frac{1}{\Delta}\{2b(P_1-P_3)(Q_1-Q_3)-d[(P_1-P_3)(Q_2-Q_3)+(P_2-P_3)(Q_1-Q_3)]\}$$

$$n_3^2 = 1-n_1^2-n_2^2 \tag{14}$$

where indices denote principal components, while the second order tensor **P** is defined as:

$$\mathbf{P} = \mathbf{Q}+\mathbf{B}. \tag{15}$$

Note that tensor **P** is co-axial with the stress tensor, because tensors **B** [see eqn (8)] and **Q** are also. The co-axiality is necessary to obtain the solution for **n** in the principal stress component system. The scalars Δ, d and b are:

$$\Delta = -[(Q_1-Q_3)(P_2-P_3)-(Q_2-Q_3)(P_1-P_3)]^2 \tag{16}$$

$$d = Q_1(P_1-P_3)+P_1(Q_1-Q_3)+\nu[Q_2(P_1-P_3)+P_2(Q_1-Q_3)] \tag{17}$$

$$b = Q_2(P_2-P_3)+P_2(Q_2-Q_3)+\nu[Q_1(P_2-P_3)+P_1(Q_2-Q_3)] \tag{18}$$

where ν is Poisson's ratio (depending on $\boldsymbol{\varepsilon}^p$).

(ii) One of the components of versor **n** is zero:

$$n_i^2 = (1-\nu)\frac{P_iQ_i-P_jQ_j}{(P_i-P_j)(Q_i-Q_j)}-\frac{Q_j-\nu\,\mathrm{tr}\,\mathbf{Q}}{2(Q_i-Q_j)}-\frac{P_j-\nu\,\mathrm{tr}\,\mathbf{P}}{2(P_i-P_j)}$$

$$n_j^2 = 1-n_i^2, \quad n_k = 0 \tag{19}$$

where the indices, denoting principal components and ranging between 1 and 3, are not summed. The symbol tr indicates the trace of a tensor.

(iii) Two components of versor **n** are zero:

$$n_i^2 = 1, \quad n_j^2 = n_k^2 = 0. \tag{20}$$

The actual solution for strain localization corresponds to such a set of components of **n** from among the cases (i), (ii) and (iii), which maximizes the hardening modulus:

$$H^1(\mathbf{n}) = 2G\{2\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{Q}\cdot\mathbf{n}-(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})-\mathbf{P}:\mathbf{Q} -\frac{\nu}{1-\nu}(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{P})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{Q})\} \tag{21}$$

where G is the elastic shear modulus (depending on $\boldsymbol{\varepsilon}^p$).

4. TRIAXIAL EXPERIMENT

The above criteria for the zero second order work and strain localization will now be applied to a particular form of coupled elastoplastic model, developed for fine sand and silt. An application to an axially symmetric, constant isotropic pressure, triaxial test is then discussed.

A set of constitutive assumptions are now made: the yield function is that of Drucker–Prager with a tension cut-off, i.e. for $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$:

$$f = \sqrt{J'_2} - \alpha p - k = 0 \tag{22}$$

where compressive stresses are taken as positive, following the soil mechanics sign convention. p is the mean stress, i.e. $p = \operatorname{tr} \boldsymbol{\sigma}/3$, α is a constant constitutive parameter, related to the internal friction angle, and J'_2 is the second invariant of the deviatoric stress $\mathbf{s}$:

$$J'_2 = \tfrac{1}{2}\mathbf{s} : \mathbf{s}. \tag{23}$$

The isotropic hardening rule is described by a scalar variable k, the rate of which is assumed as a function of plastic deviatoric strain rate tensor, $\dot{\mathbf{e}}^p$:

$$\dot{k} = \dot{k}(\dot{\mathbf{e}}^p). \tag{24}$$

The choice of the hardening function $\dot{k}$ is made in such a way that the resulting hardening modulus is a monotonically decreasing function of the deviatoric plastic strain and has an asymptote when the deviatoric plastic strain is sufficiently large:

$$\dot{k} = \sigma_0 \{\exp[-a(\varepsilon_q^p - \varepsilon_q^c)] - 1\} \dot{\varepsilon}_q^p / \sqrt{3} \tag{25}$$

where σ_0 and a are material parameters and ε_q^p is the deviatoric plastic strain invariant defined as:

$$\varepsilon_q^p = \sqrt{\tfrac{2}{3}\mathbf{e}^p : \mathbf{e}^p}. \tag{26}$$

The hardening modulus reaches zero at a critical plastic deviatoric strain ε_q^c, which is a function of the sole isotropic stress p:

$$\varepsilon_q^c = \frac{1}{a} \ln(2 - p/p_b) \tag{27}$$

where p_b is a hypothetical isotropic stress for which the hardening modulus is zero right at the onset of plastic deviatoric strain. In this way the possibility arises to model a behavior in which softening may be preceded by hardening, as is the case for most granular, cohesive and brittle geologic materials as well as for concrete (see, e.g., Bieniawski, 1971; Carpinteri, 1985; Di Tommaso, 1984; Hillerborg, 1984; Ko and Scott, 1967).

The variation of elastic moduli in sand was found to be a function of plastic dilatancy (Cavalera and Hueckel, 1985). The variation of the bulk and shear moduli, K and G respectively, for sand may suitably be represented by the following functions:

$$K = K_0/(1 + \psi \varepsilon_v^p) \quad \text{and} \quad G = G_0 \exp(-\chi \varepsilon_v^p) \tag{28a, b}$$

where ε_v^p is the plastic volumetric strain, K_0 and G_0 are respectively the initial moduli and ψ and χ are material parameters.

Substituting eqn (28) into (8), the tensor $\mathbf{B}$ becomes:

$$\mathbf{B} = [C_1(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{I} + C_2\boldsymbol{\sigma}]\,\mathrm{tr}\,\mathbf{Q} \tag{29}$$

where $\mathbf{I}$ is the identity tensor and the scalars C_1 and C_2 are the rates of change of elastic moduli:

$$C_1 = \frac{\partial}{\partial \varepsilon_v^p}\left[-\frac{\lambda}{2G(3\lambda+2G)}\right] \tag{30}$$

$$C_2 = \frac{\partial}{\partial \varepsilon_v^p}\left[\frac{1}{4G}\right] \tag{31}$$

where λ is the Lamé modulus (depending on $\boldsymbol{\varepsilon}^p$).

The critical hardening modulus (12) (corresponding to zero second order work) for the material described by the above constitutive functions is obtained as:

$$\begin{aligned} H_{cr}^u = {} & \tfrac{1}{2}\{[G+\alpha^2(\lambda+2G/3)][G+\alpha^2(\lambda+2G/3)[1+C_3^2(\mathrm{tr}\,\boldsymbol{\sigma})^2] \\ & +4G\alpha^2C_2^2J_2'+2\alpha^2(\lambda+2G/3)C_3\,\mathrm{tr}\,\boldsymbol{\sigma}-4G\alpha C_2\sqrt{J_2'}]\}^{0.5} \\ & -\tfrac{1}{2}[G+\alpha^2(\lambda+2G/3)[1+C_3\,\mathrm{tr}\,\boldsymbol{\sigma}]-2G\alpha C_2\sqrt{J_2'}] \end{aligned} \tag{32}$$

where $C_3 = 3C_1 + C_2$.

To calculate H_{cr}^u at the given stress state, the plastic volumetric strain must be determined on which G and λ, C_1 and C_2 depend explicitly. Also, the calculation of the critical modulus for localization requires a prior determination of the components of the versor $\mathbf{n}$ [(14)–(20)] for given tensors $\mathbf{Q}$ and $\mathbf{B}$, which are functions of the plastic strain history. This has been performed numerically for the integrated expression for the hardening parameter k and the yield surface (22) to simulate a triaxial experiment on sand.

In two examined tests at constant isotropic stress equal to 45 and 100 kPa, sand was subjected to prior initial isotropic consolidation at $\varepsilon_0^p = 0.003$ and $\varepsilon_0^p = 0.008$ of compressive plastic volumetric strain, respectively. Two cases are studied, in which the bulk modulus varies either according to eqn (28a) or remains constant ($\psi = 0$), while the shear modulus varies according to eqn (28b).

Figures 2 and 3 present the actual hardening modulus as a function of deviatoric strain,

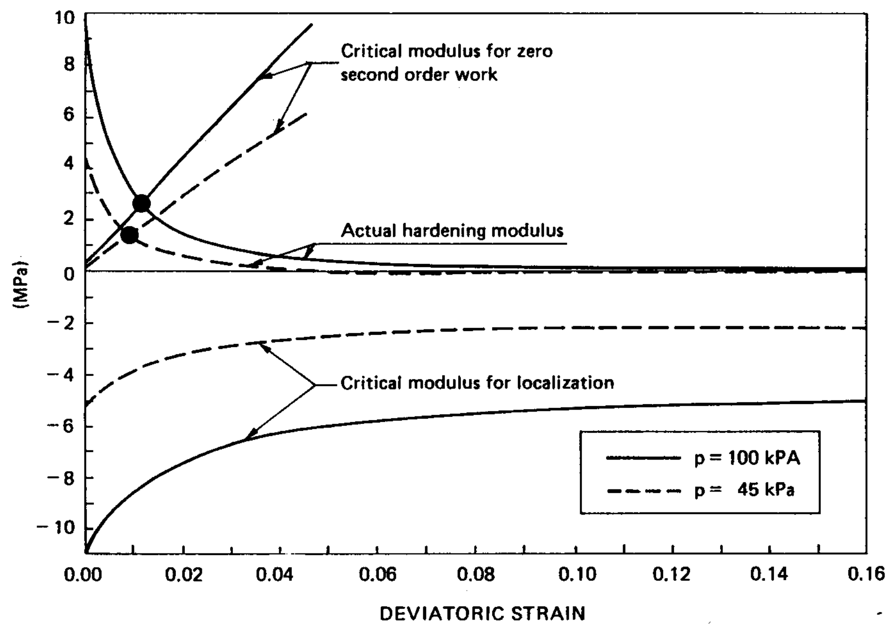

Fig. 2. Critical and actual hardening moduli. $\sigma_0 = 200$ kPa, $p_0 = -0.68$ kPa, $\alpha = 1.24$, $a = 281$, $G_0 = 3333.3$ kPa, $\chi = -152.71$, $K_0 = 21621.00$ kPa. (●) Zero second order work for $\psi = 16.89$.

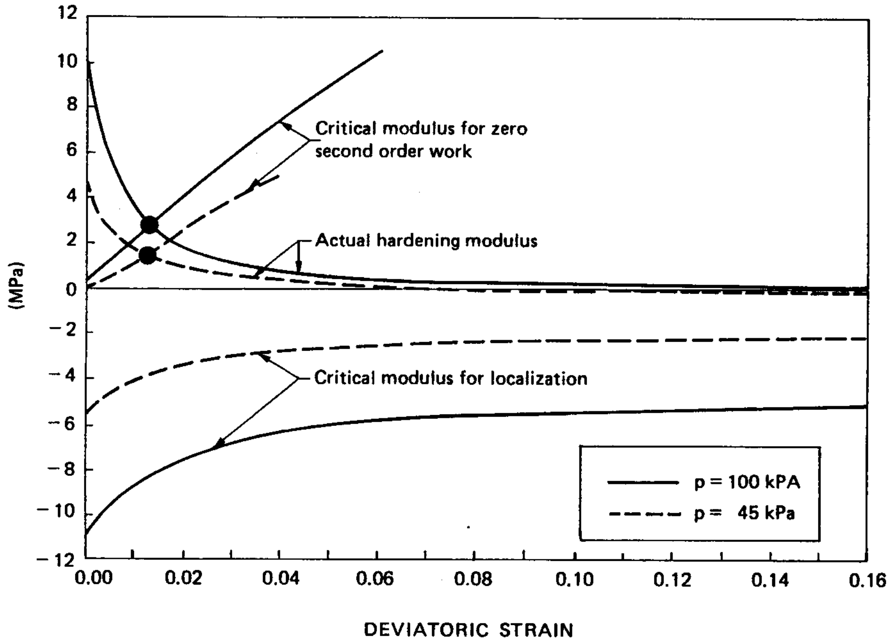

Fig. 3. Critical and actual hardening moduli. $\sigma_0 = 200$ kPa, $p_0 = -0.68$ kPa, $\alpha = 1.24$, $a = 281$, $G_0 = 3333.3$ kPa, $\chi = -152.71$, $K_0 = 21621.00$ kPa. (●) Zero second order work for $\psi = 0$ (constant bulk modulus).

starting respectively at $q_0 = 52$ kPa, and $q_0 = 120$ kPa, where q_0 indicates the deviatoric stress at the onset of plastic shear straining. In the figures the evolution of critical hardening moduli, both for zero second order work and for localization, are also presented.

It may be seen that the actual hardening modulus decreases exponentially, eventually reaching small negative values for 45 kPa, as is typical for medium dense sand. The critical moduli both for uniqueness and for localization increase with deviatoric deformation. The former modulus grows almost linearly, while that for localization appears to approach a constant value which depends on the confining stress. The ultimate value of the hardening modulus necessary for localization is about -2.5 MPa for an isotropic stress of 45 kPa, whereas it is about -5 MPa for an isotropic stress of 100 kPa. However, the actual hardening modulus never reached the localization threshold in any of the considered cases. Thus the shear band is predicted not to form, even at a strain of 16%, and actually has not formed in the experiments (Cavalera and Hueckel, 1985). This is despite the fact that, in all cases, loss of positiveness of the second order work (and thus possibility of loss of uniqueness of rate response) occurred at a relatively low strain (about 1%). The value of the modulus at loss of positiveness of second order work is in all cases positive and increases with the confining stress.

In Figs 4–6, the loss of positiveness of second order work is marked on the corresponding stress–strain curves and on volumetric vs deviatoric strain curves. The deviatoric strain at the respective critical modulus is almost the same for both confining stress values. The stress–strain curve is steeper at the critical point for the higher confining pressure test. For the constant bulk modulus, the critical point occurs at larger strain than for the variable modulus. Thus the variation of the bulk modulus appears to have a destabilizing effect. The shear modulus at the critical point is almost half of the initial modulus.

5. CONCLUSIONS

The second order work criterion and the strain localization criterion were investigated for an elastoplastic material with varying moduli of elasticity. The explicit form of the localization criterion allows for an almost immediate evaluation of the related threshold, even for the case of a complex and plastic strain-dependent criterion.

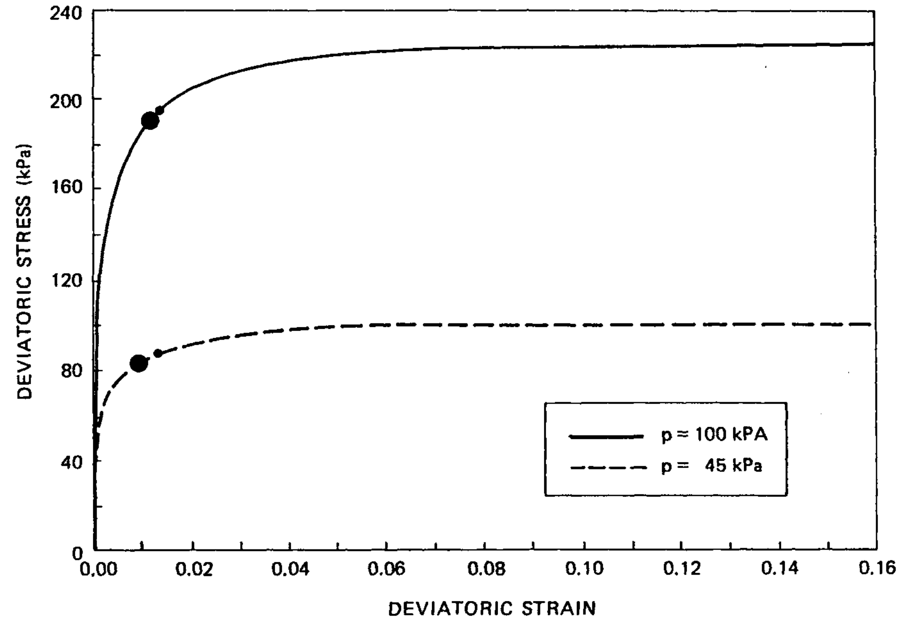

Fig. 4. Deviatoric stress–strain curve. $\sigma_0 = 200$ kPa, $p_0 = -0.68$ kPa, $\alpha = 1.24$, $a = 281$, $G_0 = 3333.3$ kPa, $\chi = -152.71$, $K_0 = 21621.00$ kPa. (●) Zero second order work for $\psi = 16.89$. (•) Zero second order work for $\psi = 0$ (constant bulk modulus).

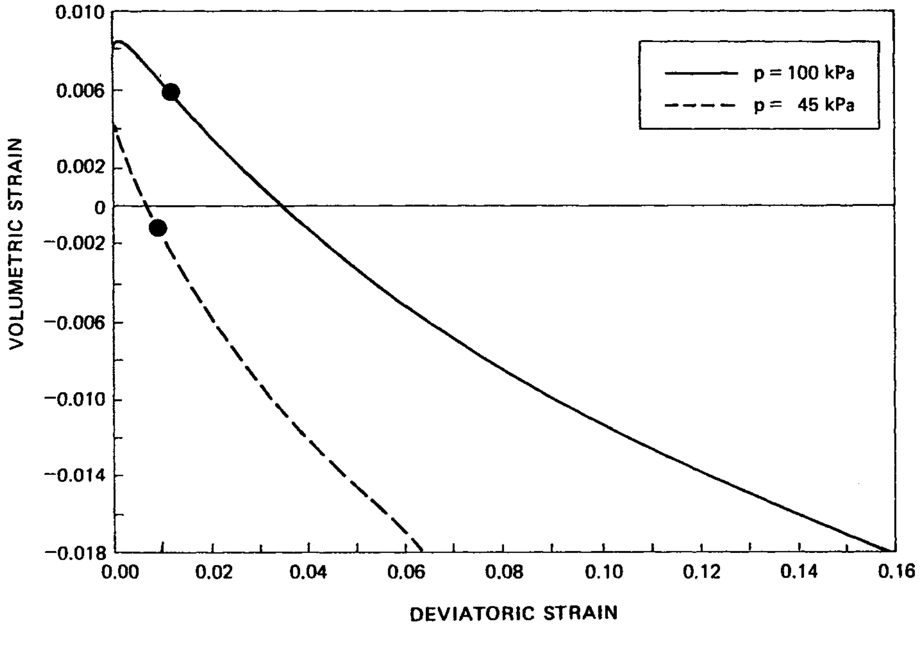

Fig. 5. Volumetric vs deviatoric strain. $\sigma_0 = 200$ kPa, $p_0 = -0.68$ kPa, $\alpha = 1.24$, $a = 281$, $G_0 = 3333.3$ kPa, $\chi = -152.71$, $K_0 = 21621.00$ kPa. (●) Zero second order work for $\psi = 16.89$.

Two examples show that for the discussed model in three-dimensional unconstrained conditions, localization did not develop even at significant strain, despite a very marked degradation of elasticity (namely a drop in the shear modulus to less than one-tenth of the initial value) and despite the resulting very pronounced non-associativity. This confirms a

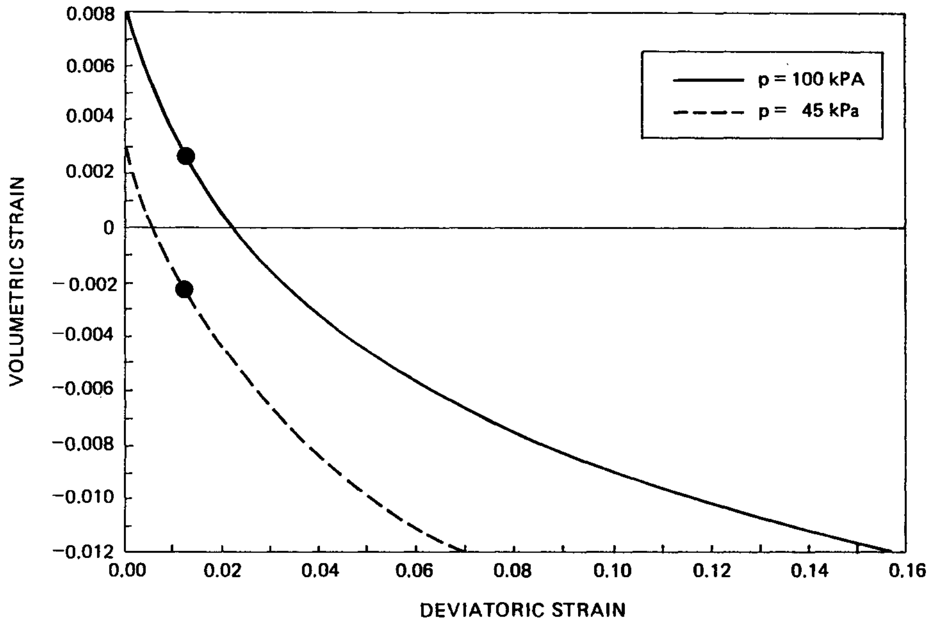

Fig. 6. Volumetric vs deviatoric strain. $\sigma_0 = 200$ kPa, $p_0 = -0.68$ kPa, $\alpha = 1.24$, $a = 281$, $G_0 = 3333.3$ kPa, $\chi = -152.71$, $K_0 = 21621.00$ kPa. (●) Zero second order work for $\psi = 0$ (constant bulk modulus).

similar conclusion obtained for other types of non-associative elastoplastic models (Rudnicki and Rice, 1975). However, the above mentioned destabilizing effects produce an early possibility of loss of uniqueness because of the loss of second order work positiveness, thus leading to possible diffuse bifurcation modes (bulging or barreling).

While the curves for critical modulus for zero second order work are sensitive to the change in elastic bulk modulus, as seen by comparing Fig. 2 for $\psi \neq 0$ and Fig. 3 for $\psi = 0$, the critical modulus for localization seems to be almost unaffected by the change of elastic bulk modulus.

Some of the above findings are known from experiments. Various diffuse bifurcation forms develop in triaxial specimens of medium dense sand (such as barreling and bulging), as was pointed out by Roscoe *et al.* (1963), Hettler and Vardoulakis (1984), and Vardoulakis and Drescher (1985). This was also the case in the presented simulation. However, no shear band localization appeared on triaxial specimens of medium dense sand up to 16% of axial strain (Hettler and Vardoulakis, 1984). This fact is linked by Vardoulakis and Drescher (1985) to the lack of significant softening in this material. However, sands at such advanced strain show both elasticity degradation and marked non-associativity (Tatsuoka and Ishihara, 1974; Cavalera and Hueckel, 1985).

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