Bifurcation and Instability of Non-Associative Elastoplastic Solids

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Abstract. Global and local uniqueness and stability criteria for elastoplastic solids with non-associative flow rules are presented. Hill's general theory is developed in the form generalized by Raniecki to non-associativity. Local stability criteria are presented and systematically discussed in a critical way. These are: positive definiteness and non-singularity of the constitutive operator, and positive definiteness (strong ellipticity) and non-singularity (ellipticity) of the acoustic tensor. The former criteria are particularly relevant for homogeneous deformation of solids subject to all-round controlled nominal surface tractions. Dually, the latter criteria are particularly relevant for homogeneous deformation of solids subject to displacements prescribed on the entire boundary. Flutter instability as related to complex conjugate eigenvalues of the acoustic tensor is also briefly discussed.

1 Introduction

Due to the activation of different micromechanisms at various structural levels -for instance, sliding on microfissures or on intergranular contact surfaces, and pore or joint interactions- a broad class of materials is characterized by an *internal friction*, roughly obeying a Coulomb law. As a consequence, pressure-sensitive yielding and plastic dilatancy become dominant phenomena (Nikolaevskii and Rice, 1979). This class of materials embraces porous and particulate reinforced metals, polymers, ceramics, powders, asphalts, granular materials, (traditional, fiber reinforced, and high-strength) concretes, rocks, and soils.

In the terminology of elastoplasticity, the Coulomb law of friction is intrinsically *non-associative*, in the sense that adopting the normality rule overestimates the plastic dilatancy (Drucker, 1954). Therefore, a non-associative flow-rule is generally considered the remedy for correctly modelling the inelastic response of the above-mentioned materials (Hill, 1967b; Mandel, 1966; Mróz, 1963; 1966)[1]. An important consequence of flow rule non-associativity is the lack of symmetry of the tangent constitutive operator, thus precluding the possibility of defining strain rate potentials.

Material instabilities start to grow at a point in a body and are strongly influenced by lack of symmetry of the tangent constitutive operator. For instance, flutter instability

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[1] Lack of symmetry of the tangent constitutive operator may result also as a consequence of *elastoplastic coupling* (Hueckel, 1976). Also, in elastoplastic corner models symmetry is lost even for an associative flow-rule when the hardening moduli matrix is not symmetric.

is excluded in the case of symmetry. Bifurcations in diffuse modes, strain localization, and elastoplastic cavitation may occur at high strain hardening for non-associative flow laws.

There are different ways of modelling the occurrence of material instabilities. One is to consider the effects of inevitably existing small defects (imperfections) in a material which may be viewed as homogeneous at a certain scale. Another is to consider the macroscopic, inelastic, constitutive response of a homogeneous material element and seek specific instabilities and bifurcations. The latter description of material instabilities is due to Rice (1977) and will be followed here. Material instabilities may be detected with mechanical tests. However, mechanical tests on material elements often consist of imposing displacements (resultant forces) on certain external surfaces of a finite volume of material and measuring resultant forces (displacements) on the same surfaces. Under these conditions, stress and strain cannot be directly controlled. Therefore, instabilities occurring in a mechanical test should always be referred to a specific boundary value problem. As a consequence, material instabilities should be presented within the general context of bifurcation and instability theory for elastoplastic material. This theory was mainly developed by Hill (1958, 1959, 1978) for elastoplastic solids with an associative flow rule. In these notes, this theory will be presented with reference to the generalization to non-associative flow rules given by Raniecki (1979) and Raniecki and Bruhns (1981). In this general framework, several local criteria of stability and bifurcation will be introduced. These are: positive definiteness and non-singularity of the constitutive operator, strong ellipticity and ellipticity of the acoustic tensor and, finally, flutter instability (complex conjugate eigenvalues of the acoustic tensor). Emphasis is given to the connections between these criteria and to their mechanical interpretation.

2   Notation and preliminaries

We refer generally to Gurtin's (1972, 1981) notation. In particular, boldface minuscules $(\mathbf{a}, \mathbf{b}, ...)$ and majuscules $(\mathbf{A}, \mathbf{B}, ...)$ denote vectors (or vector fields) and second-order tensors (or tensor fields), respectively. The space of vectors is denoted by $\mathcal{V}$, the set of second-order tensors by Lin and its symmetric restriction by Sym. The inner products of two vectors $\mathbf{a}$ and $\mathbf{b}$ and two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ are designated by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{3} a_k b_k =: a_k b_k \quad \text{and} \quad \mathbf{A} \cdot \mathbf{B} = \sum_{h,k=1}^{3} A_{hk} B_{hk} =: A_{hk} B_{hk},$$

respectively. The product $\mathbf{AB}$ of two second-order tensors is defined by composition, namely, for every vector $\mathbf{a}$,

$$(\mathbf{AB})\mathbf{a} = \mathbf{A}(\mathbf{Ba}), \quad \text{or} \quad (\mathbf{AB})_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} =: A_{ik} B_{kj}.$$

The tensor product $\mathbf{a} \otimes \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined for every vector $\mathbf{v}$ by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v}), \quad \text{or} \quad (\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j.$$

Moreover, $|\cdot|$ denotes the euclidean norm, $\mathbf{I}$ the second-order identity tensor, a superscript T denotes transpose and tr the trace operator, i.e., for every $\mathbf{A}, \mathbf{B}$, $\mathrm{tr}(\mathbf{AB}) = \mathbf{A}^T \cdot \mathbf{B}$. We will use the symbol $\propto$ between second-order tensors, i.e. $\mathbf{A} \propto \mathbf{B}$, to mean that a scalar ρ exists such that $\mathbf{A} = \rho\mathbf{B}$.

Two symmetric, second-order tensors $\mathbf{A}$ and $\mathbf{B}$ are defined to be *coaxial* when their product commutes, $\mathbf{AB} = \mathbf{BA}$. Note that two coaxial tensors share at least one principal reference system (Appendix A).

Fourth-order tensors are denoted by sans-serif majuscules, as for instance, the elasticity tensor $\mathsf{E}[\cdot]$. These are linear mappings assigning to each second-order tensor $\mathbf{A}$ a second-order tensor $(\mathsf{E}[\mathbf{A}])_{ij} = \mathsf{E}_{ijhk}A_{hk}$.

The product EH of two fourth-order tensors is defined, analogously to second-order tensors, by composition, namely, for every $\mathbf{A} \in \mathrm{Lin}$,

$$(\mathsf{EH})[\mathbf{A}] = \mathsf{E}[\mathsf{H}[\mathbf{A}]], \quad \text{or} \quad (\mathsf{EH})_{ijhk} = \mathsf{E}_{ijst}\mathsf{H}_{sthk}.$$

Two tensorial products will be employed, denoted by symbols $\otimes$ and $\boxtimes$. These are defined, for every $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{Lin}$, as (Del Piero, 1979)

$$(\mathbf{A} \otimes \mathbf{B})[\mathbf{C}] = (\mathbf{B} \cdot \mathbf{C})\mathbf{A}, \quad \text{or} \quad (\mathbf{A} \otimes \mathbf{B})_{ijhk} = A_{ij}B_{hk},$$

and

$$(\mathbf{A}\boxtimes\mathbf{B})[\mathbf{C}] = \mathbf{A}\mathbf{C}\mathbf{B}^T, \quad \text{or} \quad (\mathbf{A}\boxtimes\mathbf{B})_{ijhk} = A_{ih}B_{jk}.$$

Note that, with the above definition, $\mathbf{I}\boxtimes\mathbf{I}$ is the fourth-order identity tensor. Defining the transpose of a fourth-order tensor (for every $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$) as $\mathbf{B} \cdot \mathsf{E}^T[\mathbf{A}] = \mathbf{A} \cdot \mathsf{E}[\mathbf{B}]$, we say that E has the major symmetry whenever $\mathsf{E} = \mathsf{E}^T$. Moreover, we note that

$$(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{B} \otimes \mathbf{A} \quad \text{and} \quad (\mathbf{A}\boxtimes\mathbf{B})^T = \mathbf{A}^T\boxtimes\mathbf{B}^T.$$

The special symbol $\mathbf{I}\overline{\otimes}\mathbf{I}$ is reserved for the fourth-order tensor which associates to every second-order tensor $\mathbf{X}$ its symmetric part:

$$\mathbf{I}\overline{\otimes}\mathbf{I}[\mathbf{X}] = \frac{1}{2}\left(\mathbf{X} + \mathbf{X}^T\right).$$

Obviously $\mathbf{I}\overline{\otimes}\mathbf{I}$ is singular (because it associates the null tensor to every skew symmetric tensor), but its restriction to Sym is invertible and the inverse is the tensor itself.

For any given smooth vector field $\mathbf{a}$, the divergence of $\mathbf{a}$ is defined as

$$\mathrm{div}\,\mathbf{a} = \mathrm{tr}(\mathrm{grad}\,\mathbf{a}),$$

where $\mathrm{grad}\,\mathbf{a}$ is the gradient of $\mathbf{a}$ at $\mathbf{x}$ (in rectangular, cartesian components $(\mathrm{grad}\,\mathbf{a})_{ij} = a_{i,j}$ and $\mathrm{div}\,\mathbf{a} = a_{k,k}$). The divergence operator of a tensor field $\mathbf{A}$ is defined as the unique vector field $\mathrm{div}\mathbf{A}$ which for every constant vector $\mathbf{a}$ satisfies $(\mathrm{div}\mathbf{A}) \cdot \mathbf{a} = \mathrm{div}(\mathbf{A}^T\mathbf{a})$. In (rectangular, cartesian) components:

$$(\mathrm{div}\mathbf{A})_i = A_{ij,j}.$$

A dot over a symbol denotes rate of change (right hand derivative) of the quantity at a fixed material point with respect to a scalar time-like parameter (which will govern the deformation process).

Finally, we note that the symbol $\mathcal{C}(\cdot)$ is reserved for a nonlinear constitutive operator, assigning to each second-order tensor $\mathbf{A}$ the second-order tensor $\mathcal{C}(\mathbf{A})$.

3 Incremental boundary value problem

Quasi-static deformation of an inviscid solid body is assumed to be governed by a time-like parameter (abbreviated as 'time'). In a loading program, displacements and nominal surface tractions (mixed boundary conditions) are prescribed; these are assumed to be sufficiently regular functions of place and time over specific portions $\partial\Omega_\xi^0$ and $\partial\Omega_\sigma^0$ of the boundary in the reference configuration ($\partial\Omega^0 = \partial\Omega_\xi^0 \cup \partial\Omega_\sigma^0$). For simplicity, we limit the presentation to *controlled nominal surface tractions* on $\partial\Omega_\sigma^0$ (in other words, deformation-sensitive loadings are not considered), so that the boundary conditions are

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{x}_0, t), \quad \text{on } \partial\Omega_\xi^0; \quad \mathbf{S}\mathbf{n}_0 = \boldsymbol{\sigma}(\mathbf{x}_0, t) \quad \text{on } \partial\Omega_\sigma^0, \tag{1}$$

where $\mathbf{x}$ and $\mathbf{x}_0$ are the places occupied by the material points in the current and reference configuration, respectively, $\mathbf{n}_0$ is the outward unit vector to $\partial\Omega_\sigma^0$ and $\mathbf{S} \in$ Lin is the first Piola-Kirchhoff stress[2], defined with respect to the Cauchy stress $\mathbf{T} \in$ Sym by

$$\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}, \quad J = \det\mathbf{F}, \tag{2}$$

where $\mathbf{F} \in$ Lin is the deformation gradient.

At a generic stage of the loading program, i.e. at a generic time, we assume that the current geometry and the state of the body are known, and we analyze the response to a prescribed small perturbation of the boundary conditions. In other words, in a series expansion of all quantities specifying deformation of the body we analyze the first-order terms: velocity, stress and strain rates. This is the so-called velocity problem[3], in which velocities and traction rates are prescribed on complementary, regular subsurfaces of the boundary

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\xi}}(\mathbf{x}_0, t), \quad \text{on } \partial\Omega_\xi^0; \quad \dot{\mathbf{S}}\mathbf{n}_0 = \dot{\boldsymbol{\sigma}}(\mathbf{x}_0, t) \quad \text{on } \partial\Omega_\sigma^0. \tag{3}$$

The superposed dot in eqn. (3) denotes the material time derivative (i.e. the time derivative at fixed $\mathbf{x}_0$), $\dot{\mathbf{x}}$ is the material description of the velocity. The velocity field is *assumed to be spatially continuous*. From the Lagrangean standpoint the first-order rate equations are, in the *absence of body forces*:

$$\text{Div}\,\dot{\mathbf{S}} = \mathbf{0}, \quad \text{in } \Omega^0 \setminus \Sigma^0, \tag{4}$$

where $\mathbf{S}$ is the first Piola-Kirchhoff stress. It may be important to note that the divergence operator Div in (4) is referred to material points $\mathbf{x}_0$. Moreover, Σ^0 represents any possible surface of discontinuity of $\dot{\mathbf{S}}$, in the reference description, and the notation in (4) indicates all of Ω^0 excluding Σ^0. Across this surface, the nominal traction rate must remain continuous (Hill, 1961; Chadwick and Powdrill, 1965):

$$\left[\!\left[\dot{\mathbf{S}}\right]\!\right]\mathbf{m}_0 = \mathbf{0}, \quad \text{on } \Sigma^0, \tag{5}$$

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[2] The first Piola-Kirchhoff stress tensor is the transpose of the nominal stress tensor used, among others, by Hill (1978) and Ogden (1984). Note also that these authors use another definition of Div, namely $(\text{Div}\,\mathbf{A})_j = A_{ij,i}$.

[3] Higher-order problems can also be analyzed; see Petryk and Thermann (1985), Nguyen and Triantafyllidis (1989), Cheng and Lu (1993), Bigoni (1996).

where the symbol $[\![\cdot]\!]$ denotes a jump of the relevant argument, i.e. $[\![\dot{\mathbf{S}}]\!] = \dot{\mathbf{S}}^+ - \dot{\mathbf{S}}^-$, and $\mathbf{m}_0$ is the unit normal vector to Σ^0 directed toward $+$ and away from $-$. In weak form, eqs (3), (4), (5) are equivalent to

$$\int_{\Omega^0} \dot{\mathbf{S}} \cdot \nabla \mathbf{w} - \int_{\partial \Omega_\sigma^0} \dot{\boldsymbol{\sigma}} \cdot \mathbf{w} = 0, \tag{6}$$

for every (continuous and piecewise continuously twice differentiable) variation $\mathbf{w}$ of the velocity. In particular, $\nabla \mathbf{w} = \operatorname{Grad} \mathbf{w}$ is the gradient (with respect to material points $\mathbf{x}_0$) of a field $\mathbf{w}$ defined in the reference configuration and taking null values on the portions of the boundary where displacements (and velocities) are prescribed.

4 Constitutive Equations

Elastoplastic, isothermal and time independent material behaviour of a solid subject to large strains is described in this section. We present a broad constitutive framework in which many existing elastoplastic models may fit. In the interest of generality, certain details are intentionally left unspecified.

Adopting Ogden's (1984) notation, let us consider a pair of symmetric, Lagrangean, stress $\mathbf{T}^{(m)}$ and strain $\mathbf{E}^{(m)}$ measures, work-conjugate in the Hill sense (1968, 1978), so that

$$\mathbf{T}^{(m)} \cdot \dot{\mathbf{E}}^{(m)} \tag{7}$$

gives the stress power density (per unit volume of Ω^0) independently of the positive[4], integer exponent m. In particular, if we take

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \tag{8}$$

where $\mathbf{U}$ is the right stretch tensor, related to the deformation gradient $\mathbf{F}$ through $\mathbf{U} = (\mathbf{F}^T \mathbf{F})^{1/2}$, then the conjugate stress $\mathbf{T}^{(m)}$ is obtained by imposing the equality:

$$\mathbf{S} \cdot \dot{\mathbf{F}} = \mathbf{T}^{(m)} \cdot \dot{\mathbf{E}}^{(m)}. \tag{9}$$

For instance, when $m = 2$ we obtain the Green-Lagrange strain and the second Piola-Kirchhoff stress tensors, respectively:

$$\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}), \quad \mathbf{T}^{(2)} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}. \tag{10}$$

Following Hill and Rice (1973) and Hill (1978) but without introducing elastic potentials, inelastic materials are considered that may at any stage of deformation exhibit a purely elastic response for appropriate loading. For these materials, elastic response is

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[4] Negative or null exponents can also be introduced (Ogden, 1984). The notation $\mathbf{T}^m = \underbrace{\mathbf{T}\mathbf{T}...\mathbf{T}}_{m\ times}$ should not be confused with $\mathbf{T}^{(m)}$.

assumed to be a one-to-one relation between $\mathbf{T}^{(m)}$ and $\mathbf{E}^{(m)}$, though depending on the prior inelastic history, i.e.

$$\mathbf{T}^{(m)} = \hat{\mathbf{T}}^{(m)}(\mathbf{E}^{(m)}, \mathcal{K}), \quad \mathbf{E}^{(m)} = \hat{\mathbf{E}}^{(m)}(\mathbf{T}^{(m)}, \mathcal{K}), \tag{11}$$

where $\hat{\mathbf{T}}^{(m)}$ and $\hat{\mathbf{E}}^{(m)}$ are functionals of the prior history of inelastic deformation through the unspecified set $\mathcal{K}$ of variables of generic tensorial nature (thus embracing second-order tensors and scalars). For a purely elastic deformation rate (in other words, at fixed $\mathcal{K}$) we have

$$\dot{\mathbf{T}}^{(m)} = \mathsf{E}[\dot{\mathbf{E}}^{(m)}], \quad \dot{\mathbf{E}}^{(m)} = \mathsf{M}[\dot{\mathbf{T}}^{(m)}], \tag{12}$$

where

$$\mathsf{E}(\mathbf{E}^{(m)}, \mathcal{K}) = \frac{\partial \hat{\mathbf{T}}^{(m)}}{\partial \mathbf{E}^{(m)}}, \quad \mathsf{M}(\mathbf{T}^{(m)}, \mathcal{K}) = \frac{\partial \hat{\mathbf{E}}^{(m)}}{\partial \mathbf{T}^{(m)}}, \tag{13}$$

and obviously

$$\mathsf{E} = \mathsf{M}^{-1}. \tag{14}$$

For an increment involving elastic and inelastic strain rates, we may write

$$\dot{\mathbf{T}}^{(m)} = \mathsf{E}[\dot{\mathbf{E}}^{(m)}] - \dot{\Lambda}\mathsf{E}[\mathbf{P}], \quad \dot{\mathbf{E}}^{(m)} = \mathsf{M}[\dot{\mathbf{T}}^{(m)}] + \dot{\Lambda}\mathbf{P}, \tag{15}$$

where $\mathbf{P} \in \text{Sym}$,

$$\dot{\Lambda}\mathbf{P} = \frac{\partial \hat{\mathbf{E}}^{(m)}}{\partial \mathcal{K}}[\dot{\mathcal{K}}] \tag{16}$$

and the scalar $\dot{\Lambda} \geq 0$, called the plastic multiplier, is null when $\dot{\mathcal{K}} = 0$. A yield surface is assumed at each $\mathcal{K}$. This may be alternatively expressed as $f_{\mathbf{T}^{(m)}}(\mathbf{T}^{(m)}, \mathcal{K}) \leq 0$ or as $f_{\mathbf{E}^{(m)}}(\mathbf{E}^{(m)}, \mathcal{K}) \leq 0$, thus defining regions of the $\mathbf{T}^{(m)}$ or $\mathbf{E}^{(m)}$ space, respectively, within which the response is elastic. Prager's consistency condition requires $\dot{f}_{\mathbf{T}^{(m)}} = \dot{f}_{\mathbf{E}^{(m)}} = 0$, when inelastic strain rate is different from zero. As a consequence, employing the stress space representation, the elastoplastic incremental constitutive equations can be written as

$$\dot{\mathbf{T}}^{(m)} = \begin{cases} \mathsf{E}[\dot{\mathbf{E}}^{(m)}] - \dfrac{1}{g} < \mathbf{Q} \cdot \mathsf{E}[\dot{\mathbf{E}}^{(m)}] > \mathsf{E}[\mathbf{P}] & \text{if } f_{\mathbf{T}^{(m)}}(\mathbf{T}^{(m)}, \mathcal{K}) = 0, \\[2mm] \mathsf{E}[\dot{\mathbf{E}}^{(m)}] & \text{if } f_{\mathbf{T}^{(m)}}(\mathbf{T}^{(m)}, \mathcal{K}) < 0, \end{cases} \tag{17}$$

where the operator $< \cdot >$ denotes the Macaulay brackets, i.e. $\forall \alpha \in \Re, < \alpha > = (\alpha + |\alpha|)/2$. Moreover, $\mathbf{Q} = \partial f_{\mathbf{T}^{(m)}}/\partial \mathbf{T}^{(m)} \in \text{Sym}$ is the yield function gradient and the plastic modulus

$$g = h + \mathbf{Q} \cdot \mathsf{E}[\mathbf{P}], \tag{18}$$

is assumed to be strictly positive (a negative plastic modulus would correspond to a so-called locking material, not investigated here). In the Hill (1967b) notation, the hardening modulus h in (18) describes *hardening* when positive, *softening* when negative and *perfect plasticity* when null. It is defined as

$$\dot{\Lambda}h = -\frac{\partial f_{\mathbf{T}^{(m)}}}{\partial \mathcal{K}} \cdot \dot{\mathcal{K}}. \tag{19}$$

As Hill (1967b) remarks, hardening and softening are not measure-invariant concepts, in the sense that h depends on the choice of $\mathbf{T}^{(m)}$ and $\mathbf{E}^{(m)}$. Therefore, the nomenclature is, to some extent, arbitrary. Moreover, we remark that, in addition to h, also $\mathbf{Q}$, $\mathbf{P}$ and E are measure-dependent. On the contrary, the plastic modulus g can be shown to be measure-independent (Hill, 1967b; Petryk, 1999). Note also that all quantities appearing in the rate equations (17) fully depend on the entire path of deformation reckoned from some ground state.

The scalar product of the first equation in (17) with $\mathbf{Q}$ gives

$$\mathbf{Q} \cdot \dot{\mathbf{T}}^{(m)} = \mathbf{Q} \cdot \mathsf{E}[\dot{\mathbf{E}}^{(m)}] - \frac{\mathbf{Q} \cdot \mathsf{E}[\mathbf{P}]}{g} < \mathbf{Q} \cdot \mathsf{E}[\dot{\mathbf{E}}^{(m)}] > . \tag{20}$$

In the case when $h > 0$, we note that

$$\mathrm{sign}(\mathbf{Q} \cdot \mathsf{E}[\dot{\mathbf{E}}^{(m)}]) = \mathrm{sign}(\mathbf{Q} \cdot \dot{\mathbf{T}}^{(m)}).$$

Therefore, assuming $h > 0$ and using (20), we obtain the inverse constitutive equations

$$\dot{\mathbf{E}}^{(m)} = \begin{cases} \mathsf{M}[\dot{\mathbf{T}}^{(m)}] + \dfrac{1}{h} < \mathbf{Q} \cdot \dot{\mathbf{T}}^{(m)} > \mathbf{P} & \text{if } f(\mathbf{T}^{(m)}, \mathcal{K}) = 0 , \\ \mathsf{M}[\dot{\mathbf{T}}^{(m)}] & \text{if } f(\mathbf{T}^{(m)}, \mathcal{K}) < 0 , \end{cases} \tag{21}$$

It may be important to remark that *all possible choices of* $\mathbf{T}^{(m)}$ *and* $\mathbf{E}^{(m)}$ *in (17) or (21) are equivalent and that all resulting constitutive equations respect the requirement of material frame indifference* (Truesdell and Noll, 1965).

It is particularly convenient to write the constitutive equation (17) in terms of the material time derivative of the first Piola-Kirchhoff stress and of the deformation gradient. This can be done for any choice of $\mathbf{T}^{(m)}$ and $\mathbf{E}^{(m)}$. For instance, in the case of the second Piola-Kirchhoff stress tensor $\mathbf{T}^{(2)}$ and the Green-Lagrange strain tensor $\mathbf{E}^{(2)}$, the following relations hold true

$$\dot{\mathbf{S}} = \mathbf{F}\dot{\mathbf{T}}^{(2)} + J\mathbf{L}\mathbf{T}\mathbf{F}^{-T}, \quad \dot{\mathbf{E}}^{(2)} = \mathbf{F}^T\mathbf{D}\mathbf{F}, \tag{22}$$

where $\mathbf{L} = \mathrm{grad}\,\mathbf{v}$ is the velocity gradient and $\mathbf{D}$ its symmetric part, i.e. the rate of deformation. The minor symmetries of E and the relation $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ imply

$$\mathsf{E}[\mathbf{F}^T\mathbf{D}\mathbf{F}] = \frac{1}{2}\mathsf{E}[\mathbf{F}^T\dot{\mathbf{F}} + (\mathbf{F}^T\dot{\mathbf{F}})^T] = \mathsf{E}[\mathbf{F}^T\dot{\mathbf{F}}]. \tag{23}$$

Moreover, from the equality $\mathbf{F}^{-T}\mathbf{B} \cdot \mathbf{F}\mathbf{A} = \mathbf{B} \cdot \mathbf{A}$, holding for every $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$, we conclude that (17) can be written as

$$\dot{\mathbf{S}} = \begin{cases} \mathsf{G}[\dot{\mathbf{F}}] - \dfrac{1}{g} < (\mathbf{F}^{-T}\mathbf{Q}) \cdot \mathsf{B}[\dot{\mathbf{F}}] > \mathsf{B}[\mathbf{F}^{-T}\mathbf{P}] & \text{if } f_S(\mathbf{S}, \mathcal{K}) = 0 , \\ \mathsf{G}[\dot{\mathbf{F}}] & \text{if } f_S(\mathbf{S}, \mathcal{K}) < 0 , \end{cases} \tag{24}$$

where

$$\mathsf{B} = (\mathbf{F}\boxtimes\mathbf{I})\mathsf{E}(\mathbf{F}\boxtimes\mathbf{I})^T, \quad \mathsf{G} = \mathsf{B} + \mathbf{I}\boxtimes\mathbf{S}^T\mathbf{F}^{-T}. \tag{25}$$

It should be noted that neither B nor G have the minor symmetries, and that both B and G have the major symmetry only in the case of Green elasticity, i.e. when E has the major symmetry, too[5]. Moreover, the yield function $f_S(S, \mathcal{K})$ in (24) has been expressed in terms of the first Piola-Kirchhoff stress tensor; its gradient with respect to S is $F^{-T}Q$. This follows from the chain rule of differentiation

$$\frac{\partial f_S}{\partial S} \cdot A = \frac{\partial f_{T^{(2)}}}{\partial T^{(2)}} \cdot \frac{\partial T^{(2)}}{\partial S}[A] = F^{-T}Q \cdot A, \tag{26}$$

for every $A \in$ Lin. When the yield criterion is satisfied, $f_S(S, \mathcal{K}) = 0$, constitutive equations (24) define a piecewise-linear function of $\dot{F}$

$$\dot{S} = \mathcal{C}(\dot{F}), \tag{27}$$

i.e. an incrementally nonlinear constitutive equation with two branches, corresponding to plastic loading and elastic unloading. This constitutive equation may obviously be written in the following form, useful for subsequent analysis:

$$\boxed{\dot{S} = G[\dot{F}] - \frac{1}{g} < N \cdot \dot{F} > M,} \tag{28}$$

where $N = B^T[F^{-T}Q] \in$ Lin and $M = B[F^{-T}P] \in$ Lin are the yield surface and plastic potential normals, respectively, in strain space. In particular, expressing the yield function in terms of the deformation gradient, i.e. $f_F(F, \mathcal{K})$, its gradient with respect to F is exactly N; in fact

$$\frac{\partial f_F}{\partial F} \cdot A = \frac{\partial f_{T^{(2)}}}{\partial T^{(2)}} \cdot \frac{\partial T^{(2)}}{\partial E^{(2)}} \left[\frac{\partial E^{(2)}}{\partial F}[A] \right] = B^T[F^{-T}Q] \cdot A, \tag{29}$$

for every $A \in$ Lin. Note also that $P = Q$ is equivalent to $M = N$ only when B has the major symmetry, i.e. for Green elasticity. In other words, simultaneous normality in stress and strain spaces is not assured for Cauchy elasticity (Hill, 1978). In the following, we will refer to associative flow rule when $P = Q$ and B has the major symmetry, so that normality is preserved in the strain space also, $M = N$.

As far as the choice of P and Q is concerned, this is to some extent arbitrary. However, experiments show that many materials exhibit a peculiar kind of non-associativity, involving only the volumetric part of plastic deformation. This case of special interest corresponds to so-called *deviatoric associativity*, where the deviatoric parts of P and Q are aligned. This may be generically defined as

$$P = \chi_1 \hat{S} + \frac{\chi_2}{3} I, \qquad Q = \psi_1 \hat{S} + \frac{\psi_2}{3} I, \tag{30}$$

where $\hat{S} \in$ Sym is traceless, χ_1 and ψ_1 are assumed strictly positive and χ_2 and ψ_2 are assumed to be non-negative. The parameters ψ_2 and χ_2 describe the pressure-sensitivity

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[5] Note that $S^T F^{-T} = J F^{-1} T F^{-T} \in$ Sym. Thus, $G - B = I \boxtimes S^T F^{-T}$ has always the major symmetry.

and the dilatancy of the material, respectively. For instance, in the case of the constitutive model proposed by Rudnicki and Rice (1975)

$$\chi_1 = \psi_1 = \frac{1}{2\sqrt{J_2}}, \quad \hat{\mathbf{S}} = \text{dev}\,\mathbf{T}, \tag{31}$$

where $\text{dev}\,\mathbf{T} = \mathbf{T} - (\text{tr}\mathbf{T}/3)\mathbf{I}$ is the stress deviator and $J_2 = (\text{dev}\mathbf{T} \cdot \text{dev}\mathbf{T})/2$. Note that, obviously, deviatoric associativity implies coaxiality of $\mathbf{Q}$ and $\mathbf{P}$.

4.1 The infinitesimal theory

In the special case of the infinitesimal theory, equations (11) simplify to

$$\mathbf{T} = \mathsf{E}[\mathbf{E} - \mathbf{E}_p], \quad \mathbf{E} = \mathsf{M}[\mathbf{T}] + \mathbf{E}_p, \tag{32}$$

where $\mathbf{E} = \frac{1}{2}[\text{grad}\,\mathbf{u} + (\text{grad}\,\mathbf{u})^T]$ is the infinitesimal strain tensor with $\mathbf{E}_p$ being its plastic part. The fourth-order tensor $\mathsf{E} = \mathsf{M}^{-1}$ is now the elasticity tensor of the usual infinitesimal theory[6], which, in the well-known case of isotropy is

$$\mathsf{E} = \lambda\,\mathbf{I} \otimes \mathbf{I} + 2\mu\,\mathbf{I}\,\overline{\otimes}\,\mathbf{I}, \tag{33}$$

where $\mathbf{I}\,\overline{\otimes}\,\mathbf{I}$ is the fourth-order tensor which associates to every tensor $\mathbf{A} \in \text{Lin}$ its symmetric part $\mathbf{I}\,\overline{\otimes}\,\mathbf{I}[\mathbf{A}] = (\mathbf{A} + \mathbf{A}^T)/2$ and λ and μ are the Lamé constants. In the stress space formulation, the yield function can be written as $f(\mathbf{T}, \mathcal{K}) \leq 0$, where $\mathcal{K}$ is a generic set of internal variables, possibly depending on $\mathbf{E}_p$. Constitutive equations (27) therefore become (in the case $f(\mathbf{T}, \mathcal{K}) = 0$)

$$\dot{\mathbf{T}} = \mathsf{E}[\mathbf{D}] - \frac{1}{g} < \mathbf{Q} \cdot \mathsf{E}[\mathbf{D}] > \mathsf{E}[\mathbf{P}], \tag{34}$$

where $g = h + \mathbf{Q} \cdot \mathsf{E}[\mathbf{P}]$ is the plastic modulus (assumed strictly positive). Note that $\mathbf{Q} \cdot \mathsf{E}[\mathbf{D}] = \mathbf{D} \cdot \mathsf{E}^T[\mathbf{Q}]$; therefore the tangent operator is symmetric for the associative flow rule if and only if E has the major symmetry.

5 Global uniqueness and stability

5.1 Uniqueness

With reference to the Lagrangean description, the regular first-order rate problem can be stated with reference to the velocity field $\dot{\mathbf{x}}$, when the constitutive equation (27) is employed in $(3)_2$, $(4)_2$ and $(5)_2$. Therefore, we obtain:

$$\begin{aligned}
\dot{\mathbf{x}} &= \dot{\boldsymbol{\xi}}(\mathbf{x}_0, t), && \text{on } \partial\Omega_\xi^0, \\
\mathcal{C}(\dot{\mathbf{F}})\mathbf{n}_0 &= \dot{\boldsymbol{\sigma}}(\mathbf{x}_0, t), && \text{on } \partial\Omega_\sigma^0, \\
\text{Div}(\mathcal{C}(\dot{\mathbf{F}})) &= \mathbf{0}, && \text{in } \Omega^0 \setminus \Sigma^0, \\
[\![\mathcal{C}(\dot{\mathbf{F}})]\!]\,\mathbf{m}_0 &= \mathbf{0}, && \text{on } \Sigma^0,
\end{aligned} \tag{35}$$

---

[6] Usually E is constant. In the case of elastoplastic coupling, E depends on $\mathbf{E}_p$ (Hueckel, 1976). In the following it is assumed constant.

where $\dot{\mathbf{F}} = \nabla \dot{\mathbf{x}}$ in $\Omega^0 \setminus \Sigma^0$ and $[\![\dot{\mathbf{x}}]\!] = \mathbf{0}$ across Σ^0.

The above equations refer to the part of the body at yielding, i.e. to the so-called plastic zone. In the elastic zone the governing equations are the same with $\mathcal{C}$ replaced by $\mathbf{G}$.

We are now in a position to state the velocity problem, namely, given a certain state of a body, find a continuous and piecewise continuously twice differentiable (shortly, 'admissible') velocity field satisfying (35).

In order to obtain an exclusion condition for bifurcation in velocity, we follow the Hill (1950, 1958) argument[7], which generalizes to elastoplasticity the Kirchhoff theorem of linear elasticity. Suppose therefore that the velocity problem admits two solutions, say $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ both satisfying (35). Their difference $\Delta\dot{\mathbf{x}} = \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2$ defines an admissible velocity field, with gradient $\Delta\dot{\mathbf{F}}$. Also the difference in stresses $\Delta\dot{\mathbf{S}}$ satisfies equilibrium equations (4), (5). Moreover, the difference fields satisfy homogeneous conditions on the boundary. On application of the divergence theorem it follows that

$$\int_{\Omega^0} \Delta\dot{\mathbf{S}} \cdot \Delta\dot{\mathbf{F}} = 0, \tag{36}$$

where $\Delta\dot{\mathbf{S}} = \mathcal{C}(\dot{\mathbf{F}}_1) - \mathcal{C}(\dot{\mathbf{F}}_2)$. It should be stressed that *due to the nonlinearity of* $\mathcal{C}$, $\Delta\dot{\mathbf{S}}$ does not, in general, coincide with $\mathcal{C}(\Delta\dot{\mathbf{F}})$.

Therefore, *a sufficient condition to exclude bifurcations of the velocity problem* is:

$$\boxed{\int_{\Omega^0} \Delta\dot{\mathbf{S}} \cdot \Delta\dot{\mathbf{F}} > 0,} \tag{37}$$

for all pairs of distinct, admissible velocity fields taking the given values (35)$_1$ on $\partial\Omega_\xi^0$.

Note that the exclusion condition (37) would be true even replacing '$>$'with '$<$'. We will see that this possibility can be excluded in terms of instability.

Raniecki comparison solids. The difficulty in proceeding with (37) is related to the nonlinearity of the constitutive operator $\mathcal{C}$. To overcome this problem, Hill (1958) proposed to introduce a *linear comparison solid*, which, when used to replace the original constitutive operator, provides a lower bound to (37). Results of Hill were restricted to the associative flow rule, where the comparison solid turns out to coincide with a linear solid defined by the constitutive tensor corresponding to the loading branch of the elastoplastic operator (28). The Hill comparison theorem was later generalized by Raniecki (1979) and Raniecki and Bruhns (1981) to cover non-associative flow rules. In particular, Raniecki introduced a family of linear comparison solids (briefly, 'Raniecki comparison solids') defined by the following constitutive tensor having the major symmetry (for every $\psi \in \Re^+$)

$$\mathbf{R} = \mathbf{G} - \frac{1}{4\psi g}(\mathbf{M} + \psi\mathbf{N}) \otimes (\mathbf{M} + \psi\mathbf{N}), \tag{38}$$

---

[7] Hill's (1950) proof was restricted to the infinitesimal theory and was based on a theorem due to Melan (1938).

such that the following *comparison theorem* holds true

$$\boxed{\Delta\dot{\mathbf{S}} \cdot \Delta\dot{\mathbf{F}} \geq \Delta\dot{\mathbf{F}} \cdot \mathsf{R}[\Delta\dot{\mathbf{F}}],} \qquad (39)$$

for every difference of tensors $\Delta\dot{\mathbf{F}} = \dot{\mathbf{F}}_1 - \dot{\mathbf{F}}_2$ and related difference $\Delta\dot{\mathbf{S}} = \mathcal{C}(\dot{\mathbf{F}}_1) - \mathcal{C}(\dot{\mathbf{F}}_2)$. Therefore, the exclusion condition (37) is necessarily satisfied when the stronger condition

$$\boxed{\int_{\Omega^0} \dot{\mathbf{F}} \cdot \mathsf{C}^c[\dot{\mathbf{F}}] > 0} \qquad (40)$$

holds true for all (not identically zero) continuous and piecewise continuously twice differentiable velocity fields, satisfying homogeneous conditions on $\partial\Omega_\xi^0$. The comparison solid C^c in (40) is equal, by definition, to R in the current plastic zone and to G in the current elastic zone. Note that, due to the linearity of C^c, the difference fields denoted by Δ in (37) do not appear in (40).

In the case of hyperelasticity and associative flow-rule, $\mathbf{N} = \mathbf{M}$, the comparison solid (38) reduces to the Hill comparison solid by taking $\psi = 1$.

The Raniecki's comparison theorem can be proved as follows. Three cases must be analyzed:

$$
\begin{aligned}
&1)\ \mathbf{N} \cdot \dot{\mathbf{F}}_1 < 0 \quad \text{and} \quad \mathbf{N} \cdot \dot{\mathbf{F}}_2 < 0, \quad &&(\text{unloading/unloading}), \\
&2)\ \mathbf{N} \cdot \dot{\mathbf{F}}_1 > 0 \quad \text{and} \quad \mathbf{N} \cdot \dot{\mathbf{F}}_2 > 0, \quad &&(\text{loading/loading}), \\
&3)\ \mathbf{N} \cdot \dot{\mathbf{F}}_1 > 0 \quad \text{and} \quad \mathbf{N} \cdot \dot{\mathbf{F}}_2 < 0, \quad &&(\text{loading/unloading}).
\end{aligned}
$$

In all cases, it suffices to prove that

$$\Delta\dot{\mathbf{S}} \cdot \Delta\dot{\mathbf{F}} - \Delta\dot{\mathbf{F}} \cdot \mathsf{R}[\Delta\dot{\mathbf{F}}] = \tfrac{1}{4\psi g} \qquad (41)$$
$$\left(-4\psi < \mathbf{N} \cdot \dot{\mathbf{F}}_1 > \mathbf{M} \cdot \Delta\dot{\mathbf{F}} + 4\psi < \mathbf{N} \cdot \dot{\mathbf{F}}_2 > \mathbf{M} \cdot \Delta\dot{\mathbf{F}} + [(\mathbf{M} + \psi\mathbf{N}) \cdot \Delta\dot{\mathbf{F}}]^2\right) \geq 0,$$

which, taking into account that $\psi g > 0$ and analyzing Cases 1)-3), follows directly. It is important to remark that the comparison theorem (39) holds true for every G, in other words, regardless of the symmetries and the definiteness of the elastic tensor G.

Finally, it may be worth noting that the exclusion condition (40) for bifurcations of the velocity problem, may be shown to be sufficient to exclude second- and higher-order bifurcations (under specific regularity conditions, see Petryk and Thermann, 1985; Nguyen and Triantafyllidis, 1989; Cheng and Lu, 1993; Bigoni, 1996)[8].

Associative elastoplasticity. It may be interesting to keep contact with the case of the associative flow rule $\mathbf{Q} = \mathbf{P}$, and Green elasticity (E, B, and G have the major symmetry)[9]. In this case, constitutive equations admit a velocity-gradient potential. With

---

[8] Moreover, the techniques introduced in this section to exclude bifurcation can be exported to another, similar context. This is the problem of contact with friction of an elastic body with a constraint (Radi et al. 1999).

[9] A detailed presentation can be found in Petryk (1993a).

reference to (28), it follows[10] that

$$\dot{\mathbf{S}} = \frac{\partial U}{\partial \dot{\mathbf{F}}}, \quad U = \frac{\dot{\mathbf{F}} \cdot \mathbf{G}[\dot{\mathbf{F}}]}{2} - \frac{< \mathbf{N} \cdot \dot{\mathbf{F}} >^2}{2g}. \tag{42}$$

Let us consider now the following functional (Hill, 1958;1959), defined for every admissible velocity field $\mathbf{v}$ satisfying $(35)_1$ on $\partial \Omega_\xi^0$

$$J(\mathbf{v}) = \int_{\Omega^0} U(\mathbf{v}) - \int_{\partial \Omega_\sigma^0} \dot{\sigma} \cdot \mathbf{v}. \tag{43}$$

The vanishing of the first weak (Gateaux[11]) variation of $J(\mathbf{v})$ with respect to every admissible variation $\mathbf{w}$ of $\mathbf{v}$ is equivalent to (6). Therefore, a velocity field is a solution of the rate problem if and only if it assigns to the functional (43) a stationary value. Moreover, when the uniqueness condition (37) holds true, the functional $J(\mathbf{v})$ can be proved to be strictly convex. It follows that in the range where (37) holds true, the unique solution assigns to the functional $J(\mathbf{v})$ a strict, absolute minimum (Hill, 1958; 1959).

For the linear comparison solid defined by $\psi = 1$, i.e. by the loading branch of the constitutive operator, we have:

$$\dot{\mathbf{S}} = \frac{\partial U^L}{\partial \dot{\mathbf{F}}}, \quad U^L = \frac{\dot{\mathbf{F}} \cdot \mathbf{G}[\dot{\mathbf{F}}]}{2} - \frac{(\mathbf{N} \cdot \dot{\mathbf{F}})^2}{2g}, \tag{44}$$

so that the exclusion condition (40) corresponds now to the positive definiteness of the quadratic functional (where U^L in the elastic zone has to be identified with the actual elastic potential)

$$I^L(\mathbf{w}) = \int_{\Omega^0} U^L(\mathbf{w}) > 0, \tag{45}$$

for every admissible field $\mathbf{w}$ vanishing on Ω_ξ^0.

The variational basis of the rate problem for an associative flow-rule has important consequences on bifurcation. In fact, for a given deformation path, let us assume that a series of configurations continuously evolves in parameter space satisfying $I^L > 0$, and that this is terminated by a configuration for which

$$\begin{cases} I^L(\mathbf{w}) \geq 0 \text{ for every } \mathbf{w} \\ I^L(\mathbf{w}^*) = 0 \text{ for some } \mathbf{w}^* \neq \mathbf{0}, \end{cases} \tag{46}$$

---

[10] Note that $\frac{\partial < \mathbf{N} \cdot \dot{\mathbf{F}} >^2}{\partial \dot{\mathbf{F}}} = < \mathbf{N} \cdot \dot{\mathbf{F}} > \frac{\partial (\mathbf{N} \cdot \dot{\mathbf{F}} + |\mathbf{N} \cdot \dot{\mathbf{F}}|)}{\partial \dot{\mathbf{F}}} = < \mathbf{N} \cdot \dot{\mathbf{F}} > (1 + \frac{\mathbf{N} \cdot \dot{\mathbf{F}}}{|\mathbf{N} \cdot \dot{\mathbf{F}}|}) \mathbf{N} = 2 < \mathbf{N} \cdot \dot{\mathbf{F}} > \mathbf{N}.$

[11] The first weak variation of a functional J in the direction $\mathbf{w}$ at $\mathbf{v}$ is

$$\delta J(\mathbf{v}, \mathbf{w}) = \frac{d}{d\alpha} J(\mathbf{v} + \alpha \mathbf{w})|_{\alpha=0}.$$

where the admissible fields $\mathbf{w}$ vanish on Ω^0_ξ. As shown by Hill and Hutchinson (1975) and Young (1976), the first instant at which $I^L > 0$ fails to hold need not satisfy (46)[12]. However, for situations in which (46) holds true a *primary eigenstate* has been found and I^L is stationary at the minimum point $\mathbf{w}^*$. This is necessary and sufficient for $\mathbf{w}^*$ to be a solution of the homogeneous problem. Therefore, a *critical point* has been detected. This may represent *either* a true *bifurcation point* for the comparison solid *or* a so-called *limit point*. In the former case, i.e. when a bifurcation point in the comparison solid has been detected, the eigenmode $\mathbf{w}^*$ can be added to a solution $\mathbf{v}$ of the non-homogeneous problem to generate a bifurcated solution. In other words, if $\mathbf{v}$ is a solution, $\mathbf{v} + \gamma\mathbf{w}^*$, $\gamma \in \Re$ is a family of possible solutions. In this family, i.e. for certain values of γ, bifurcated solutions can usually be found that correspond to the plastic branch of the constitutive equations in the current plastic zone. Among these solutions, those which initiate a quasi-static post-bifurcation path represent genuine elastic-plastic bifurcations of the real elastoplastic solid, and may occur under broad hypotheses (Hutchinson, 1973).

'In loading comparison solid'. From the discussion relative to associative flow rules, it should be clear that failure of the Hill/Raniecki exclusion condition (40) is in general not sufficient for bifurcation even in the case of associative elastoplasticity. This becomes indeed more evident for non-associative elastoplasticity, where due to the lack of a variational structure of the governing incremental field equations, failure of (40) is far from implying bifurcation. We may note, however, that the comparison solid in the associative case plays a double role. On one hand it excludes bifurcation when used in (40) and, on the other hand, it provides a bifurcated field for the comparison solid when (40) fails, which can often be 'adjusted' for the real elastoplastic solid. In the non-associative case, the Raniecki comparison solids are effective for excluding bifurcation, but not to provide a bifurcated field useful for the real elastoplastic solid. Therefore, following Raniecki and Bruhns (1981), let us consider, even for non-associative flow rules a fictitious, incrementally-linear solid with tangent constitutive tensor

$$\mathbf{C} = \mathbf{G} - \frac{1}{g}\mathbf{M} \otimes \mathbf{N}, \tag{47}$$

corresponding to the loading branch of the constitutive operator (28) (briefly, 'in loading comparison solid'). Let us consider a deformation path in which an elastoplastic solid is deformed in the plastic range, so that the actual behaviour corresponds to the behaviour of the fictitious solid 'in loading'. The first bifurcation for this comparison solid may correspond (in the sense already explained for associative flow-rule) to a possible bifurcation of the real elastoplastic solid. Therefore, for non-associative elastoplasticity:

> *two bounds can be defined for bifurcation. The lower bound corresponds to failure of exclusion condition (40) for the optimal Raniecki comparison solid* R. *The upper bound corresponds to bifurcation in the 'in loading comparison solid' defined by the incremental tensor* C.

---

[12] In Hill and Hutchinson (1975) and Young (1976), this possibility arises in connection with the achievement of a situation where surface bifurcation modes exist of arbitrarily short wavelength.

Whether or not some bifurcation actually occurs in the real elastoplastic solid within the two bounds is still an open question. No examples are in fact yet known.

5.2 Stability

Let us consider a generic equilibrium configuration of a body at a fixed value of the loading parameter governing the deformation, so that the prescribed displacements on the boundary are fixed and the prescribed nominal surface tractions correspond, momentarily, to dead loading. Roughly speaking, the configuration is called stable when the effects of a small disturbance remain sufficiently small during the entire motion subsequent to the application of the disturbance itself. Therefore, stability analysis involves considerations about the dynamics of the system. The definition of stability becomes a precise concept when the measures of the distances and the class of perturbations are specified. However, path-dependence of inelastic material (together with certain mathematical difficulties connected to the analysis of a continuum problem) makes a rigorous analysis awkward. Therefore, we content ourselves with presenting the simple analysis which was proposed by Hill (1958) in the context of an associative flow rule. In that context, the Hill's analysis has a much more firm basis than in the case of a non-associative flow rule (Petryk, 1991; 1985b; 1993a;b).

Let us confine our attention to *direct* paths of departure from the equilibrium configuration (thus excluding arbitrary circuitous paths). In this way, a perturbed motion is a-priori assumed such that variations of the direction of the velocity field along the path are negligible. This is the so-called *directional stability*, which was analyzed by Hill in the framework of associative elastoplasticity. Along any admissible direct path starting from the equilibrium state under consideration, the work of deformation in the body can be written as

$$W = \int_{\Omega^0} \mathbf{S} \cdot \mathrm{Grad}\, \Delta \mathbf{u} + \frac{1}{2} \int_{\Omega^0} \Delta \mathbf{S} \cdot \mathrm{Grad}\, \Delta \mathbf{u} + o((\Delta t)^2), \tag{48}$$

where $\mathbf{S}$ is the first Piola-Kirchhoff stress at the equilibrium state, $\Delta \mathbf{u}$ and $\Delta \mathbf{S}$ are the increments in displacement and stress reached along the path. Finally, Δt is the increment in the time-like parameter measuring the length of the path. In view of the fact that $\Delta \mathbf{u}$ vanishes on $\partial \Omega^0_\xi$, from the principle of virtual power, the first integral in (48) is equal to the work done by external *dead* loads W^{load}. Therefore, the work difference can be written as:

$$W - W^{load} = \frac{(\Delta t)^2}{2} \int_{\Omega^0} \dot{\mathbf{S}} \cdot \nabla \dot{\mathbf{u}} + o((\Delta t)^2). \tag{49}$$

It follows from (27) that if

$$\boxed{\int_{\Omega^0} \nabla \mathbf{w} \cdot \mathcal{C}(\nabla \mathbf{w}) > 0, \quad \text{(stability in Hill's sense)}} \tag{50}$$

for every admissible (not identically zero) velocity field $\mathbf{w}$ taking zero values on $\partial \Omega^0_\xi$, then any movement from the equilibrium configuration requires some additional energy to be

supplied to the system from external sources. In this sense (50) is a *sufficient condition for directional stability* of equilibrium. We will briefly refer to this condition as 'stability in Hill's sense', even if:

- it is only a sufficient condition (for the assumed class of paths),
- Hill never proposed to use this condition in a broader context than associative elasto-plasticity.

With reference to elasticity, with '>' replaced with '≥', condition (50) was proposed by Hadamard (1903) and is usually called *infinitesimal stability*. As noticed by Truesdell and Noll (1965, Sect. 68) and Beatty (1987) the existence of a stored energy function is not essential in the above definition of stability. In the elastoplastic case, when the associative flow rule is assumed, condition (50), with '>' replaced with '≥', was proved also to be necessary for stability under broad hypotheses (Petryk, 1993a;b).

In the case of non-associative flow laws, two important points should be emphasized. First, instability as related to violation of (50) is not proven, so that (50) may not be a necessary condition for stability. Second, condition (50) should not be even considered sufficient. There are in fact certain instability phenomena, such as flutter, which may in principle occur even when (50) is satisfied.

It is worth noting that (40) implies (50), more explicitly[13]

$$\boxed{\text{exclusion condition for bifurcation} \implies \text{stability in Hill's sense,}}$$

but the converse need not be true.

6   Local conditions for uniqueness and stability

From the global uniqueness and stability criteria considered in the previous section, local conditions may be derived, which are the subject of this section. The importance of local conditions lies in the connection to material instabilities, i.e. to instabilities which can develop from a point in a continuum. For instance, we will see that loss of ellipticity corresponds to strain localization into planar bands. Five local criteria will be analyzed, namely:
1) positive definiteness (PD) and
2) non-singularity (NS)
of the constitutive operator;
3) strong ellipticity (SE);
4) ellipticity (E);
5) flutter (F).
In the following we will assume for simplicity that *the constitutive equations evolve continuously with time*. Therefore, in a continuous deformation path initiating when (PD) [or (SE)] holds, failure of (PD) [or (SE)] will be shown to be simultaneous to failure of (NS) [or (E)] in the case of an *associative flow rule*. But this will be not the case of

---

[13] This statement is not equivalent to saying that uniqueness implies stability, because the Raniecki condition is a *sufficient* condition for uniqueness.

non-associative flow rules, where (PD) and (NS) as well as (SE) and (E) are different even in the case of a continuous loading path initiating when (PD) or (SE) hold. In particular, it will be shown that

$$
(PD) \implies \begin{array}{l} \nearrow (SE) \implies (E) \\ \searrow (NS) \end{array}
$$

from which it should be noted that, except for an associative flow law, (NS) has no relation with (SE) and (E).

6.1 A local sufficient condition for uniqueness: positive definiteness of the constitutive operator

It is easy to observe that, when R is positive definite at every point of the body, (40) and (50) are both satisfied. Positive definiteness of R implies positive definiteness of G, which is assumed in this section. Following Raniecki and Bruhns (1981), we will show that

$$
\text{positive definiteness of } R^{opt} \iff \text{positive definiteness of } C \text{ (PD condition)},
$$

where R^{opt} is the Raniecki solid corresponding to an optimal value of ψ to be defined later. In other words, positive definiteness of the best chosen Raniecki solid corresponds to positive definiteness of the 'in loading comparison solid'. Obviously, positive definiteness of $\mathcal{C}$ and C are equivalent (if $\mathbf{X} \cdot \mathcal{C}(\mathbf{X}) = \mathbf{X} \cdot G[\mathbf{X}]$, it suffices to consider $-\mathbf{X}$ to obtain $\mathbf{X} \cdot \mathcal{C}(\mathbf{X}) = \mathbf{X} \cdot C[\mathbf{X}])$[14].

In order to show the equivalence between positive definiteness of R^{opt} and C, under the assumption that G is positive definite, let us consider, for every tensor $\mathbf{X} \in \mathrm{Lin}$, the following inequality:

$$
\mathbf{X} \cdot C[\mathbf{X}] \geq \mathbf{X} \cdot R[\mathbf{X}] = \mathbf{X} \cdot \tilde{G}[\mathbf{X}] - \frac{(\mathbf{X} \cdot \mathbf{R})^2}{4\psi g}, \tag{51}
$$

where $\mathbf{R} = \mathbf{M} + \psi \mathbf{N}$ and

$$
\tilde{G} = \frac{G + G^T}{2}
$$

is the symmetric (with respect to the major symmetry) part of G; only this part plays a role (in fact $\mathbf{X} \cdot G[\mathbf{X}] = \mathbf{X} \cdot \tilde{G}[\mathbf{X}]$). The Cauchy-Schwarz inequality in the metric $\tilde{G}$ (Appendix B) can be used to yield

$$
(\mathbf{X} \cdot \mathbf{R})^2 = (\mathbf{X} \cdot \tilde{G}[\tilde{G}^{-1}[\mathbf{R}]])^2 \leq (\mathbf{X} \cdot \tilde{G}[\mathbf{X}])(\mathbf{R} \cdot \tilde{G}^{-1}[\mathbf{R}]), \tag{52}
$$

which, employed in (51), gives

$$
\mathbf{X} \cdot R[\mathbf{X}] \geq \frac{\mathbf{X} \cdot \tilde{G}[\mathbf{X}]}{4\psi g} \left(4\psi g - \mathbf{R} \cdot \tilde{G}^{-1}[\mathbf{R}] \right). \tag{53}
$$

---

[14] Note that (only for a non-associative flow rule) for certain tensors $\mathbf{X}$ the elastoplastic response is stiffer than the elastic, namely, $\mathbf{X} \cdot G[\mathbf{X}] < \mathbf{X} \cdot C[\mathbf{X}]$ (Mróz, 1963, 1966; Runesson and Mróz, 1989).

From (53), we note that $\mathcal{C}$, C and R are positive definite when:

$$g > \frac{(\mathbf{M} + \psi\mathbf{N}) \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{M} + \psi\mathbf{N}]}{4\psi}. \tag{54}$$

However, when (54) is violated, R is not positive definite. In other words, (54) is a necessary and sufficient condition for positive definiteness of R. To show this it suffices to note that

$$g \leq \frac{\mathbf{R} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{R}]}{4\psi}, \quad \mathbf{X} \propto \tilde{\mathbf{G}}^{-1}[\mathbf{R}] \implies \mathbf{X} \cdot \mathbf{R}[\mathbf{X}] \leq 0.$$

We can consider now the dependence on ψ and state that

$$g > \min_{\psi > 0} \left\{ \frac{(\mathbf{M} + \psi\mathbf{N}) \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{M} + \psi\mathbf{N}]}{4\psi} \right\} \implies \mathcal{C} \text{ and C positive definite.} \tag{55}$$

The minimum problem is solved by

$$\psi = \sqrt{\frac{\mathbf{M} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{M}]}{\mathbf{N} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{N}]}},$$

which defines R^{opt} and yields the following proposition:

$$\boxed{g > g_{cr}^{PD} = \frac{1}{2} \left(\sqrt{(\mathbf{M} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{M}])(\mathbf{N} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{N}])} + \mathbf{M} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{N}] \right) \geq 0 \iff (\mathrm{PD}).} \tag{56}$$

Note that $g_{cr}^{PD} = 0 \iff \mathbf{M} \propto -\mathbf{N}$, a condition that should never be satisfied for realistic constitutive models.

In order to complete the proof of (56), it remains to show that when $g \leq g_{cr}^{PD}$ tensor C is not positive definite. To this purpose, it suffices to note that

$$g \leq g_{cr}^{PD}, \quad \mathbf{X} \propto \sqrt{\mathbf{M} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{M}]}\, \tilde{\mathbf{G}}^{-1}[\mathbf{N}] + \sqrt{\mathbf{N} \cdot \tilde{\mathbf{G}}^{-1}[\mathbf{N}]}\, \tilde{\mathbf{G}}^{-1}[\mathbf{M}] \implies \mathbf{X} \cdot \mathbf{C}[\mathbf{X}] \leq 0.$$

The condition (PD) has been expressed in terms of a critical value of the hardening modulus. In general, any condition expressed in this way is implicitly referred to a sufficiently regular deformation path in which the plastic modulus is a continuously varying function of the loading parameter.

A problem with the (PD) condition is that:

the hypothesis that G be positive definite is stronger than it may appear.

Let us analyze this point in detail. Let us express $\dot{\mathbf{S}} \cdot \dot{\mathbf{F}}$ going back to (22) and taking into account that $\dot{\mathbf{F}} = \mathbf{LF}$ as

$$\dot{\mathbf{S}} \cdot \dot{\mathbf{F}} = \mathbf{F}^T \mathbf{LF} \cdot \dot{\mathbf{T}}^{(2)} + \mathbf{LF} \cdot \mathbf{LKF}^{-T} = \mathbf{F}^T \mathbf{DF} \cdot \dot{\mathbf{T}}^{(2)} + \mathbf{LK} \cdot \mathbf{L}, \tag{57}$$

where $\mathbf{K} = J\mathbf{T} \in$ Sym is the Kirchhoff stress. Eqn. (57), using (17) but restricted to the loading branch of the constitutive operator, may be written as:

$$\mathsf{C}[\dot{\mathbf{F}}] \cdot \dot{\mathbf{F}} = \mathbf{D} \cdot \mathsf{H}[\mathbf{D}] + \underbrace{\frac{1}{2}\mathbf{D} \cdot \mathsf{W}[\mathbf{D}] - \mathbf{D} \cdot \mathsf{K}[\mathbf{W}] + \frac{1}{2}\mathbf{W} \cdot \mathsf{W}[\mathbf{W}]}_{terms\ depending\ on\ spin} - \frac{\hat{\mathbf{Q}} \cdot \mathsf{H}[\mathbf{D}]}{g}\mathbf{D} \cdot \mathsf{H}[\hat{\mathbf{P}}], \quad (58)$$

where $\mathbf{W} = \mathbf{L} - \mathbf{D}$ is the spin, $\hat{\mathbf{Q}} = \mathbf{F}^{-T}\mathbf{Q}\mathbf{F}^{-1}$, $\hat{\mathbf{P}} = \mathbf{F}^{-T}\mathbf{P}\mathbf{F}^{-1}$ and

$$\mathsf{H} = (\mathbf{F}\boxtimes\mathbf{F})\mathsf{E}(\mathbf{F}\boxtimes\mathbf{F})^T, \quad \mathsf{W} = \mathbf{K}\boxtimes\mathbf{I} + \mathbf{I}\boxtimes\mathbf{K}, \quad \mathsf{K} = \mathbf{K}\boxtimes\mathbf{I} - \mathbf{I}\boxtimes\mathbf{K}, \quad (59)$$

where

- tensor H has always the minor symmetries (because E has) and has the major symmetry in the case of a Green elastic material.
- tensor W always has the major symmetry, but not the minor symmetries, and transforms symmetric tensors into symmetric tensors and skew tensors into skew tensors. Moreover, $\mathbf{W} \cdot \mathsf{W}[\mathbf{W}] = 2\mathbf{W} \cdot \mathbf{K}\mathbf{W}$.
- tensor K has always the major symmetry, but not the minor symmetries, and transforms symmetric tensors into skew tensors and skew tensors into symmetric tensors. Moreover, $\mathbf{W} \cdot \mathsf{K}[\mathbf{W}] = \mathbf{D} \cdot \mathsf{K}[\mathbf{D}] = 0$ and $-\mathbf{D} \cdot \mathsf{K}[\mathbf{W}] = 2\mathbf{D}\mathbf{K} \cdot \mathbf{W}$. Finally, when $\mathbf{A}$ and $\mathbf{K}$ are coaxial, i.e. $\mathbf{A}\mathbf{K} = \mathbf{K}\mathbf{A}$, $\mathsf{K}[\mathbf{A}] = \mathbf{0}$.

The spin $\mathbf{W}$ and the rate of deformation $\mathbf{D}$ are *independent* tensors. Therefore, a necessary condition for $\dot{\mathbf{S}} \cdot \dot{\mathbf{F}} > 0$, i.e. for C to be positive definite, is that $\mathbf{W} \cdot \mathsf{W}[\mathbf{W}] > 0$ for every $\mathbf{W}$, in other words:

$$\boxed{(\text{PD}) \implies (\text{tr}\mathbf{T})\mathbf{I} - \mathbf{T}\ \text{pos. def.} \iff T_1 + T_2 > 0,\ T_1 + T_3 > 0,\ T_2 + T_3 > 0.} \quad (60)$$

Positive definiteness of $(\text{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$ is equivalent to saying that the restriction of $\mathbf{W}$ to the space of skew tensors is positive definite. This condition was obtained by Hill (1967a) and it *holds true for every constitutive assumption*. In other words, (60) reflects a purely 'geometrical effect'. Now, let us assume positive definiteness of $(\text{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$. Under this condition, the function (at fixed $\mathbf{D}$)

$$z(\mathbf{W}) = -\mathbf{D} \cdot \mathsf{K}[\mathbf{W}] + \frac{1}{2}\mathbf{W} \cdot \mathsf{W}[\mathbf{W}], \quad (61)$$

is strictly convex and therefore it admits a minimum point, which corresponds to

$$\frac{\partial z(\mathbf{W})}{\partial \mathbf{W}} = \mathsf{W}[\mathbf{W}] - \mathsf{K}[\mathbf{D}] = \mathbf{0}. \quad (62)$$

Due to positive definiteness of $(\text{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$, the restriction of W to the space of all skew tensors is invertible[15] and therefore condition (62) defines the minimum point of $z(\mathbf{W})$ in terms of $\hat{\mathbf{W}}$

$$\hat{\mathbf{W}} = \mathsf{W}^{-1}\mathsf{K}[\mathbf{D}], \quad (63)$$

---

[15] This statement means that

$$\mathsf{W}\mathsf{W}^{-1} = \mathbf{I}\boxtimes\mathbf{I} - \mathbf{I}\,\underline{\otimes}\,\mathbf{I},$$

so that $(\mathbf{I}\boxtimes\mathbf{I} - \mathbf{I}\,\underline{\otimes}\,\mathbf{I})[\mathbf{A}] = (\mathbf{A} - \mathbf{A}^T)/2$ associates to every tensor its skew symmetric part (Del Piero, 1979).

as

$$z_{min} = -\frac{1}{2}\mathbf{D} \cdot \left(\mathsf{KW}^{-1}\mathsf{K}\right)[\mathbf{D}]. \tag{64}$$

The components of tensor $\mathsf{KW}^{-1}\mathsf{K}$ can be easily calculated in the principal reference system of $\mathbf{K}$. This calculation reveals that $\mathsf{KW}^{-1}\mathsf{K}$ has both the major and minor symmetries and has only the following non-null components

$$(\mathsf{KW}^{-1}\mathsf{K})_{ijij} = (\mathsf{KW}^{-1}\mathsf{K})_{jiij} = \frac{(K_i - K_j)^2}{2(K_i + K_j)}, \quad i \neq j; \, i,j \in [1,3]. \tag{65}$$

As a conclusion, the following inequality is obtained:

$$\mathsf{C}[\dot{\mathbf{F}}] \cdot \dot{\mathbf{F}} \geq \mathbf{D} \cdot \mathsf{L}[\mathbf{D}] - \frac{1}{g}(\mathbf{D} \cdot \mathsf{H}[\hat{\mathbf{P}}])(\mathbf{D} \cdot \mathsf{H}^T[\hat{\mathbf{Q}}]), \tag{66}$$

(with the equality holding true when $\mathbf{W} = \hat{\mathbf{W}}$) where

$$\mathsf{L} = \mathsf{H} + \frac{1}{2}\mathsf{W} - \frac{1}{2}\mathsf{KW}^{-1}\mathsf{K}, \tag{67}$$

therefore, under condition that $(\mathrm{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$ be positive definite,

> *positive definiteness of the restriction of* L *to the space of all symmetric tensors*
> *is equivalent to positive definiteness of* G *on Lin.*

Therefore, assuming L positive definite, it is possible to show that

$$\boxed{\begin{array}{l} T_1 + T_2 > 0, \; T_1 + T_3 > 0, \; T_2 + T_3 > 0, \; \& \; g > g_{cr}^{PD} \geq 0, \\[1mm] g_{cr}^{PD} = \frac{1}{2}\left(\sqrt{(\hat{\mathbf{P}} \cdot \mathsf{H}^T\tilde{\mathsf{L}}^{-1}\mathsf{H}[\hat{\mathbf{P}}])(\hat{\mathbf{Q}} \cdot \mathsf{H}\tilde{\mathsf{L}}^{-1}\mathsf{H}^T[\hat{\mathbf{Q}}])} + \hat{\mathbf{Q}} \cdot \mathsf{H}\tilde{\mathsf{L}}^{-1}\mathsf{H}[\hat{\mathbf{P}}]\right) \iff \text{(PD)}, \end{array}} \tag{68}$$

where $\tilde{\mathsf{L}} = (\mathsf{L} + \mathsf{L}^T)/2$ and $\tilde{\mathsf{L}}^{-1}\tilde{\mathsf{L}} = \mathbf{I} \overline{\otimes} \mathbf{I}$. Condition (68) was obtained by Raniecki and Bruhns (1981) in the case of L having the major symmetry. Condition (68) is equivalent to (56), but more explicit in the sense that it singles out the effect of $(\mathrm{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$.

Uniaxial tension. In the case of uniaxial tension along axis 1, $T_1 = \sigma > 0$ and $T_2 = T_3 = 0$, thus $(\mathrm{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$ is positive semi-definite and G is not invertible. Let us develop this point in detail. In this case, by direct calculation we obtain

$$z(\mathbf{W}) = \sigma\left(W_{12}^2 + W_{13}^2 - 2D_{12}W_{12} - 2D_{13}W_{13}\right), \tag{69}$$

which is zero when $\mathbf{D} = \mathbf{0}$ and the only non-null component of $\mathbf{W}$ is W_{23}, representing a spin about axis 1. The minimum of $z(\mathbf{W})$ is

$$z_{min} = -\sigma\left(D_{12}^2 + D_{13}^2\right). \tag{70}$$

It follows that

$$\dot{\mathbf{F}} \cdot \mathbf{C}[\dot{\mathbf{F}}] \geq \mathbf{D} \cdot \mathbf{H}[\mathbf{D}] - \frac{1}{g}(\mathbf{D} \cdot \mathbf{H}^T[\hat{\mathbf{Q}}])(\mathbf{D} \cdot \mathbf{H}[\hat{\mathbf{P}}]) + \sigma D_{11}^2, \qquad (71)$$

and therefore, when the restriction of

$$\mathbf{H} + \sigma \, \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{g}\mathbf{H}[\hat{\mathbf{P}}] \otimes \mathbf{H}^T[\hat{\mathbf{Q}}], \qquad (72)$$

to Sym is positive definite, condition (PD) is verified, except for a velocity gradient having W_{23} as the unique non-null component. However, the integrals (40) and (50) can vanish only if $W_{23} \neq 0$ is the only non-null component of the velocity gradient everywhere in the body and this corresponds to a loss of uniqueness consisting in arbitrary rigid rotations about the axis of tension (which is also an axis of neutral stability). This circumstance becomes particularly clear if we consider a bar pulled in tension and therefore subject to homogeneous stress and all-round dead loading (Fig. 1 a). Analogously, $(\mathrm{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$ is negative semi-definite (and thus G is indefinite) in the important case of a bar subject to dead loading of uniaxial compression, even for a vanishing small value of axial force. This correctly corresponds to a well-known instability due to rigid-body rotation (Fig. 1b).

(a)

(b)

Fig. 1. a-neutral rotational equilibrium of a rod under tensile dead load; b-rotational instability of a rod under compressive dead load.

Using (72), condition (PD) —except for rigid-body rotations about the axis of tension— can be written in explicit as:

$$\boxed{\begin{aligned} &T_1 > 0, \; T_2 = T_3 = 0, \; \& \; g > g_{cr}^{PD} \geq 0, \; \mathbf{L} = \mathbf{H} + \sigma \, \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1, \\ &g_{cr}^{PD} = \tfrac{1}{2}\left(\sqrt{(\hat{\mathbf{P}} \cdot \mathbf{H}^T\tilde{\mathbf{L}}^{-1}\mathbf{H}[\hat{\mathbf{P}}])(\hat{\mathbf{Q}} \cdot \mathbf{H}\tilde{\mathbf{L}}^{-1}\mathbf{H}^T[\hat{\mathbf{Q}}])} + \hat{\mathbf{Q}} \cdot \mathbf{H}\tilde{\mathbf{L}}^{-1}\mathbf{H}[\hat{\mathbf{P}}]\right) \iff (\mathrm{PD}) \end{aligned}} \qquad (73)$$

where $\tilde{\mathbf{L}}$ is the symmetric part of $\mathbf{L}$, assumed positive definite.

The infinitesimal theory. In the infinitesimal theory, where we refer to (34), it may be interesting to write g_{cr}^{PD} in terms of a critical value of the hardening modulus h_{cr}^{PD}. For a symmetric E, this becomes

$$h_{cr}^{PD} = \frac{1}{2}\left(\sqrt{(\mathbf{P}\cdot\mathsf{E}[\mathbf{P}])(\mathbf{Q}\cdot\mathsf{E}[\mathbf{Q}])} - \mathbf{P}\cdot\mathsf{E}[\mathbf{Q}]\right) \geq 0, \tag{74}$$

which is never negative and becomes null for associative elastoplasticity. Therefore, in the infinitesimal theory (and assuming E symmetric), (PD) is always lost before softening. Critical modulus (74) was obtained by Mróz (1963), Maier and Hueckel (1979) and Raniecki (1979). A derivation of the critical hardening modulus for plane strain and plane stress situations was given by Bigoni and Hueckel (1991a).

In closing this section, we note that in under the hypotheses of homogeneity and all-round controlled nominal surface tractions, failure of (PD) implies failure of the Hill sufficient conditions for stability (50) and uniqueness (40). This situation is critical in the case of the associative flow rule, where instability and bifurcation may occur at loss of (PD) under broad hypotheses (Hill, 1967; Miles, 1973). The situation is however still not clear both for stability and bifurcation in non-associative elastoplasticity. In that context, examples are not known in which any real bifurcation has been found in coincidence with failure of (PD).

6.2 Singularity of the constitutive operator

Loss of (PD) at a point of a body during a loading program is not in general a sufficient condition for bifurcation. There are however certain situations where loss of (PD) may become close to critical. These have been touched on in the closure of the previous section. Let us consider therefore a special class of problems where:
1) controlled nominal tractions are prescribed on the entire boundary and
2) material properties and deformation (and therefore stress) are homogeneous,
during a given loading path. These situations have been analyzed by Hill (1967a), Miles (1973), Raniecki and Bruhns (1981) and Ogden (1985). Under the above hypotheses, conditions (40) and (50) are equivalent to (PD). In other words, exclusion of bifurcation and stability in Hill's sense fail to hold when (PD) is lost, i.e. when at least one $\mathbf{X}^* \neq \mathbf{0}$ exists such that

$$\mathbf{X}^* \cdot \mathsf{C}[\mathbf{X}^*] = 0. \tag{75}$$

Except for the associative case, this condition does not mean that C is singular, in other words $\mathsf{C}[\mathbf{X}^*] \neq \mathbf{0}$ should be in general expected. Therefore, loss of (PD) is not directly connected to a bifurcation, even in the above special hypotheses. On the other hand, if we define the non-singularity condition

$$\boxed{\mathsf{C}[\mathbf{X}] \neq \mathbf{0}, \text{ for every } (\text{non} - \text{zero}) \; \mathbf{X} \in \text{Lin} \quad (\text{NS condition}),} \tag{76}$$

we can understand that, in a sense, failure of this condition is critical for bifurcation of the homogeneous problem with controlled nominal surface tractions on the entire boundary.

We note, in passing, that obviously (PD) implies (NS) and that when (PD) fails the first time in a continuous loading path (and for continuous dependence of constitutive equations on time) (NS) also fails for associative, hyperelastic-plastic solids.

With respect to the constitutive operator (28) where G is assumed invertible[16], we note that, assuming $\mathbf{N} \cdot \mathbf{G}^{-1}[\mathbf{M}] > 0$

$$\dot{\mathbf{S}} = \mathbf{0} \iff \dot{\mathbf{F}} \propto \mathbf{G}^{-1}[\mathbf{M}], \quad g = \mathbf{N} \cdot \mathbf{G}^{-1}[\mathbf{M}]. \tag{77}$$

Therefore

$$\boxed{g \neq g_{cr}^{NS} = \mathbf{N} \cdot \mathbf{G}^{-1}[\mathbf{M}], \iff \mathcal{C} \text{ is not singular. (NS condition)}} \tag{78}$$

Note that $g_{cr}^{NS} < 0$ when $\mathbf{N} \cdot \mathbf{G}^{-1}[\mathbf{M}] < 0$. In this case, loss of (NS) does not occur (in the present constitutive framework).

Loss of (NS) is critical in the sense explained by Raniecki and Bruhns (1981), namely, assuming $\mathbf{C}[\mathbf{X}^*] = \mathbf{0}$, if $\dot{\mathbf{F}}$ is a solution of the problem in velocities (corresponding to plastic loading everywhere in the body) it satisfies:

$$\mathcal{C}(\dot{\mathbf{F}}) = \mathsf{C}[\dot{\mathbf{F}}] \quad \text{and} \quad \mathsf{C}[\dot{\mathbf{F}}]\mathbf{n}_0 = \dot{\boldsymbol{\sigma}} \quad \text{on} \quad \partial \Omega^0,$$

where $\mathbf{n}_0$ is the unit outward normal to $\partial \Omega^0$. It follows that if $\mathcal{C}(\dot{\mathbf{F}} + \gamma \mathbf{X}^*) = \mathsf{C}[\dot{\mathbf{F}}] + \gamma \mathsf{C}[\mathbf{X}^*]$ for at least some $\gamma \neq 0$,

$$\mathcal{C}(\dot{\mathbf{F}} + \gamma \mathbf{X}^*)\mathbf{n}_0 = \mathsf{C}[\dot{\mathbf{F}}]\mathbf{n}_0 = \dot{\boldsymbol{\sigma}} \quad \text{on} \quad \partial \Omega^0,$$

and therefore either a *bifurcation* or a *load maximum* has been reached in the loading program (Hill, 1967a).

Condition (78) may be further elaborated, assuming $T_1 + T_2 > 0$, $T_1 + T_3 > 0$, $T_2 + T_3 > 0$, as follows. Let us observe that

$$\dot{\mathbf{S}}\mathbf{F}^T = \mathsf{H}[\mathbf{D}] + \mathbf{LK} - \frac{1}{g}(\hat{\mathbf{Q}} \cdot \mathsf{H}[\mathbf{D}])\mathsf{H}[\hat{\mathbf{P}}], \tag{79}$$

where we may note that

$$\mathbf{LK} = \frac{1}{2}(\mathbf{W} - \mathbf{K})[\mathbf{D}] + \frac{1}{2}(\mathbf{W} - \mathbf{K})[\mathbf{W}]. \tag{80}$$

For $\mathbf{W}$ in the form (63), i.e. $\mathbf{W} = \mathsf{W}^{-1}\mathsf{K}[\mathbf{D}]$, and L defined by (67), eqn. (79), using (80), becomes

$$\dot{\mathbf{S}}\mathbf{F}^T = \mathsf{L}[\mathbf{D}] - \frac{1}{g}(\hat{\mathbf{Q}} \cdot \mathsf{H}[\mathbf{D}])\mathsf{H}[\hat{\mathbf{P}}]. \tag{81}$$

Assuming L invertible, from (81) we conclude that, under hypothesis (63), $\dot{\mathbf{S}} = \mathbf{0} \iff g = \hat{\mathbf{Q}} \cdot (\mathsf{H}\mathsf{L}^{-1}\mathsf{H})[\hat{\mathbf{P}}]$ and $\mathbf{D} \propto \mathsf{L}^{-1}\mathsf{H}[\hat{\mathbf{P}}]$. Vice-versa, assuming $\dot{\mathbf{S}} = \mathbf{0}$ in (79), the minor symmetries of H imply $\mathbf{W} = \mathsf{W}^{-1}\mathsf{K}[\mathbf{D}]$. We obtain therefore the following proposition:

$$\boxed{\begin{array}{l} T_1 + T_2 > 0, \ T_1 + T_3 > 0, \ T_2 + T_3 > 0 : \\[4pt] g \neq g_{cr}^{NS} = \hat{\mathbf{Q}} \cdot (\mathsf{H}\mathsf{L}^{-1}\mathsf{H})[\hat{\mathbf{P}}], \iff \mathcal{C} \text{ is not singular. (NS condition)} \end{array}} \tag{82}$$

---

[16] Singularity of G implies singularity of $\mathcal{C}$.

Obviously, conditions (78) and (82) are equivalent, under the condition that $(\mathrm{tr}\mathbf{T})\mathbf{I} - \mathbf{T}$ is positive definite.

Uniaxial tension. For uniaxial tension along axis 1, $T_1 = \sigma > 0$ and $T_2 = T_3 = 0$. Direct calculations on (79) yield that the critical condition is again given by (82), but with

$$\mathbf{L} = \mathbf{H} + \sigma\,\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1.$$

The infinitesimal theory. In the case of the infinitesimal theory (34) with $\mathbf{E}$ symmetric and positive definite, the condition (NS) becomes $h \neq 0$, and:

$$\text{for } h = 0 \quad \text{and} \quad \mathbf{X}^* \propto \mathbf{P}, \quad \mathbf{C}[\mathbf{X}^*] = \mathbf{0}.$$

As a conclusion and with the exception of associative plasticity, (PD) and (NS) clearly do not coincide in the infinitesimal theory, where (PD) is lost before softening and (NS) is always lost in the perfectly plastic case ($h = 0$).

6.3 Strong ellipticity

In the previous section we have analyzed the special condition of a homogeneous body subject to all-round controlled nominal surface tractions. In that case, the (PD) and (NS) conditions play a special role. Now we analyze a dual case in which a homogeneous body is deformed in an homogeneous way under prescribed displacements on the entire boundary. As a result, we will show that strong ellipticity (SE) and ellipticity (E) play a role similar to those shown for (PD) and (NS) in the previous section.

Let us begin by considering the 'in loading comparison solid', i.e. the fictitious solid defined by the constitutive tensor $\mathbf{C}$. For this material, we will show the validity of the following *uniqueness theorem for the velocity problem, due to van Hove (1947)*:

For a homogeneous and homogeneously deformed body, characterized by an incrementally linear constitutive operator (here $\mathbf{C}$) and subject to prescribed velocity over the entire boundary, the strong ellipticity condition

$$\boxed{\mathbf{g} \cdot \mathbf{C}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n} > 0, \qquad \text{(SE condition)}} \tag{83}$$

for every non-zero unit vector[17] $\mathbf{n}$ and vector $\mathbf{g}$, implies that the velocity problem has at most one solution.

The condition of strong ellipticity may be expressed, in a different notation, as the positive definiteness of the acoustic tensor[18] $\mathbf{A}(\mathbf{n})$ defined, for every unit vector $\mathbf{n}$ and $\mathbf{g} \in \mathcal{V}$, as

$$\boxed{\mathbf{A}(\mathbf{n})\mathbf{g} = \mathbf{C}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}.} \tag{84}$$

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[17] We prefer to introduce the acoustic tensor directly with respect to unit vectors $\mathbf{n}$.

[18] In our definition of acoustic tensor we refer to a unit mass density.

With reference to the 'in loading' comparison solid (47), the acoustic tensor becomes:

$$\boxed{\mathbf{A}(\mathbf{n}) = \mathbf{A}_E(\mathbf{n}) - \frac{1}{g}\mathbf{M}\mathbf{n} \otimes \mathbf{N}\mathbf{n},} \tag{85}$$

where $\mathbf{A}_E(\mathbf{n})$ is the elastic acoustic tensor

$$\mathbf{A}_E(\mathbf{n})\mathbf{g} = \mathbb{G}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n},$$

assumed *positive definite* in the following.

For the proof of the van Hove theorem, we follow Hayes (1966). Let us consider the functional in (40) with $\mathbb{C}^c$ replaced by $\mathbb{C}$. When this functional is positive, for every admissible velocity field $\dot{\mathbf{x}}$ vanishing on $\partial\Omega^0$, bifurcation is excluded (in the incrementally linear solid defined by $\mathbb{C}$). We can extend the definition of $\dot{\mathbf{x}}$ from $\bar{\Omega}^0$ to all Euclidean space $\mathcal{E}$ by simply defining $\dot{\mathbf{x}} = \mathbf{0}$ on $\mathcal{E} \setminus \bar{\Omega}^0$. The resulting field is admissible and its gradient is discontinuous only on $\partial\Omega^0$. Therefore, both $\dot{\mathbf{x}}$ and $\nabla\dot{\mathbf{x}}$ possess the three-dimensional Fourier transforms

$$\dot{\mathbf{x}}^*(\mathbf{x}_0) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{\mathcal{E}} e^{i(\mathbf{x}_0 \cdot \mathbf{y})}\dot{\mathbf{x}}(\mathbf{y})dv_{\mathbf{y}}, \tag{86}$$

$$\mathbf{X}(\mathbf{x}_0) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{\mathcal{E}} e^{i(\mathbf{x}_0 \cdot \mathbf{y})}\nabla\dot{\mathbf{x}}(\mathbf{y})dv_{\mathbf{y}}. \tag{87}$$

Before continuing, it should be noted that (86), (87) and the divergence theorem yield

$$\mathbf{X} = -i\,\dot{\mathbf{x}}^* \otimes \mathbf{x}_0, \tag{88}$$

where $i = \sqrt{-1}$ is the imaginary unit. Since homogeneity is satisfied in Ω^0, the generalized Parseval theorem gives

$$\int_{\mathcal{E}} \nabla\dot{\mathbf{x}} \cdot \mathbb{C}[\nabla\dot{\mathbf{x}}] = C_{ijhk}\int_{\mathcal{E}} \dot{x}_{i,j}\dot{x}_{h,k} = C_{ijhk}\int_{\mathcal{E}} X_{ij}\overline{X}_{hk} = \int_{\mathcal{E}} \mathbf{X} \cdot \mathbb{C}[\overline{\mathbf{X}}], \tag{89}$$

where $\overline{\mathbf{X}}$ is the complex conjugate of $\mathbf{X}$. Now, we note that (88) implies

$$\mathbf{X}_R = \dot{\mathbf{x}}_I^* \otimes \mathbf{x}_0, \quad \mathbf{X}_I = -\dot{\mathbf{x}}_R^* \otimes \mathbf{x}_0, \tag{90}$$

where the indices R and I stand for the real and imaginary parts, respectively. Using (90) in (89), it may be found that

$$\int_{\Omega^0} \nabla\dot{\mathbf{x}} \cdot \mathbb{C}[\nabla\dot{\mathbf{x}}] = \int_{\mathcal{E}}(\dot{\mathbf{x}}_R^* \otimes \mathbf{x}_0) \cdot \mathbb{C}[\dot{\mathbf{x}}_R^* \otimes \mathbf{x}_0] + \int_{\mathcal{E}}(\dot{\mathbf{x}}_I^* \otimes \mathbf{x}_0) \cdot \mathbb{C}[\dot{\mathbf{x}}_I^* \otimes \mathbf{x}_0]. \tag{91}$$

As a conclusion, when (SE) holds, the above integral is greater than zero, and uniqueness follows.

Obviously, the van Hove theorem holds true for the Raniecki family of incrementally linear solids. Defining therefore the acoustic tensor relative to the generic Raniecki solid

$$\mathbf{A}_R(\mathbf{n}, \psi)\mathbf{g} = \mathbb{R}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}, \tag{92}$$

we may conclude that for a homogeneous elastoplastic body subject to prescribed velocities on the entire boundary, strong ellipticity of at least one of the Raniecki solids (SE$_R$) is sufficient for uniqueness. As a conclusion, we can state that

$$\boxed{(\text{SE}_R) \implies \text{uniqueness for an } \textit{elastoplastic solid} \text{ for van Hove's b.v.p.}}$$

$$\boxed{(\text{SE}) \implies \text{uniqueness for the 'in loading comparison solid' for van Hove's b.v.p.}}$$

where by 'van Hove b.v.p.' we mean boundary value problems satisfying van Hove's conditions. An immediate consequence of the Raniecki comparison theorem is the following implication

$$(\text{PD}) \implies (\text{SE}_R) \implies (\text{SE}). \tag{93}$$

Now we will determine the critical plastic moduli for loss of (SE) and loss of (SE$_R$).

In order to write (SE) of C in terms of a critical value of the plastic modulus, let us consider, for every vector x and unit vector n, the following inequality:

$$\mathbf{x} \cdot \mathbf{A}(\mathbf{n})\mathbf{x} \geq \mathbf{x} \cdot \mathbf{A}_R(\mathbf{n}, \psi)\mathbf{x} = \mathbf{x} \cdot \tilde{\mathbf{A}}_E(\mathbf{n})\mathbf{x} - \frac{(\mathbf{x} \cdot \mathbf{Rn})^2}{4\psi g}, \tag{94}$$

where $\tilde{\mathbf{A}}_E(\mathbf{n}) = [\mathbf{A}_E(\mathbf{n}) + \mathbf{A}_E^T(\mathbf{n})]/2$, is the symmetric part of the elastic acoustic tensor.

We proceed analogously to the proof of Raniecki and Bruhns for (PD), using the Cauchy-Schwarz inequality in the metric $\tilde{\mathbf{A}}_E(\mathbf{n})$ (Appendix B) (the dependence of $\tilde{\mathbf{A}}_E$ on n is omitted for simplicity in the following)

$$(\mathbf{x} \cdot \mathbf{Rn})^2 = (\mathbf{x} \cdot \tilde{\mathbf{A}}_E \tilde{\mathbf{A}}_E^{-1} \mathbf{Rn})^2 \leq (\mathbf{x} \cdot \tilde{\mathbf{A}}_E \mathbf{x})(\mathbf{Rn} \cdot \tilde{\mathbf{A}}_E^{-1} \mathbf{Rn}). \tag{95}$$

After calculations that parallel those reported in Sect. 6.1, we find that

$$g > \min_{\psi > 0} \left\{ \frac{(\mathbf{Mn} + \psi \mathbf{Nn}) \cdot \tilde{\mathbf{A}}_E^{-1}(\mathbf{Mn} + \psi \mathbf{Nn})}{4\psi} \right\} \implies \mathbf{A}(\mathbf{n}) \text{ pos. def.} \tag{96}$$

The minimum problem is solved by

$$\psi(\mathbf{n}) = \sqrt{\frac{\mathbf{Mn} \cdot \tilde{\mathbf{A}}_E^{-1} \mathbf{Mn}}{\mathbf{Nn} \cdot \tilde{\mathbf{A}}_E^{-1} \mathbf{Nn}}}, \tag{97}$$

which yields the following proposition:

$$\boxed{g > g_{cr}^{SE}(\mathbf{n}) \iff \mathbf{A}(\mathbf{n}) \text{ pos. def.} \quad \text{(at fixed n)},} \tag{98}$$

where

$$\boxed{g_{cr}^{SE}(\mathbf{n}) = \frac{1}{2}\left(\sqrt{(\mathbf{Mn} \cdot \tilde{\mathbf{A}}_E^{-1} \mathbf{Mn})(\mathbf{Nn} \cdot \tilde{\mathbf{A}}_E^{-1} \mathbf{Nn})} + \mathbf{Mn} \cdot \tilde{\mathbf{A}}_E^{-1} \mathbf{Nn} \right) \geq 0.} \tag{99}$$

Note that $g_{cr}^{SE} = 0 \iff \mathbf{N} \propto -\mathbf{M}$, a condition which should never be satisfied for realistic constitutive models.

In order to complete the proof of (98), it may be checked that at $g = g_{cr}^{SE}(\mathbf{n})$, tensor $\mathbf{A}(\mathbf{n})$ loses positive definiteness for vectors $\mathbf{x}^*$ defined as

$$\mathbf{x}^* \propto \left(\sqrt{\mathbf{Mn} \cdot \tilde{\mathbf{A}}_E^{-1}\mathbf{Mn}}\right)\tilde{\mathbf{A}}_E^{-1}\mathbf{Nn} + \left(\sqrt{\mathbf{Nn} \cdot \tilde{\mathbf{A}}_E^{-1}\mathbf{Nn}}\right)\tilde{\mathbf{A}}_E^{-1}\mathbf{Mn},$$

and that for $g \leq g_{cr}^{SE}(\mathbf{n})$, $\mathbf{x}^* \cdot \mathbf{A}(\mathbf{n})\mathbf{x}^* \leq 0$.

It may be important to note from (99) that for every $\mathbf{n}$, failure of (SE) necessarily occurs for $g_{cr}^{SE}(\mathbf{n})$ positive.

All the above holds at $\mathbf{n}$ fixed, and proves that for a given $\mathbf{n}$, loss of (PD) in the 'in loading' comparison solid C and in the optimal Raniecki solid defined by (97) are equivalent. However, loss of (SE) in C will actually occur when g equals the maximum of $g_{cr}^{SE}(\mathbf{n})$ as a function of $\mathbf{n}$

$$g > g_{cr}^{SE} = \max_{\mathbf{n}, |\mathbf{n}|=1} g_{cr}^{SE}(\mathbf{n}) \iff \text{(SE condition)}. \tag{100}$$

Therefore, differently from g_{cr}^{PD}, the critical value of the plastic modulus for loss of strong ellipticity, g_{cr}^{SE}, is not given in explicit terms, but as the solution of a constrained maximization problem.

Let us consider now strong ellipticity for Raniecki solids. From (94) and (95) we obtain that

$$\mathbf{A}_R(\mathbf{n}, \psi) \text{ pos. def.} \iff g > g_{cr}^{SE_R}(\mathbf{n}, \psi) = \frac{(\mathbf{Mn} + \psi\mathbf{Nn}) \cdot \tilde{\mathbf{A}}_E^{-1}(\mathbf{Mn} + \psi\mathbf{Nn})}{4\psi}, \tag{101}$$

Taking into consideration the optimal ψ, we conclude that (SE$_R$) can be written as

$$g > g_{cr}^{SE_R} = \inf_{\psi>0} \max_{\mathbf{n}, |\mathbf{n}|=1} g_{cr}^{SE_R}(\mathbf{n}, \psi), \iff \text{(SE$_R$ condition)} \tag{102}$$

The above critical plastic moduli were obtained by Bigoni and Zaccaria (1992a;b). It may be worth noting that (93) is equivalent to

$$g_{cr}^{PD} \geq g_{cr}^{SE_R}(\mathbf{n}, \psi) \geq g_{cr}^{SE}(\mathbf{n}), \tag{103}$$

holding for every ψ and $\mathbf{n}$. Moreover, we have seen that

$$g_{cr}^{SE}(\mathbf{n}) = \min_{\psi>0} g_{cr}^{SE_R}(\mathbf{n}, \psi). \tag{104}$$

Therefore, (SE) and (SE$_R$) are equivalent criteria whenever

$$\inf_{\psi>0} \max_{\mathbf{n}, |\mathbf{n}|=1} g_{cr}^{SE_R}(\mathbf{n}, \psi) = \max_{\mathbf{n}, |\mathbf{n}|=1} \min_{\psi>0} g_{cr}^{SE_R}(\mathbf{n}, \psi), \tag{105}$$

a condition which was proved only under very restrictive assumptions (infinitesimal theory, isotropic elasticity and coaxiality of $\mathbf{P}$ and $\mathbf{Q}$).

Untill this point we have given an interpretation of the (SE_R) and (SE) conditions as sufficient conditions for uniqueness of a special class of problems (van Hove hypotheses)[19]. In particular, the former condition is sufficient for uniqueness of the velocity problem for the elastoplastic solid, and the latter for the fictitious solid corresponding to the loading branch of the constitutive tangent operator. However, as stressed by Bigoni and Zaccaria (1992 b),

> the strong ellipticity condition has a meaning even when a generic situation of inhomogeneous deformation and mixed boundary conditions is considered.

In order to explain this point, let us introduce the semi-strong ellipticity condition (SSE), which is defined as in (83) except that '>' is replaced with '≥'. In other words, (SSE) is the condition of semi-positive definiteness of the acoustic tensor. In nonlinear elasticity, the theorem of Cattaneo (1946) (see also Truesdell and Noll, 1965, Sect. 68bis) proves that (SSE) is a necessary condition for infinitesimal stability. In non-associative elastoplasticity the theorem of Cattaneo was generalized by Ryzhak (1987)[20]. Assuming positive definiteness of $A_E(n)$, the Ryzhak theorem states that (SSE) is a necessary condition for semi-stability in Hill's sense, namely

$$\boxed{\text{semi-stability in Hill sense} \implies \mathsf{C} \text{ is (SSE)},}$$

where with the term 'semi-stability', we intend the sufficient stability condition (50) where '>' is replaced with '≥'. The (SSE) is a local criterion and when it fails during a loading program of a generic (inhomogeneous) boundary value problem (with mixed boundary conditions) Hill's sufficient stability condition does not hold.

The infinitesimal theory. In the special case of infinitesimal theory (34) and Green elasticity, the critical plastic modulus (99) simplifies to

$$g_{cr}^{SE}(\mathbf{n}) = \frac{1}{2}\left(\sqrt{(\mathsf{E}[\mathbf{P}]\mathbf{n} \cdot \mathbf{A}_E^{-1}\mathsf{E}[\mathbf{Pn}])(\mathsf{E}[\mathbf{Q}]\mathbf{n} \cdot \mathbf{A}_E^{-1}\mathsf{E}[\mathbf{Q}]\mathbf{n})} + \mathsf{E}[\mathbf{P}]\mathbf{n} \cdot \mathbf{A}_E^{-1}\mathsf{E}[\mathbf{Q}]\mathbf{n} \right) \geq 0, \tag{106}$$

where once again we note that $g_{cr}^{SE} = 0$ is equivalent to $\mathbf{P} \propto -\mathbf{Q}$, a condition which is excluded for realistic constitutive models. Assuming isotropic elasticity (33), the acoustic tensor can be written in the well-known form

$$\mathbf{A}_E(\mathbf{n}) = (\lambda + \mu)\mathbf{n} \otimes \mathbf{n} + \mu\mathbf{I}, \tag{107}$$

with the inverse

$$\mathbf{A}_E^{-1}(\mathbf{n}) = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\mathbf{n} \otimes \mathbf{n} + \frac{1}{\mu}\mathbf{I}. \tag{108}$$

Under the above hypotheses, and assuming in addition coaxiality of $\mathbf{P}$ and $\mathbf{Q}$, Bigoni and Zaccaria (1992, a,b) have proved (105) so that:

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[19] The van Hove theorem has been generalized in special cases by Ryzhak (1993; 1994)

[20] A simpler, alternative proof can be inferred from (Petryk, 1985a; 1992), as an application of Graves' theorem. The discussion there was confined to symmetry of the constitutive operator, but in the (SSE) condition only the symmetric part of the constitutive operator plays a role.

in the case of the infinitesimal theory, with isotropic elasticity and $\mathbf{P}$ *and* $\mathbf{Q}$ *coaxial, (SE) and (SE$_R$) are equivalent criteria.*

Moreover, they obtained an explicit solution for (100).

6.4 Strain localization

For associative hyperelastic-plastic behaviour, $\mathbf{N} = \mathbf{M}$, in a continuous deformation evolution initiating when (SE) holds, the acoustic tensor becomes singular as soon as (SE) fails. This is no longer true in the non-associative case, where (SE) is usually lost while the acoustic tensor is still non-singular. Therefore, we can introduce a condition analogous to (NS), but on the acoustic tensor (defined here relatively to the 'in loading' comparison solid). This is the condition of ellipticity

$$\boxed{\det \mathbf{A}(\mathbf{n}) \neq 0, \quad \text{for all } \mathbf{n} \in \mathcal{V}, |\mathbf{n}| = 1 \quad \text{(E condition)}.} \tag{109}$$

In a (sufficiently regular) deformation path, starting from a situation in which $\det \mathbf{A}(\mathbf{n}) > 0$ is satisfied, the acoustic tensor $\mathbf{A}(\mathbf{n})$ becomes singular when the plastic modulus reaches a critical value $g_{cr}^E(\mathbf{n})$. This critical hardening modulus was derived by Rice (1977) as follows. The condition

$$\det \left(\mathbf{A}_E(\mathbf{n}) - \frac{1}{g} \mathbf{Mn} \otimes \mathbf{Nn} \right) > 0,$$

assuming $\mathbf{A}_E$ *positive definite*, can be written as

$$\det \mathbf{A}_E(\mathbf{n}) \det \left(\mathbf{I} - \frac{1}{g} \mathbf{A}_E^{-1}(\mathbf{n}) \mathbf{Mn} \otimes \mathbf{Nn} \right) > 0.$$

Making use of the identity $\det(\mathbf{I} + \mathbf{a} \otimes \mathbf{b}) = 1 + \mathbf{a} \cdot \mathbf{b}$, holding for every $\mathbf{a}$ and $\mathbf{b} \in \mathcal{V}$, we obtain the critical hardening modulus for loss of (E) in the direction $\mathbf{n}$

$$\boxed{g_{cr}^E(\mathbf{n}) = \mathbf{Nn} \cdot \mathbf{A}_E^{-1}(\mathbf{n}) \mathbf{Mn},} \tag{110}$$

which is greater than zero for associative elastoplasticity, but may exceptionally result negative for a non-associative flow rule[21]. When the acoustic tensor is singular, i.e. when $g = g_{cr}^E$, the eigenvector corresponding to the null eigenvalue is

$$\mathbf{g} \propto \mathbf{A}_E^{-1}(\mathbf{n}) \mathbf{Mn}. \tag{111}$$

As for the case of the (SE) condition, the critical plastic modulus for loss of (E) is the solution of the constrained maximization problem

$$\boxed{g > g_{cr}^E = \max_{\mathbf{n}, |\mathbf{n}|=1} g_{cr}^E(\mathbf{n}) \iff \det \mathbf{A}(\mathbf{n}) > 0, \text{ for all } \mathbf{n} \in \mathcal{V}, |\mathbf{n}| = 1 \implies \text{(E condition)}.}$$
$$\tag{112}$$

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[21] In the case $g_{cr}^E < 0$, localization is excluded for strictly positive values of g, i.e. in the present context.

The condition of loss of ellipticity admits a particularly nice mechanical interpretation, namely, *localization of deformation into a planar band* becomes possible at failure of (E). This has been known since Hadamard (1903), but was investigated in the case of elastoplasticity by Nadai (1931, 1950), Hill (1952, 1962), Prager (1954), Thomas (1953, 1961), Mandel (1966), Rudnicki and Rice (1975), Rice (1977), Rice and Rudnicki (1980). Strain localization may be directly linked to the initiation and growth of slip mechanisms and fractures in solids. It is experimentally observed in a wide range of materials (including metals, polymers, concretes, geomaterials) and is one of the most explored research fields, since Rice's (1977) paper.

It is important to understand how failure of ellipticity is connected to the emergence of localized deformations. To this purpose, let us consider an infinite body, subject to remote boundary conditions sufficient to impose continued quasi-static, homogeneous deformation. At any instant of the deformation process, the uniform stress field trivially satisfies the equilibrium equations. At a certain point of the deformation process, let us assume that a non-trivial incremental solution becomes possible, consisting of a velocity gradient that is uniform except across a planar band, where it is discontinuous. Inside and outside the band the incremental stress and strain fields remain uniform, so that equilibrium and compatibility are satisfied. If the band has normal n_0 in the material description, the nominal traction must remain continuous, (5), across the band:

$$\left[\!\left[\dot{\mathbf{S}} \right]\!\right] \mathbf{n}_0 = 0; \tag{113}$$

moreover, the jump in velocity gradient across the band must satisfy the *Maxwell compatibility conditions*[22]

$$\left[\!\left[\dot{\mathbf{F}} \right]\!\right] = \mathbf{g} \otimes \mathbf{n}_0. \tag{114}$$

If we express via constitutive equations (28) the Piola-Kirchhoff stress rate in (113) in terms of jump of the gradient of velocity written using (114), we arrive at

$$\left(\mathcal{C}(\dot{\mathbf{F}} + \mathbf{g} \otimes \mathbf{n}_0) - \mathcal{C}(\dot{\mathbf{F}}) \right) \mathbf{n}_0 = 0. \tag{115}$$

This is a necessary condition for strain localization into a planar band. Four cases need to be examined, corresponding to conditions of plastic loading (or elastic unloading) inside and outside the band, and plastic loading inside (or outside) and elastic unloading outside

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[22] The compatibility conditions follow from continuity of velocity across the band, $[\![\dot{\mathbf{x}}]\!] = 0$. This can be easily shown, defining as $[\![\nabla \dot{\mathbf{x}}]\!]\,\mathbf{m}$ the directional derivative of the velocity jump in the generic direction singled out by the unit vector $\mathbf{m}$. Due to continuity of $\dot{\mathbf{x}}$, the derivative along $\mathbf{n}_0$, i.e. $[\![\nabla \dot{\mathbf{x}}]\!]\,\mathbf{n}_0$, remains unrestricted, but the derivative orthogonal to $\mathbf{n}_0$ must vanish: $[\![\nabla \dot{\mathbf{x}}]\!]\,\mathbf{t} = 0$, for every unit vector $\mathbf{t}$ orthogonal to $\mathbf{n}_0$. This condition implies that, when projected onto a basis $\mathbf{n}_0$, $\mathbf{t}$, $\mathbf{s} = \mathbf{n}_0 \times \mathbf{t}$, tensor $[\![\nabla \dot{\mathbf{x}}]\!]$ has only the three non-null components $[\![\nabla \dot{x}]\!]_{n_0 n_0}$, $[\![\nabla \dot{x}]\!]_{t n_0}$ and $[\![\nabla \dot{x}]\!]_{s n_0}$. Therefore, defining a vector $\mathbf{g}$ as

$$\mathbf{g} = [\![\nabla \dot{x}]\!]_{n_0 n_0}\,\mathbf{n}_0 + [\![\nabla \dot{x}]\!]_{t n_0}\,\mathbf{t} + [\![\nabla \dot{x}]\!]_{s n_0}\,\mathbf{s},$$

it is readily obtained that $[\![\nabla \dot{\mathbf{x}}]\!] = \mathbf{g} \otimes \mathbf{n}_0$, eqn. (114).

(or inside) the band:

1) $\left(\mathcal{C}(\dot{\mathbf{F}} + \mathbf{g} \otimes \mathbf{n}_0) - \mathcal{C}(\dot{\mathbf{F}})\right) \mathbf{n}_0 = C[\mathbf{g} \otimes \mathbf{n}_0]\mathbf{n}_0,$ $\qquad$ (plastic/plastic),

2) $\left(\mathcal{C}(\dot{\mathbf{F}} + \mathbf{g} \otimes \mathbf{n}_0) - \mathcal{C}(\dot{\mathbf{F}})\right) \mathbf{n}_0 = G[\mathbf{g} \otimes \mathbf{n}_0]\mathbf{n}_0,$ $\qquad$ (elastic/elastic),

3) $\left(\mathcal{C}(\dot{\mathbf{F}} + \mathbf{g} \otimes \mathbf{n}_0) - \mathcal{C}(\dot{\mathbf{F}})\right) \mathbf{n}_0 = C[\mathbf{g} \otimes \mathbf{n}_0]\mathbf{n}_0 + (C - G)[\dot{\mathbf{F}}]\mathbf{n}_0,$ (plastic/elastic),

4) $\left(\mathcal{C}(\dot{\mathbf{F}} + \mathbf{g} \otimes \mathbf{n}_0) - \mathcal{C}(\dot{\mathbf{F}})\right) \mathbf{n}_0 = G[\mathbf{g} \otimes \mathbf{n}_0]\mathbf{n}_0 - (C - G)[\dot{\mathbf{F}}]\mathbf{n}_0,$ (elastic/plastic).

It is important to note that only Case 1) corresponds to violation of (E), eqn. (109), i.e. $\det \mathbf{A}(\mathbf{n}) = 0$. Case 2) corresponds to a purely elastic loss of (E), $\det \mathbf{A}_E(\mathbf{n}) = 0$. However, we assume for simplicity $\mathbf{A}_E(\mathbf{n})$ positive definite and therefore Case 2) is not considered. We report now the Rice and Rudnicki (1980) proof that

if $\det \mathbf{A}(\mathbf{n}) > 0$, so that Case 1) is excluded, Cases 3) and 4) are also excluded.

Therefore, we assume that (the index 0 of $\mathbf{n}_0$ is omitted for conciseness)

$$\det \mathbf{A}(\mathbf{n}) > 0 \quad \Longleftrightarrow \quad 1 - \frac{\mathbf{Nn} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}}{g} > 0. \tag{116}$$

Let us analyze Case 3). If (115) is satisfied under Conditions 3), we have:

$$\mathbf{N} \cdot \dot{\mathbf{F}} \leq 0, \quad \mathbf{N} \cdot \dot{\mathbf{F}} + \mathbf{g} \cdot \mathbf{Nn} \geq 0, \quad \mathbf{g} = \frac{\mathbf{N} \cdot \dot{\mathbf{F}} + \mathbf{g} \cdot \mathbf{Nn}}{g} \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}, \tag{117}$$

from which it is immediate that $\mathbf{g} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn} \geq 0$. But (excluding the trivial case $\mathbf{N} \cdot \dot{\mathbf{F}} + \mathbf{g} \cdot \mathbf{Nn} = 0$) the determinant

$$\det \left\{ \mathbf{I} - \frac{\mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}}{g} \otimes \left(\mathbf{Nn} + \frac{\mathbf{N} \cdot \dot{\mathbf{F}}}{\mathbf{g}^2} \mathbf{g} \right) \right\}$$

$$= 1 - \frac{\mathbf{Nn} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}}{g} - \frac{\mathbf{N} \cdot \dot{\mathbf{F}}}{g\mathbf{g}^2} \mathbf{g} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}, \tag{118}$$

is positive, so that (115) is not satisfied. Let us analyze Case 4). If (115) is satisfied under Conditions 4), we have:

$$\mathbf{N} \cdot \dot{\mathbf{F}} \geq 0, \quad \mathbf{N} \cdot \dot{\mathbf{F}} + \mathbf{g} \cdot \mathbf{Nn} \leq 0, \quad \mathbf{g} = -\frac{\mathbf{N} \cdot \dot{\mathbf{F}}}{g} \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}. \tag{119}$$

Using (119)$_3$ and (119)$_1$ in (119)$_2$, (excluding the trivial case $\mathbf{N} \cdot \dot{\mathbf{F}} = 0$) we get

$$1 - \frac{\mathbf{Nn} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbf{Mn}}{g} \leq 0, \tag{120}$$

which is a contradiction to (116). It follows from the above that, assuming $\mathbf{A}_E(\mathbf{n})$ positive definite and a continuous dependence of constitutive equations on time:

In a loading program controlled by a (regularly) varying parameter and starting from a situation of ellipticity with $\det \mathbf{A}(\mathbf{n}) > 0$, *the first possibility of localization always occurs at failure of ellipticty,* $\det \mathbf{A}(\mathbf{n}) = 0$, *in the comparison solid corresponding to the loading branch of the constitutive operator* C.

The infinitesimal theory. In the case of infinitesimal theory (34) and isotropic elasticity (33), eqn. (108) allows a simplification of (110), which becomes

$$g^E_{cr} = \max_{\mathbf{n}, \, |\mathbf{n}|=1} \left\{ -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}(\mathbf{n} \cdot \mathsf{E}[\mathbf{Q}]\mathbf{n})(\mathbf{n} \cdot \mathsf{E}[\mathbf{P}]\mathbf{n}) + \frac{1}{\mu}\mathsf{E}[\mathbf{Q}]\mathbf{n} \cdot \mathsf{E}[\mathbf{P}]\mathbf{n} \right\}, \qquad (121)$$

where E has the isotropic form (33). The constrained maximization problem, (121), was solved under various hypotheses on the form of **P** and **Q** by Rudnicki and Rice (1975), Needleman and Ortiz (1991), Ottosen and Runesson (1991). It was solved under the sole hypothesis of coaxiality of **P** and **Q** by Boehler and Willis (1991) and Bigoni and Hueckel (1990, 1991a,b). Moreover, generalization of the above solution to incorporate the effects of geometrical terms were given by Rudnicki and Rice (1975) and Szabó (1994), and to incorporate anisotropic elasticity by Bigoni and Loret (1999) and Bigoni et al. (2000).

6.5   Flutter instability

In the case of non-associative elastoplasticity (or when the elastic tensor does not possess the major symmetry), the acoustic tensor is non-symmetric and there is, in principle, the possibility of a particular type of instability. This is the so-called *flutter instability* (Rice, 1977) and corresponds to the occurrence of two complex conjugate eigenvalues of the acoustic tensor or, in other words, when (for at least one unit vector **n**):

> **A(n)** has complex eigenvalues $\Longleftrightarrow$ flutter instability   (F condition).

This instability is for many reasons not still completely understood. In particular, the following points will be addressed:
1) when flutter instability may occur,
2) what is known about its mechanical interpretation.

Onset of flutter. There are two possible approaches to the problem. One is simply to consider flutter instability to be excluded when the eigenvalues of the acoustic tensor are real. Another is to consider the coalescence of two eigenvalues as a critical condition. In fact, when two eigenvalues are coincident, an appropriate, infinitesimally small disturbance may induce complex eigenvalues. The latter point of view was suggested by An and Schaeffer (1990). Proceeding with it, we should restrict the class of possible disturbances, otherwise we will end up considering unstable a linear, isotropic elastic material (in which the two eigenvalues μ of the acoustic tensor always coincide).

Different classes of disturbances have been considered: Loret (1992) analyzed perturbations in the plastic flow direction, Bigoni and Zaccaria (1994a) and Bigoni (1995) considered perturbations due to effects of large strains. Recently, Bigoni and Loret (1999)

have analyzed perturbations in terms of a small (hyper)elastic anisotropy superimposed on the usual isotropic-elastic, plastic models. The latter perturbation is perhaps the more convincing, because it is *symmetric* and has a clear physical interpretation. The result is that flutter may be triggered by such a vanishing small perturbation (for non-associative flow rules), when two eigenvalues of the acoustic tensor coincide. The same perturbation has no effects when superimposed on a linear elastic law or on an associative elastoplastic model. These results are not still conclusive, but *suggest* that:

> *for non-associative elastoplasticity the condition of coalescence of eigenvalues of the acoustic tensor could be considered critical for flutter even in cases in which complex eigenvalues are excluded in the absence of any perturbation.*

As a conclusion, the onset of flutter can be defined simply by finding the conditions of coalescence of eigenvalues. However, this may be not an easy task for elastoplasticity at finite strain. We will therefore limit the investigation to elastic-plastic solids with isotropic elasticity and subject to small strains. The results are mainly due to Loret et al. (1990); however, we follow Bigoni and Zaccaria (1994a) and consider the acoustic tensor corresponding to the plastic branch of the constitutive equation (34)

$$\mathbf{A(n)} = (\lambda + \mu)\mathbf{n} \otimes \mathbf{n} + \mu\mathbf{I} - \frac{1}{g}\mathbf{p} \otimes \mathbf{q}, \tag{122}$$

where

$$\mathbf{q} = \lambda(\mathrm{tr}\mathbf{Q})\mathbf{n} + 2\mu\mathbf{Qn}, \quad \mathbf{p} = \lambda(\mathrm{tr}\mathbf{P})\mathbf{n} + 2\mu\mathbf{Pn}, \tag{123}$$

are linear functions of $\mathbf{n}$. Assuming that $\mathbf{n} \times \mathbf{q} \neq \mathbf{0}$[23] let us consider the following non-orthogonal dual bases of $\mathcal{V}$, so that $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$

$$\mathbf{e}_1 = \mathbf{n}, \qquad\qquad \mathbf{e}_2 = \mathbf{q}, \qquad\qquad \mathbf{e}_3 = \frac{\mathbf{n}\times\mathbf{q}}{|\mathbf{n}\times\mathbf{q}|},$$

$$\mathbf{e}^1 = \frac{(\mathbf{q}^2)\mathbf{n}-(\mathbf{q}\cdot\mathbf{n})\mathbf{q}}{\mathbf{q}^2-(\mathbf{q}\cdot\mathbf{n})^2}, \quad \mathbf{e}^2 = \frac{(\mathbf{q}\cdot\mathbf{n})\mathbf{n}-\mathbf{q}}{\sqrt{\mathbf{q}^2-(\mathbf{q}\cdot\mathbf{n})^2}}, \quad \mathbf{e}^3 = \mathbf{e}_3. \tag{124}$$

Projected onto the bases (124), the eigenvalue problem for (122) gives the characteristic equation

$$\det\begin{pmatrix} \lambda + 2\mu - \eta & -\frac{1}{g}\mathbf{p}\cdot\mathbf{n} & 0 \\ (\lambda+\mu)\mathbf{q}\cdot\mathbf{n}\,\mu - \frac{1}{g}\mathbf{p}\cdot\mathbf{q} - \eta & 0 \\ 0 & -\frac{1}{g}\mathbf{p}\cdot\mathbf{e}_3 & \mu - \eta \end{pmatrix} = 0, \tag{125}$$

where η is the generic eigenvalue of the acoustic tensor (122). The solutions of the characteristic equation are the eigenvalue μ and the two roots of the polynomial equation:

$$\eta^2 - \left(\lambda + 3\mu - \frac{1}{g}\mathbf{p}\cdot\mathbf{q}\right)\eta + (\lambda + 2\mu)\left(\mu - \frac{1}{g}\mathbf{p}\cdot\mathbf{q}\right) + \frac{1}{g}(\lambda+\mu)(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) = 0. \tag{126}$$

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[23] In the special case $\mathbf{n} \times \mathbf{q} = \mathbf{0}$ the final results do not change. This particular case is straightforward and analyzed in Bigoni and Zaccaria (1994a) and Bigoni (1995).

Flutter corresponds to negative values of the discriminant of the second-order polynomial in (126). This can be written as

$$\Delta = \left(\lambda + 3\mu - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} \right)^2 - 4\mu(\lambda + 2\mu) \left(1 - \frac{g_{cr}^E(\mathbf{n})}{g} \right), \tag{127}$$

where $g_{cr}^E(\mathbf{n})$ is the critical plastic modulus for strain localization (121) at fixed $\mathbf{n}$, i.e.

$$g_{cr}^E(\mathbf{n}) = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}(\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) + \frac{\mathbf{p} \cdot \mathbf{q}}{\mu}. \tag{128}$$

Therefore, we conclude that (Bigoni and Zaccaria, 1994a):

> for a given direction $\mathbf{n}$, flutter is always excluded for values of the plastic modulus less than or equal to the critical plastic modulus for localization in a band orthogonal to that direction $\mathbf{n}$[24].

Straightforward manipulation of the discriminant (127) allows one to obtain the necessary and sufficient conditions for flutter

$$\boxed{(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q}) > 0, \quad \& \quad (\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q}) - \mathbf{p} \cdot \mathbf{q} > 0, \quad \& \quad g \in (g_1, g_2),} \tag{129}$$

where

$$\boxed{\left. \begin{array}{c} g_1 \\ g_2 \end{array} \right\} = \frac{1}{\lambda + \mu} \left(\sqrt{(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q})} \pm \sqrt{(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p} \cdot \mathbf{q}} \right)^2.} \tag{130}$$

If we assume deviatoric associativity (30), a simple calculation shows that $(129)_2$ is never satisfied, therefore we reach the conclusion (Loret et al., 1990):

> For elastic-plastic solids in the presence of isotropic elasticity and deviatoric associativity (30) with χ_1 and ψ_1 strictly positive, complex eigenvalues of the acoustic tensor are excluded.

However, coincident eigenvalues are possible. These may be determined by requiring that the discriminant (127) be null[25]. This occurs when one of the following two conditions is satisfied

$$(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q}) = 0, \quad \& \quad g = g_{cr}^C(\mathbf{n}) = -\frac{\mathbf{p} \cdot \mathbf{q}}{\lambda + \mu}, \tag{131}$$

or

$$(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}, \quad \& \quad g = g_{cr}^C(\mathbf{n}) = \frac{\mathbf{p} \cdot \mathbf{q}}{\lambda + \mu}. \tag{132}$$

Assuming isotropic elasticity (33) and deviatoric associativity (30), it is easy to obtain that

$$\mathbf{p} \cdot \mathbf{q} - (\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q}) = 4\mu^2 \chi_1 \psi_1 \left(\hat{\mathbf{S}}\mathbf{n} \cdot \hat{\mathbf{S}}\mathbf{n} - (\mathbf{n} \cdot \hat{\mathbf{S}}\mathbf{n})^2 \right) \geq 0. \tag{133}$$

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[24] This does not mean that flutter always occurs before strain localization. Rather, it means that flutter at a certain $\mathbf{n}$ always occurs before localization corresponding to that $\mathbf{n}$.

[25] We are thus interested in determining coincident roots of (126) and do not consider the situation when (126) has different roots but one coincides with μ.

Therefore

$$(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \mathbf{q}) = 0 \quad \Longrightarrow \quad \mathbf{p} \cdot \mathbf{q} \geq 0 \quad \Longrightarrow \quad g_{cr}^{C}(\mathbf{n}) \leq 0, \tag{134}$$

so that Case (131) is not interesting. In Case (132), (126) gives two coincident solutions equal to μ. Therefore:

The acoustic tensor (for certain $\mathbf{n}$) has an eigenvalue equal to μ, with multiplicity 3, when condition (132) is satisfied.

Now we determine the critical plastic modulus for such a coalescence. We begin noting that the following condition holds true at coalescence

$$\hat{\mathbf{S}}\mathbf{n} \cdot \hat{\mathbf{S}}\mathbf{n} = (\mathbf{n} \cdot \hat{\mathbf{S}}\mathbf{n})^{2}, \tag{135}$$

and is verified if and only if $\mathbf{n}$ is an eigenvector of $\hat{\mathbf{S}}$, but in this case $\mathbf{n}$ is also an eigenvector of $\mathsf{E}[\mathbf{P}]$ and $\mathsf{E}[\mathbf{Q}]$[26]. Therefore, the critical plastic modulus for coalescence of eigenvalues is

$$\boxed{g_{cr}^{C} = \max_{i=1,2,3} \frac{(\mathsf{E}[\mathbf{P}])_{i}(\mathsf{E}[\mathbf{Q}])_{i}}{\lambda + \mu},} \tag{136}$$

where E has the isotropic form (33) and the index i, not summed, denotes principal components of $\mathsf{E}[\mathbf{P}]$ and $\mathsf{E}[\mathbf{Q}]$ (in the same reference system).

To summarize with the above specific example at hand, we may note that for deviatoric associativity, complex eigenvalues of the acoustic tensor are excluded, but coalescence of three eigenvalues may occur. When coalescence occurs and with reference to the above example, Bigoni and Loret (1999) have shown that a perturbation in terms of a small (appropriate) elastic anisotropy superimposed on the isotropic elastic law is sufficient to trigger flutter. Therefore, even if for this model complex eigenvalues are excluded, flutter as induced by physically motivated perturbations is possible and the critical condition corresponds to coalescence of the three eigenvalues of the acoustic tensor.

When coalescence is considered, Bigoni and Loret (1999) have shown that this may occur without relation to the other criteria, namely (PD), (NS), (SE) and (E). In conclusion:

coalescence of eigenvalues of the acoustic tensor and therefore flutter may occur even when (PD) is verified. Therefore, flutter instability is not excluded by the Hill sufficient condition of stability.

Physical meaning and consequences of flutter. As far as the mechanical interpretation of flutter instability is concerned, results are scarce. In particular, Bigoni and Willis (1994) have analyzed a particular wave propagation problem, showing that flutter may correspond to an oscillating motion of material particles, which blows up with time. However, the analysis is valid only for an incrementally *linear* material. As pointed out by Rice (1977), in the case of an elastoplastic solid, an oscillation may yield a crossing

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[26] Vice-versa, all eigenvectors of $\mathsf{E}[\mathbf{P}]$ and $\mathsf{E}[\mathbf{Q}]$ are also eigenvectors of $\hat{\mathbf{S}}$ because χ_1 and ψ_1 have been assumed strictly positive.

of the loading-branch constitutive cone and may thus invalidate an analysis based on a linear material. Only a partial answer has been given to this point. This is the numerical analysis of Simões (1997), in which a growth and decay of an oscillation is found in a plane strain intentation of a Drucker-Prager, ideally-plastic material with non-associative flow law. In this problem, flutter is excluded in the sense intended in the present section, but it may occur as connected to the presence of a free boundary. However, we may conclude that the real mechanical meaning of flutter is still not completely understood and more investigation is needed.

6.6   Other types of instabilities

The reader should be aware at this point that there may occur many types of instabilities in elastoplastic solids, with different mechanical consequences. However, a number of these instabilities were not analyzed for simplicity here. Some of these are briefly mentioned below.

Occurrence of a particular condition was noted by Ottosen and Runesson (1991), Brannon and Drugan (1993) and Bigoni (1995). It corresponds to the situation in which the acoustic tensor has two coincident eigenvalues with geometric multiplicity smaller than algebraic multiplicity. This is possible when the acoustic tensor is not symmetric. As an example, let us consider the Jordan block

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which has an eigenvalue equal to 1 with geometric multiplicity 1 and algebraic multiplicity 2 (in other words the matrix is defective). This possibility is again excluded in the case of isotropic-elastic, plastic solid with deviatoric associativity. But it may occur in other, more general, circumstances. Whether or not this occurrence may correspond to a material instability is presently not known.

A number of material instabilities may occur at a point of a boundary. These were investigated by Benallal et al. (1990), as related to the possibility of a surface instability (Biot, 1965; Hill and Hutchinson, 1975). The possibility of flutter instability in terms of complex conjugate velocities of propagation of Rayleigh waves at a free boundary of an elastoplastic solid was discovered by Loret et al. (1995).

As far as the author is aware, internal instability in the sense of Biot (1965) was not systematically investigated for elastoplastic solids.

Finally, cavitation has been thoroughly investigated for nonlinear elastic solids [see Horgan and Polignone (1995) and references cited therein], but only scarcely analyzed in the case of elastoplastic solids (Huang et al., 1991). For elastoplasticity, the effects of flow rule non-associativity should be dominant [see the related analysis by Bigoni and Laudiero (1989)].

7 Examples

Examples are presented relative to the application of the criteria described in the previous sections. We begin with the simple case of the infinitesimal theory and then we analyze a situation of uniaxial tension and compression of a solid bar, subject to finite strains.

7.1 The infinitesimal theory

We begin with some words of caution as to the use of small strain assumption in any bifurcation and stability analysis. Usually, 'geometrical effects' are crucially important in these problems. To convince oneself of this fact it suffices to recall Euler buckling of rods. Roughly speaking, with respect to the small strain approximation, the various local stability thresholds presented in Sect. 6 contain 'geometrical terms' on the order of stress over elastic shear modulus. These terms become important when the critical hardening moduli become comparable to a representative stress level (Hill, 1958; 1978; Rudnicki and Rice, 1975).

Proceeding now with the case in which all 'geometrical terms' are neglected in the equations, we assume, for simplicity, isotropic elasticity (33) and deviatoric associativity in the form (30)-(31). In other words, $\mathbf{Q}$ and $\mathbf{P}$ are the normals to yield and plastic potential surfaces, respectively, of the Drucker-Prager type. The model reduces therefore to von Mises plasticity when $\mathrm{tr}\mathbf{P} = \mathrm{tr}\mathbf{Q} = 0$. The threshold for loss of (PD) is given by (74) and the threshold for loss of (NS) is simply[27] $h_{cr}^{NS} = 0$. The critical hardening modulus for coalescence of the three eigenvalues of the acoustic tensor, condition (C), can be easily calculated using (136). Evaluation of thresholds for loss of (SE) and (E) requires solution of a constrained maximization problem for (106) and (121). As already mentioned, these maximization problems admit an analytical solution, which can be found in (Rudnicki and Rice, 1975; Bigoni and Hueckel, 1990; 1991a; Ortiz and Needleman, 1991; Ottosen and Runesson, 1991; Bigoni and Zaccaria, 1992a;b). A few values of the critical hardening moduli (divided by the elastic shear modulus μ) are reported in Tab. 1, relative to uniaxial tension. A material with Poisson's ratio $\nu = 0$ and 0.3 is considered, for several values of pressure-sensitivity $\mathrm{tr}\mathbf{Q}$ and dilatancy $\mathrm{tr}\mathbf{P}$.

Note that in the column (E), corresponding to loss of ellipticity, the angle (in degrees) has been reported in parenthesis between the normal to the band and the direction orthogonal to the axis of tension. Obviously, in the uniaxial stress problem analyzed, infinite bands become possible at strain localization, with normals describing a cone about the tension axis. Additional numerical results can be found in (Rudnicki and Rice, 1975[28]; Bigoni and Zaccaria, 1992a,b; Bigoni and Loret, 1999) and need not be reported here.

The results presented in the table are sufficient to draw the following conclusions.

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[27] A complete discussion on the eigenvalues of the elastoplastic tangent operator for isotropic elasticity may be found in Bigoni and Zaccaria (1994b).

[28] Rudnicki and Rice (1975) neglected a term in the analytical solution for h_{cr}^{E}, thus obtaining imprecise numerical results. The complete soultion can be found in (Bigoni and Hueckel, 1990).

- As expected, in the case of an associative flow rule it is $h_{cr}^{PD} = h_{cr}^{NS} = 0$. Moreover, loss of (E), coincident with loss of (SE), is always excluded at positive hardening.
- For non-associative flow laws, the critical hardening modulus for (PD) is always positive. The critical hardening moduli for loss of (SE) and (E) are different and may take any sign, even positive. Strain localization may therefore occur at positive hardening, before loss of (NS), occurring at $h = 0$.
- Loss of (PD), (NS) and (SE) always occur for a positive plastic modulus, i.e. before the snap-back limit $g = 0$ (not considered in the present notes). On the other hand, loss of (E) could be excluded for strictly positive values of the plastic modulus g (even if this does not occur in the case treated by the table).

Table 1. Uniaxial tension. Values of h_{cr}/μ defining loss of positive definiteness (PD), strong ellipticity (SE) and ellipticity (E) of the elastic-plastic constitutive operator and coalescence of three eigenvalues of the acoustic tensor (C), for Poisson's ratio $\nu = 0$ and 0.3, and several values of pressure-sensitivity $\mathrm{tr}\,\mathbf{Q}$, and plastic dilatancy $\mathrm{tr}\,\mathbf{P}$. θ in degrees is given in parenthesis.

tr Q	tr P	ν	(PD)	(SE)	(E) (θ)	(C)
0	0	0	0	-0.167	-0.167 (54.7°)	0.333
		0.3	0	-0.217	-0.217 (48.8°)	-0.467
0.3	0	0	0.015	-0.101	-0.104 (58.3°)	0.564
		0.3	0.047	-0.106	-0.130 (53.2°)	-0.166
0.3	0.3	0	0	-0.071	-0.071 (62.2°)	0.775
		0.3	0	-0.093	-0.093 (57.8°)	0.108
0.6	0	0	0.057	-0.021	-0.031 (62.2°)	0.795
		0.3	0.167	0.054	-0.018 (57.8°)	0.133
0.6	0.3	0	0.013	-0.027	-0.028 (66.4°)	0.986
		0.3	0.034	-0.020	-0.031 (62.9°)	0.382
0.6	0.6	0	0	-0.016	-0.016 (71.3°)	1.177
		0.3	0	-0.020	-0.020 (68.6°)	0.630
0.9	0	0	0.120	0.070	0.052 (66.4°)	1.026
		0.3	0.330	0.239	0.117 (62.9°)	0.434
0.9	0.3	0	0.049	0.029	0.024 (71.3°)	1.197
		0.3	0.115	0.082	0.054 (68.6°)	0.656
0.9	0.6	0	0.011	0.007	0.007 (77.8°)	1.368
		0.3	0.022	0.017	0.015 (76.1°)	0.878
0.9	0.9	0	0	-0.0005	-0.0005 (90.0°)	1.539
		0.3	0	-0.001	-0.001 (90.0°)	1.101
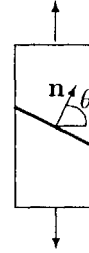
- Coalescence of eigenvalues of the acoustic tensor seems to have no relation to the other stability criteria. It may occur at positive or negative hardening, depending strongly on the value of the Poisson's ratio. It may occur before (PD), or after (E), or between these thresholds.

− All stability thresholds are strongly influenced by
- state of stress,
- constitutive parameters (dilatancy, pressure-sensitivity, Poisson's ratio),
- constitutive features not investigated in the example [yield surface curvature or vertex, non-coaxiality of **P** and **Q**, elastic or plastic anisotropy (Bigoni and Loret, 1999; Bigoni et al. 2000)].

7.2 A simple elastoplastic model including 'geometrical effects'

Bifurcations in simple boundary value problems such as compression or tension of cylinders or blocks deformed in plane strain have been thoroughly analyzed from an analytical point of view for hyperelastic materials (see for instance, Biot, 1965; Hill and Hutchinson, 1975). These analyses are often relevant to associative elastoplasticity, where they represent bifurcations of the comparison solid corresponding to the loading branch of the constitutive operator. As explained in Sect. 5, bifurcations detected in such a comparison solid may also represent genuine elastoplastic bifurcations under broad hypotheses (Hutchinson, 1973). As an example, we consider the analysis of Hutchinson and Miles (1974) in which axisymmetric bifurcations are investigated during uniaxial tension of a solid bar. The bar is incrementally linear and hyperelastic, but the results are also pertinent to associative elastoplasticity.

The situation of non-associative elastoplasticity is much more complicated. In particular, we have seen that the search for bifurcations in an elastoplastic solid with a non-associative flow rule is replaced by the search for bifurcations in two linear comparison solids. One can be any member of the family of Raniecki comparison solids and the other is simply the linear solid defined by the loading branch of the constitutive operator. Bifurcation in the former solid determines a lower bound to bifurcation stresses and bifurcation in the latter provides an upper bound. Obviously, it will be convenient to determine the optimal Raniecki solid as a function of parameter ψ with respect to the bifurcation problem under consideration.

Needleman (1979), Vardoulakis (1981; 1983), Chau and Rudnicki (1990), and Chau (1992; 1995) have investigated bifurcations in the 'in loading comparison solid' of an elastoplastic material with non-associative flow rule, similar, in essence, to the Rudnicki and Rice (1975) model. The sole analytical analysis of the Raniecki bounds (of which the author is aware) is that given by Bruhns and Raniecki (1982), whereas numerical results were presented by Tvergaard (1982), Kleiber (1984; 1986) and Tomita et al. (1988).

We develop below a simple example involving uniaxial tension and compression of a cylindrical bar. For this example, we give the (PD), (NS), (SE) and (E) thresholds and investigate the Raniecki bounds for assigned, axisymmetric mode of bifurcation. It is important to realize that *priority is given to mathematical simplicity*, therefore the example should be considered as a prototype, not intended to properly model any real material. Nevertheless, the main constitutive features such as pressure-sensitivity and plastic dilatancy are taken into account. We base the analysis on the simple constitutive model suggested by Hill (1962) for an associative flow rule (Hutchinson, 1973; Neale, 1981). This model has the structure (24) in the relative Lagrangean description (i.e.

using the current configuration as reference, $\mathbf{F} = \mathbf{I}$)

$$\dot{\mathbf{S}} = \mathsf{E}[\mathbf{D}] + \mathbf{LK} - \frac{1}{g} < \mathbf{D} \cdot \mathsf{E}[\mathbf{Q}] > \mathsf{E}[\mathbf{P}], \tag{137}$$

where E has the isotropic form (33). When λ and μ are not considered *constants* and have a special dependence on deformation, Christoffersen (1991) has proved hyperelastic behaviour for the elastic part of (137). However, we will consider λ and μ constant in the following. Moreover, tensors $\mathbf{P}$ and $\mathbf{Q}$ are selected in the form (30)-(31) proposed by Rudnicki and Rice (1975), with *isochoric plastic deformation*

$$\mathbf{Q} = \frac{\operatorname{dev} \mathbf{T}}{2\sqrt{J_2}} + \frac{\alpha}{3}\mathbf{I}, \quad \mathbf{P} = \frac{\operatorname{dev} \mathbf{T}}{2\sqrt{J_2}}. \tag{138}$$

Model (137) differs from that analyzed in the case of the infinitesimal theory only by the presence of the 'geometrical term' $\mathbf{LK}$. The elastic acoustic tensor $\mathbf{A}_E(\mathbf{n})$ corresponding to (137) is

$$\mathbf{A}_E(\mathbf{n}) = (\lambda + \mu)\mathbf{n} \otimes \mathbf{n} + \bar{\mu}\mathbf{I}, \tag{139}$$

where $\bar{\mu} = \mu + \mathbf{n} \cdot \mathbf{Kn}$ and has the inverse

$$\mathbf{A}_E^{-1}(\mathbf{n}) = -\frac{\lambda + \mu}{\bar{\mu}(\lambda + \mu + \bar{\mu})}\mathbf{n} \otimes \mathbf{n} + \frac{1}{\bar{\mu}}\mathbf{I}. \tag{140}$$

Flutter instability for the model (137) was analyzed by Bigoni and Zaccaria (1994a). They found that complex eigenvalues are excluded for deviatoric associativity (30)-(31), but coalescence of eigenvalues may still occur. This will not be examined below.

Axisymmetric bifurcations of a solid bar under uniaxial stress.

Let us consider a circular cylindrical specimen of radius R and height L subject to axisymmetric deformation with uniaxial stress $\mathbf{T} = \sigma \mathbf{e}_z \otimes \mathbf{e}_z$ aligned along the cylinder axis $\mathbf{e}_z$ and traction-free lateral surface. The relative Lagrangean description is assumed, in which the current configuration is taken as reference ($\mathbf{F} = \mathbf{I}$). A cylindrical coordinate system (r, θ, z) is adopted with the z-axis coincident with the axis of the cylinder. The ends ($z = 0, L$) are subject to flat, frictionless, rigid constraints keeping null the nominal tangential traction, i.e. $S_{rz} = 0$. The material constitutive model is assumed to correspond to the loading branch of constitutive equation (137) and is therefore given by

$$\dot{\mathbf{S}} = \mathsf{D}[\mathbf{D}] + \mathbf{LK}, \quad \mathsf{D} = \mathsf{E} - \frac{1}{g}\mathsf{E}[\mathbf{Q}] \otimes \mathsf{E}[\mathbf{P}]. \tag{141}$$

We will also analyze bifurcations in the Raniecki solid, which can be obtained simply by replacing D in (141) with

$$\mathsf{D}^R = \mathsf{E} - \frac{1}{4\psi g}(\mathsf{E}[\mathbf{P}] + \psi\mathsf{E}[\mathbf{Q}]) \otimes (\mathsf{E}[\mathbf{P}] + \psi\mathsf{E}[\mathbf{Q}]). \tag{142}$$

Incremental equilibrium equations *for axisymmetric deformations* (in cylindrical co-ordinates) are:

$$\dot{S}_{rr,r} + \dot{S}_{rz,z} + \frac{1}{r}(\dot{S}_{rr} - \dot{S}_{\theta\theta}) = 0,$$

$$\dot{S}_{zr,r} + \dot{S}_{zz,z} + \frac{1}{r}\dot{S}_{zr} = 0. \tag{143}$$

Bifurcations are sought in terms of incremental fields satisfying

$$v_z = 0, \quad \dot{S}_{rz} = 0 \quad \text{at} \quad z = 0, L,$$

$$\dot{S}_{rr} = 0, \quad \dot{S}_{zr} = 0 \quad \text{at} \quad r = R. \tag{144}$$

The method of solution parallels that proposed by Chau (1995) and is based on the introduction of the velocity potential $\Phi(r, z)$ so that

$$v_r = \Phi_{,rz}, \quad v_z = -\frac{1}{A}[B\mathcal{N}(\Phi) + C\Phi_{,zz}], \tag{145}$$

where $\mathcal{N}(\Phi) = (r\Phi_{,r})_{,r}/r$ and

$$A = \mathsf{D}_{rrzz} + \mathsf{D}_{rzrz}, \quad B = \mathsf{D}_{rrrr}, \quad C = \mathsf{D}_{rzrz} + \sigma. \tag{146}$$

For subsequent reference, we introduce three additional quantities:

$$D = \mathsf{D}_{zzzz} + \sigma, \quad E = \mathsf{D}_{rzrz}, \quad F = \mathsf{D}_{zzrr} + \mathsf{D}_{rzrz}. \tag{147}$$

We note, in passing, that different but equivalent potentials can be introduced in the bifurcation analysis, as discussed by Miles and Nuwahyid (1985). In the axisymmetric problem under consideration, the non-vanishing components of velocity gradient $\mathbf{L}$ are:

$$L_{rr} = v_{r,r}, \quad L_{\theta\theta} = \frac{v_r}{r}, \quad L_{zz} = v_{z,z}, \quad L_{rz} = v_{r,z}, \quad L_{zr} = v_{z,r}. \tag{148}$$

Substitution of the constitutive law (141) into the equilibrium equations (143) reveals that $(143)_1$ is identically satisfied, while $(143)_2$ gives:

$$\left[\mathcal{N}(\cdot) - \rho_1^2(\cdot)_{,zz}\right]\left[\mathcal{N}(\cdot) - \rho_2^2(\cdot)_{,zz}\right]\Phi = 0, \tag{149}$$

where ρ_1 and ρ_2 satisfy the condition:

$$EB\rho_i^4 + (EC + DB - AF)\rho_i^2 + DC = 0, \quad (i = 1, 2). \tag{150}$$

The nature of the roots of (150) defines the regime classification as follows

 — two ρ_i complex conjugate pairs in the elliptic complex regime,
 — four ρ_i pure imaginary in the elliptic imaginary regime,
 — four ρ_i real in the hyperbolic regime,
 — two ρ_i real and two pure imaginary in the parabolic regime.

The interest in the above classification lies in the fact that, when the boundary of the elliptic regime is touched, strain localization may occur. In other words, loss of ellipticity, which is completely equivalent to failure of condition (109), occurs when at least two roots ρ_i become real. Diffuse bifurcated solutions are sought now *in the elliptic regime* of the form:

$$\Phi(r, z) = \phi(r) \sin \eta z, \tag{151}$$

where $\eta = k\pi/L$ and $k = 1, 2, \ldots n$. The field (151) satisfies boundary conditions (144) at $z = 0, L$. The equilibrium equation (149) becomes:

$$\left[\overline{\mathcal{N}}(\cdot) + \rho_1^2\eta^2\right]\left[\overline{\mathcal{N}}(\cdot) + \rho_2^2\eta^2\right]\phi = 0, \tag{152}$$

where $\overline{\mathcal{N}}(\phi) = (d/dr)(r(d\phi/dr))/r$. The general solution of (152) for the cylinder can be expressed in terms of the Bessel function of order 0, $J_0(x)$:

$$\phi(r) = c_1 J_0(\eta r \rho_1) + c_2 J_0(\eta r \rho_2), \tag{153}$$

where constants c_1 and c_2 are, in general, complex.

Imposing boundary conditions at $r = R$ yields a 2×2 linear, homogeneous system. Bifurcation occurs when non-trivial solutions of this system are possible. This condition simply means that bifurcation is possible when $\det M_{ij} = 0$, where M_{ij} is defined as

$$M_{1j} = (\mathsf{D}_{rrrr} - \mathsf{D}_{rr\theta\theta})\rho_j J_1(\eta R \rho_j) + (\mathsf{D}_{rrzz}(\frac{B}{A}\rho_j^2 + \frac{C}{A}) - \mathsf{D}_{rrrr}\rho_j^2)\eta R J_0(\eta R \rho_j)$$

$$M_{2j} = \rho_j(1 - \frac{C}{A} - \frac{B}{A}\rho_j^2)J_1(\eta R \rho_j). \qquad (j = 1, 2). \tag{154}$$

Results. In the interest of simplicity, we assume $\lambda = 0$ (corresponding to a null Poisson's ratio). Therefore, the model (137) reduces to

$$\frac{\dot{\mathsf{S}}}{2\mu} = \mathsf{D} + \frac{\mathsf{LK}}{2\mu} - \frac{2\mu}{g} < \mathsf{D} \cdot \mathsf{Q} > \mathsf{P}. \tag{155}$$

where P and Q take the form (138), in which the stress is uniaxial

$$\mathsf{P} = \frac{\text{sign}\,\sigma}{2\sqrt{3}} (2\mathbf{e}_z \otimes \mathbf{e}_z - \mathbf{e}_r \otimes \mathbf{e}_r - \mathbf{e}_\theta \otimes \mathbf{e}_\theta), \quad \mathsf{Q} = \mathsf{P} + \frac{\alpha}{3}\mathbf{I}. \tag{156}$$

The Raniecki comparison solids are defined by

$$\frac{\dot{\mathsf{S}}}{2\mu} = \mathsf{D} + \frac{\mathsf{LK}}{2\mu} - \frac{\mu}{2cg}(\mathsf{D} \cdot \mathsf{P} - c\mathsf{D} \cdot \mathsf{Q})(\mathsf{P} + c\mathsf{Q}). \tag{157}$$

moreover, we select for simplicity the value of c suggested by Bruhns and Raniecki (1982)

$$c = \sqrt{\frac{\mathsf{P} \cdot \mathsf{P}}{\mathsf{Q} \cdot \mathsf{Q}}} = \sqrt{\frac{1}{1 + \frac{2}{3}\alpha^2}}. \tag{158}$$

which is independent of the current stress level, and it is therefore not optimal both for (PD) and (SE$_R$).

The critical plastic modulus for failure of (PD) can be obtained from (73) by noting that

$$\mathsf{L} = \mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \frac{\sigma}{2\mu}\mathbf{e}_z \otimes \mathbf{e}_z \otimes \mathbf{e}_z \otimes \mathbf{e}_z,\tag{159}$$

has the inverse

$$\mathsf{L}^{-1} = \mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} - \frac{\sigma}{2\mu + \sigma}\mathbf{e}_z \otimes \mathbf{e}_z \otimes \mathbf{e}_z \otimes \mathbf{e}_z.\tag{160}$$

For uniaxial tension, σ takes positive values and L is always positive definite. Note also that, due to coaxiality of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{T}$, failure of (NS) occurs for a mode with null spin and always corresponds to a *maximum load*.

Failure of (SE) and (E) can be obtained from (98) and (110), on the basis of the elastic acoustic tensor (139) and its inverse (140). Note that, for $\lambda = 0$, the acoustic tensor has eigenvalues equal to $\bar{\mu}$ and $\mu + \bar{\mu}$, which are always positive for tension and vanish in compression for the first time when $\mathbf{n} = \mathbf{e}_z$ and $\sigma = -\mu$. When these values are reached, a purely elastic loss of ellipticity occurs. In order to find the (SE) and (E) thresholds, the constrained maximization problems (100) and (112) have to be solved. Due to axial symmetry, this is an easy task in the present situation, where the unit vector $\mathbf{n}$ can be chosen without loss of generality in the form $\mathbf{n} = n_z\mathbf{e}_z + n_r\mathbf{e}_r$. Failure of (SE$_R$) can be obtained from (102) by solving the inf-max problem for $\mathbf{n}$ and ψ. However, for the present case of uniaxial tension, we have numerically proven the coincidence of (SF) and (SE$_R$), which holds for a certain optimal parameter ψ, which is a function of the current stress. Therefore, only (SE) will be reported in the figures. It is important to remark that both (PD) and (SE)$\equiv$(SE$_R$) are not relevant (except at $\sigma = 0$) for the Raniecki solid defined by the fixed value of ψ given by (158). This solid has its own peculiar curves for loss of (PD) and (SE) —not reported in the figures—, which, by the way, are close to the curves (PD) and (SE) relative to the solid 'in loading' (coincident with the optimal Raniecki solid).

Curves corresponding to failure of (PD), (NS), (SE) and (E) are reported in Figs. 2-4, together with the curves corresponding to axisymmetric bifurcation for a given bifurcation mode ηR. These curves are referred to a plane defined by axes $\sigma/2\mu$ (the axial stress divided by 2μ) and μ/g (the inverse of the plastic modulus, multiplied by μ). Note that the curves corresponding to (PD) and (NS) are reported only for tension, because tensor $\mathbf{G}$ is not positive definite in compression. Note also that the reported range of variation of $\sigma/(2\mu)$ is very large, and probably not sensible with the assumed simplified model. However, we prefer to show a picture as complete as possible. The range of variation of μ/g goes from 0, representative of purely elastic behaviour, i.e. $g \to \infty$, to softening, i.e. beyond the perfectly plastic limit $h = 0$, corresponding to $\mu/g = 1$. Note that the relation between h and g in this simple model is just

$$\frac{g}{\mu} = 1 + \frac{h}{\mu}.$$

Let us begin the discussion with Fig. 2, relative to the associative case, $\alpha = 0$. In this case, the plasticity is referred to a von Mises yield surface. Here the (PD) and (NS)

thresholds and also the (SE) and (E) thresholds coincide. The (PD)≡(NS) curve approaches the horizontal axis at $\mu/g = 1$, corresponding to $h = 0$, which is the value of loss of (PD)≡(NS) for the infinitesimal theory. This is obtained when the 'geometrical term' $\sigma/(2\mu)$ is set equal to zero. The curve relative to loss of (E)≡(SE) crosses the horizontal axis at $\mu/g = 1.2$, corresponding to $h/\mu = -0.167$, again the value relative to the infinitesimal theory (see Tab. 1). The curve continues indefinitely in tension, but terminates at $\sigma/(2\mu) = -0.5$ in compression, where failure of ellipticity of the elastic tensor occurs. Two other curves are repoted in the graph, relative to axisymmetric bifurcations of the cylindrical bar with prescribed modes $\eta R = 1$ and $\eta R = 2$. These cross (continuously) the horizontal axis a few decimals on the right side of the value corresponding to loss of (PD)≡(NS). As sketched in the figure, the bifurcation modes are necking modes for tension and a barrelling modes for compression. Points situated at positive values of $\sigma/(2\mu)$ and on the left side of curve (PD)≡(NS) are representative of states for which (PD) is satisfied and *bifurcation is a-priori excluded*. Points situated at $\sigma/(2\mu) > -0.5$ and on the left side of curve (SE)≡(E) are representative of *elliptic* states, for which *strain localization is excluded*.
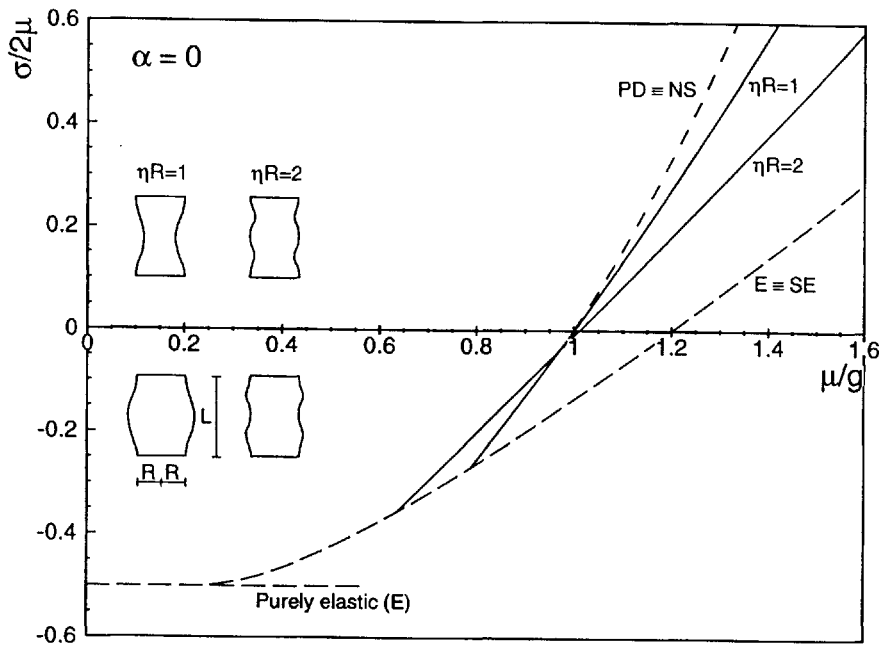

Fig. 2. Axisymmetric bifurcations and local stability criteria for a cylindrical bar subject to uniaxial stress. Associative flow-rule.

Let us assume that g is some continuous function of the stress state, which is infinity at the beginning of plastic flow and continuously decreases when the stress increases (or

decreases, in compression). Let us consider now a monotonic loading program of uniaxial tension of a cylindrical specimen starting from some unloaded state, represented by the origin in the diagram of Fig. 2.

At the beginning of the loading program, the material is in the elastic range and the point representative of the state simply moves up along the axis $\sigma/(2\mu)$, at $\mu/g = 0$. When the yield surface is reached, plastic deformation starts and the point moves toward right in the diagram. Before the point touches the (PD)$\equiv$(NS) curve, the response is unique. As soon as the point reaches the (PD)$\equiv$(NS) curve, a maximum load occurs, and the loading program can be continued only if axial displacement is prescribed (for a situation in which axial dead loading would be prescribed, instability would occur). After the (PD)$\equiv$(NS) curve is passed, the point representative of the state will touch the curve corresponding to necking bifurcation (with mode $\eta R = 1$). This bifurcation terminates the homogeneous deformation of the specimen. This might not be the first bifurcation encountered, because an earlier bifurcation might occur in another -not investigated- mode (represented by a curve lying between the two curves (PD)$\equiv$(NS) and that corresponding to the mode $\eta R = 1$). The other bifurcation mode, $\eta R = 2$ occurs after that corresponding to $\eta R = 1$ in tension (but before in compression). It is clear from the graph that there is no way of touching the (SE)$\equiv$(E) curve in tension without encountering a bifurcation into a diffuse mode. Now, let us analyze compression. Here (PD) is not defined, and in
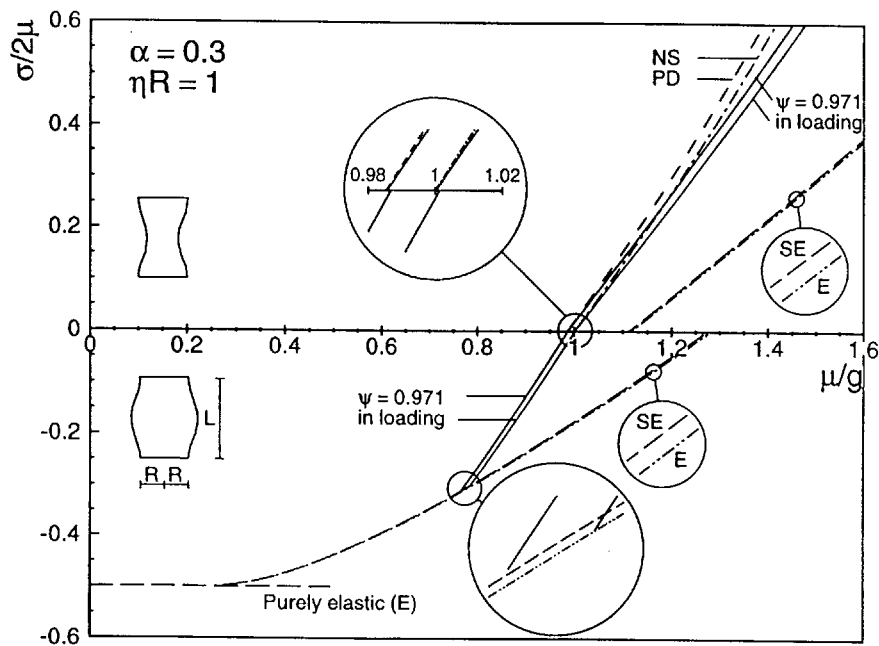

Fig. 3. Bounds to axisymmetric bifurcations and local stability criteria for a cylindrical bar subject to uniaxial stress. Non-associative flow-rule.

fact bifurcation can occur immediately, even at vanishing small stress. This is the well-known case of Euler buckling of a very slender beam, which would be recovered here by analyzing *antisymmetric bifurcation modes*. However, if we concentrate on axisymmetric bifurcation at fixed modes $\eta R = 1, 2$ and consider a loading program at prescribed axial displacements, we see that there is only a range in which bifurcation with $\eta R = 2$ can be attained. Outside this range, loss of (SE)≡(E) will terminate the homogeneous response of the specimen. Once again, we stress that the picture of bifurcation is not complete, because we expect that other axisymmetric or antisymmetric modes can be encountered before that corresponding to $\eta R = 2$.

We are now in a position to analyze the case of non-associativity. Here, the (PD), (NS), (SE), and (E) curves are separated and (SE) and (E) do not cross the horizontal axis with continuity[29]. This reflects the fact that the assumed model has a Drucker-Prager yield surface. The normal to this surface in tension has a different inclination than in compression, and different values of thresholds (SE) and (E) result. For the same reason, the curves representative of diffuse bifurcation modes also do not cross the horizontal axis with continuity (even if the jump is so small that it cannot really be appreciated at the scale of the figure, see detail in Fig. 3). Fig. 3 is relative to $\alpha = 0.3$ and Fig. 4
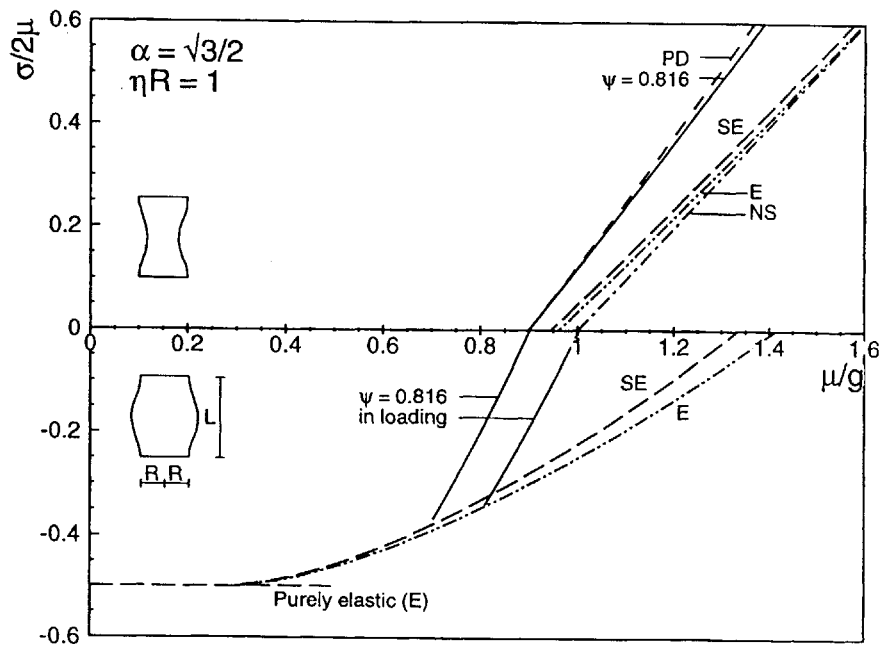

Fig. 4. Bounds to axisymmetric bifurcations and local stability criteria for a cylindrical bar subject to uniaxial stress. Non-associative flow-rule.

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[29] This would also occur for the associative case with pressure-sensitivity.

to $\alpha = \sqrt{3}/2$. As for the case of an associative flow rule, the points where the (PD), (NS), (SE) and (E) curves approach the horizontal axis correspond to the values known from the infinitesimal theory (and reported in Tab. 1, for $\alpha = 0.3$ and tension stress state). Points situated at positive values of $\sigma/(2\mu)$ and on the left side of the curve (PD) correspond to situations in which bifurcation is a-priori excluded. The curve (NS) signals that a maximum load has been reached. Curve (E) is relative to strain localization and is terminated in the negative part of the graph by the horizontal line corresponding to $\sigma/(2\mu) = -0.5$. The same line also terminates the curve (SE). When the curve (SE) is crossed, the sufficient stability condition of Hill is certainly lost, independently of the specific boundary conditions.

In Fig. 3, loss of (PD) occurs before (NS), which precedes (SE) and (E). This is completely different in Fig. 4, where failure of (PD) occurs before (SE) and (E), but (NS) is a line and follows (E) [the curve (NS) crosses the curve (E) where $\sigma/(2\mu) = 1$]. It should also be noted that the curves representative of the Raniecki lower bounds terminate in compression when loss of ellipticity occurs in the specific comparison solid (defined by $\psi = 0.971$ for Fig. 3 and $\psi = 0.816$ for Fig. 4), which does not coincide with the curve (SE) (relative to an optimal choice of ψ, see detail in Fig. 3).

Both the Raniecki bounds are reported in Figs. 3 and 4, for fixed mode $\eta R = 1$. They are separated and are observed both in tension and compression.

Regarding Fig. 3, we note that in tension, the lower bound is represented by a curve starting between (PD) and (NS) and later crossing (NS). The upper bound is close to the right side of (NS). These bifurcations in tension occur well before the loss of (SE) and (E). In compression, just considering the fixed mode of bifurcation $\eta R = 1$, diffuse modes may go beyond the elliptic range, so that there is a region in which (SE) and (E) are lost before barrelling bifurcation.

The curve corresponding to the lower bound $\psi = 0.816$ in Fig. 4 initiates in compression close (but does not touch) to the (SE) curve and continues (with a discontinuity crossing the horizontal axis) in tension, between (PD) and (NS). The bifurcation curve relative to the upper bound, with $\eta R = 1$, is not found in tension in the elliptic regime.

8   A concluding remark

Deformations of solids are limited by the occurrence of different failure modes taking place at different scales. As an example, let us recall the behaviour of a metallic, ductile bar pulled in tension. Necking bifurcation is the first threshold for which homogeneity is lost. Subsequently, elastoplastic cavitation occurs in the necked zone. This is a prelude to strain localization and, eventually, fracture propagates along the localized bands yielding complete failure. All these phenomena may be described as instabilities which interact and may be in cooperation/competition to reach global failure.

Many of the instabilities occurring in laboratory tests, such as necking, barrelling, shear banding, and cavitation, can be at least qualitatively described in terms of the classical theory of elastoplasticity, referring to smooth yield surfaces and the normality rule. Behaviour of non-metallic materials, however, is more complicated than the classical theory of plasticity may predict. Therefore, flow rule non-associativity was advocated as

a more accurate constitutive description. The result is that non-associativity not only promotes instabilities which may be already qualitatively described in the framework of normality, but opens possibility of new types of instabilities, such as flutter. These instabilities are not yet fully understood. If, on one hand, these may merely reflect a general deficiency of the non-associative models, on the other hand, these may model new phenomena, not yet fully experimentally detected.

References

An, L. and Schaeffer, D. (1990). The flutter instability in granular flow. *J. Mech. Phys. Solids* 40:683-698.

Beatty, M.F. (1987). Topics in finite elasticity: hyperelasticity of rubber, elastomers, and biological tissues-with examples. *Appl. Mech. Rev.* 40:1699-1734.

Benallal, A. Billardon, R. and Geymonat, G. (1990). Phènoménes de localisation á la frontière d'un solide. *C. R. Acad. Sci., Paris* 310:670-684.

Bigoni, D. (1995). On flutter instability in elastoplastic constitutive models. *Int. J. Solids Structures* 32:3167-3189.

Bigoni, D. (1996). On smooth bifurcations in non-associative elastoplasticity. *J. Mech. Phys. Solids* 44:1337-1351.

Bigoni, D. and Hueckel, T. (1990). A note on strain localization for a class of non-associative plasticity rules. *Ingenieur-Archiv* 60:491-499.

Bigoni, D. and Hueckel, T. (1991a). Uniqueness and localization-I. Associative and non-associative elastoplasticity. *Int. J. Solids Structures* 28:197-213.

Bigoni, D. and Hueckel, T. (1991b). Uniqueness and localization-II. Coupled elastoplasticity. *Int. J. Solids Structures* 28:215-224.

Bigoni, D. and Laudiero, F. (1989). The quasi-static finite cavity expansion in a non-standard elastoplastic medium. *Int. J. Mech. Sci.* 31:825-837.

Bigoni, D. and Loret, B. (1999). Effects of elastic anisotropy on strain localization and flutter instability in plastic solids. *J. Mech. Phys. Solids* 47:1409-1436.

Bigoni, D., Loret, B. and Radi, E. (2000). Localization of deformation in plane elastic-plastic solids with anisotropic elasticity. *J. Mech. Phys. Solids* , Special issue dedicated to Prof. J.R. Willis, in press.

Bigoni, D. and Willis, J.R. (1994). A dynamical interpretation of flutter instability. In: Chambon, R., Desrues, J. and Vardoulakis, I., eds., *Localisation and Bifurcation of Rocks and Soils* Rotterdam: A.A. Balkema Scientific Publishers. 51-58.

Bigoni, D. and Zaccaria, D. (1992a). Strong ellipticity of comparison solids in elastoplasticity with volumetric non-associativity. *Int. J. Solids Structures* 29:2123-2136.

Bigoni, D. and Zaccaria, D. (1992b). Loss of strong ellipticity in non-associative elastoplasticity. *J. Mech. Phys. Solids* 40:1313-1331.

Bigoni, D. and Zaccaria, D. (1994a). On eigenvalues of the acoustic tensor in elastoplasticity. *Eur. J. Mechanics-A/Solids* 13:621-638.

Bigoni, D. and Zaccaria, D. (1994b). Eigenvalues of the elastoplastic constitutive operator. *ZAMM* 74:355-357.

Biot, M.A. (1965) *Mechanics of incremental deformations.* New York:Wiley.

Boehler, J.P. and Willis J.R. (1991). An analysis of localization in highly pre-deformed sheet steel. *Unpublished.*

Brannon, R.M. and Drugan, W.J. (1993). Influence of non-classical elastic-plastic constitutive features on shock wave existence and spectral solutions. *J. Mech. Phys. Solids* 41:297-330.

Bruhns, O. and Raniecki, B. (1982). Ein Schrankenverfahren bei Verzweigungsproblemen in-
elastischer Formänderungen. *ZAMM* 62:T111-T113.

Cattaneo, C. (1946). Su un teorema fondamentale nella teoria delle onde di discontinuità. *Atti
Acad. Naz. Lincei* (parts I and II) I:67-72 and 728-734.

Chadwick, P. and Powdrill B. (1965). Singular surfaces in linear thermoelasticity. *Int. J. Eng.
Science* 3:561-595.

Chau, K.T. (1992). Non-normality and bifurcation in a compressible pressure-sensitive circular
cylinder under axisymmetric tension and compression. *Int. J. Solids Structures* 29:801-824.

Chau, K.T. (1995). Buckling, barrelling, and surface instabilities of a finite, transversely isotropic
circular cylinder. *Quart. Appl. Math.* 53:225-244.

Chau, K.T. and Rudnicki, J.W. (1990). Bifurcations of compressible pressure- sensitive materials
in plane strain tension and compression. *J. Mech. Phys. Solids* 38:875-898.

Cheng, Y.S. and Lu, W.D. (1993). Uniqueness and bifurcation in elastic-plastic solids. *Int. J.
Solids Structures* 30:3073-3084.

Christoffersen, J. (1991). Hyperelastic relations with isotropic rate forms appropriate for elasto-
plasticity. *Eur. J. Mechanics-A/Solids* 10:91-99.

Del Piero, G. (1979). Some properties of the set of fourth-order tensors, with applications to
elasticty. *J. Elasticity* 9:245-261.

Drucker, D.C. (1954). Coulomb friction plasticity and limit loads. *J. Appl. Mech.* 76:71-74.

Gurtin, M.E. (1972). The linear theory of Elasticity. In Flügge, S., ed., *Encyclopedia of Physics*
VIa/2. Berlin:Springer. 1-295.

Gurtin, M.E. (1981). *An introduction to continuum mechanics.* New York:Academic Press.

Hayes, M. (1966). On the displacement boundary-value problem in linear elastostatics. *Quart.
J. Mech. Appl. Math.* XIX:151-155.

Hadamard, J. (1903). *Leçons sur la Propagation des Ondes et les Équations de l' Hydrody-
namique.* Paris:Hermann.

Hill, R. (1950). *The mathematical theory of plasticity.* Oxford:Clarendon Press.

Hill, R. (1952). On discontinuous plastic states, with special reference to localized necking in
thin sheets. *J. Mech. Phys. Solids* 1:19-30.

Hill, R. (1958). A general theory of uniqueness and stability in elastic-plastic solids. *J. Mech.
Phys. Solids* 6:236-249.

Hill, R. (1959). Some basic principles in the mechanics of solids without a natural time. *J. Mech.
Phys. Solids* 7:209-225.

Hill, R. (1961). Discontinuity relations in mechanics of solids. In Sneddon, I.N. and Hill, R.,
eds., *Progress in Solid Mechanics* II. Amsterdam:North-Holland. 247-276.

Hill, R. (1962). Acceleration waves in solids. *J. Mech. Phys. Solids* 10:1-16.

Hill, R. (1967a). Eigenmodal deformations in elastic/plastic continua. *J. Mech. Phys. Solids*
15:371-386.

Hill, R. (1967b). On the classical constitutive laws for elastic/plastic solids. In Broberg, B., ed.,
Recent Progress in Applied Mechanics, The Folke Odkvist Volume Stockholm:Almqvist &
Wiksell. 241-249.

Hill, R. (1968). On constitutive inequalities for simple materials. *J. Mech. Phys. Solids* 16:229-
242.

Hill, R. (1978) Aspects of invariance in solid mechanics. In Yih, C.-S., ed., *Advances in Applied
Mechanics* 18. New York:Academic Press. 1-75.

Hill, R. and Hutchinson, J. W. (1975). Bifurcation phenomena in the plane tension test. *J. Mech.
Phys. Solids* 23:239-264.

Hill, R. and Rice, J. R. (1973). Elastic potentials and the structure of inelastic constitutive laws.
SIAM J. Appl. Math. 25:448-461.

Horgan, C.O. and Polignone, D.A. (1995). Cavitation in nonlinearly elastic solids: A review. *Appl. Mech. Rev.* 48:471-485.

Huang, K., Hutchinson, J.W. and Tvergaard, V. (1991). Cavitation instabilities in elastic-plastic solids. *J. Mech. Phys. Solids* 39:223-241.

Hueckel, T. (1976). Coupling of elastic and plastic deformation of bulk solids. *Meccanica* 11:227-235.

Hutchinson, J. W. (1973). Post-bifurcation behavior in the plastic range. *J. Mech. Phys. Solids* 21:163-190.

Hutchinson, J. W. and Miles, J.P. (1974). Bifurcation analysis of the onset of necking in an elastic/plastic cylinder under uniaxial tension. *J. Mech. Phys. Solids* 22:61-71.

Kleiber, M. (1984) Numerical study on necking-type bifurcations in void-containing elastic-plastic material. *Int. J. Solids Structures* 20:191-210.

Kleiber, M. (1986) On plastic localization and failure in plane strain and round void containing tensile bars. *Int. J. Plasticity* 2:205-221.

Loret, B. (1992). Does deviation from deviatoric associativity lead to the onset of flutter instability? *J. Mech. Phys. Solids* 40:1363-1375.

Loret, B., Martins, J.A.C. and Simões, F.M.F. (1995). Surface boundary conditions trigger flutter instability in non-associative elastic-plastic solids. *Int. J. Solids Structures* 32:2155-2190.

Loret, B., Prevost, J.H. and Harireche, O. (1990). Loss of hyperbolicity in elastic-plastic solids with deviatoric associativity. *Eur. J. Mechanics-A/Solids* 9:225-231.

Maier, G. and Hueckel, T. (1979). Non associated and coupled flow-rules of elastoplasticity for rock-like materials. *Int. J. Rock Mech. Min. Sci.* 16:77-92.

Mandel, J. (1966). Conditions de stabilité et postulat de Drucker. In Kravtchenko, J. and Sirieys, P.M., eds., *Rheology and Soil Mechanics*. Berlin:Springer. 58-68.

Melan, E. (1938). Zur Plastizität des räumliche Kontinuums. *Ingenieur-Archiv* 9:116-126.

Miles, J.P. (1973). Fluid-pressure eigenstates and bifurcation in tension specimens under lateral pressure. *J. Mech. Phys. Solids* 21:145-162.

Miles, J.P. and Nuwayhid, U.A. (1985). Bifurcation in compressible elastic/plastic cylinders under uniaxial tension. *Appl. Sci. Res.* 42:33-54.

Mróz, Z. (1963). Non-associated flow laws in plasticity *J. de Mechanique* 2:21-42.

Mróz, Z. (1966). On forms of constitutive laws for elastic-plastic solids. *Arch. Mech. Stosowanej* 18:1-34.

Nadai, A. (1931) *Plasticity.* New York:McGraw-Hill.

Nadai, A. (1950) *Theory of flow and fracture of solids.* New York:McGraw-Hill.

Neale, K.W. (1981). Phenomenological constitutive laws in finite plasticity *SM Archives* 6:79-128.

Needleman, A. (1979). Non-normality and bifurcation in plane strain tension or compression. *J. Mech. Phys. Solids* 27:231-254.

Needleman, A. and Ortiz, M. (1991). Effects of boundaries and interfaces on shear-band localization. *Int. J. Solids Structures* 28:859-877.

Nguyen, S.Q. and Triantafyllidis, N. (1989). Plastic bifurcation and postbifurcation analysis for generalized standard continua. *J. Mech. Phys. Solids* 37:545-566.

Nikolaevskii, V.N. and Rice, J.R. (1979). Current topics in non-elastic deformation of geological materials. In Timmerhaus, K.D. and Barber, M.S., eds., *Proocedings of the Sixth AIRAPT Conference: High Pressure Science and Technology.* New York:Plenum. 2:455-464.

Ogden, R.W. (1984). *Non-linear elastic deformations.* Chichester:Ellis Horwood.

Ogden, R. W. (1985). Local and global bifurcation phenomena in plane-strain finite elasticity. *Int. J. Solids Structures* 21:121-132.

Ottosen, N.S. and Runesson, K. (1991). Acceleration waves in elastoplasticity. *Int. J. Solids Structures* 28:135-159.

Petryk, H. (1985a). On energy criteria of plastic instability. In *Plastic Instability, Proc. Considère Memorial* Paris:Ecole Nat. Ponts Chauss. Press. 215-226.

Petryk, H. (1985b). On stability and symmetry conditions in time-independent plasticity. *Arch. Mech.* 37:503-520.

Petryk, H. (1991). The energy criteria of instability in time-independent inelastic solids. *Arch. Mech.* 43:519-545.

Petryk, H. (1992). Material instability and strain-rate discontinuities in incrementally nonlinear continua. *J. Mech. Phys. Solids* 40:1227-1250.

Petryk, H. (1993a). Theory of bifurcation and instability in time-independent plasticity. In Nguyen, Q.S., ed., *CISM Lecture Notes No. 327,* Udine 1991. Wien:Springer. 95-152.

Petryk, H. (1993b). Stability and constitutive inequalities in plasticity. In Muschik, W., ed., *CISM Lecture Notes No. 336,* Udine 1992. Wien:Springer. 255-329.

Petryk, H. (1999). General conditions for uniqueness in materials with multiple mechanisms of inelastic deformation. *J. Mech. Phys. Solids in press.*

Petryk, H. and Thermann, K. (1985). Second-order bifurcation in elastic-plastic solids. *J. Mech. Phys. Solids* 33:577-593.

Prager, W. (1954). Discontinuous fields of plastic stress and flow. In *2nd Nat. Congr. Appl. Mech.,* Ann Arbor, Michigan, 21-32.

Radi, E., Bigoni, D. and Tralli, A. (1999). On uniqueness for frictional contact rate problems. *J. Mech. Phys. Solids* 47:275-296.

Raniecki, B. (1979). Uniqueness criteria in solids with non-associated plastic flow laws at finite deformations, *Bull. Acad. Polon. Sci. ser. sci. tech.* XXVII:391-399.

Raniecki, B. and Bruhns, O.T. (1981). Bounds to bifurcation stresses in solids with non-associated plastic flow law at finite strain. *J. Mech. Phys. Solids* 29:153-171.

Rice, J. R. (1977). The localization of plastic deformation. In Koiter, W.T., ed., *Theoretical and Applied Mechanics.* Amsterdam:North-Holland. 207-220.

Rice, J.R. and Rudnicki, J.W. (1980). A note on some features of the theory of localization of deformation. *Int. J. Solids Structures* 16:597-605.

Rudnicki, J.W. and Rice, J.R. (1975). Conditions for the localization of deformations in pressure-sensitive dilatant materials. *J. Mech. Phys. Solids* 23:371-394.

Runesson, K. and Mróz, Z. (1989). A note on non-associated plastic flow rules. *Int. J. Plasticity* 5:639-658.

Ryzhak, E. I. (1987). Necessity of Hadamard conditions for stability of elastic-plastic solids. *Izv. AN SSSR MTT* (Mechanics of Solids) 99-102.

Ryzhak, E. I. (1993). On stable deformation of "unstable" materials in a rigid triaxial testing machine. *J. Mech. Phys. Solids* 41:1345-1356.

Ryzhak, E. I. (1994). On stability of homogeneous elastic bodies under boundary conditions weaker than displacement conditions. *Q. Jl. Mech. appl. Math.* 47:663-672.

Simões, F.M.F. (1997). Instabilities in non-associated problems of solid mechanics. Ph.D. Thesis, Technical University of Lisbon, *in Portuguese.*

Szabó, L. (1994). Shear band formulation in finite elastoplasticity. *Int. J. Solids Structures* 31:1291-1308.

Thomas, T.Y. (1953). The effect of compressibility on the inclination of plastic slip bands in flat bars. *Proc. Nat. Acad. Sci.* 39:266-273.

Thomas, T.Y. (1961) *Plastic flows and fracture of solids.* New York:Academic Press.

Tomita, Y., Shindo, A. and Fatnassi, A. (1988). Bounding approach to bifurcation point of annular plates with nonassociated flow law subjected to uniform tension at their outer edges. *Int. J. Plasticity* 4:251-263.

Truesdell, C. and Noll, W. (1965). The non-linear field theories of mechanics. In Flügge, S., ed., *Encyclopedia of Physics:*III/3. Berlin:Springer-Verlag.

Tvergaard, V. (1982). Influence of void nucleation on ductile shear fracture at a free surface. *J. Mech. Phys. Solids* 30:399-425

van Hove, L. (1947). Sur l'extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusieurs fonctions inconnues. *Proc. Sect. Sci. K. Akad. van Wetenschappen*, Amsterdam, 50:18-23.

Vardoulakis, I. (1981). Bifurcation analysis of the plane rectilinear deformation on dry sand samples. *Int. J. Solids Structures* 11:1085-1101.

Vardoulakis, I. (1983). Rigid granular plasticity model and bifurcation in the triaxial test. *Acta Mechanica* 49:57-79.

Young, N.J.B. (1976). Bifurcation phenomena in the plane compression test. *J. Mech. Phys. Solids* 24:77-91.

APPENDIX A. A note on coaxiality.

We prove the following *proposition*:

Two symmetric second-order tensors $\mathbf{A}$, $\mathbf{B}$ commute, i.e. $\mathbf{AB} = \mathbf{BA}$ if and only if they possess at least three common, linearly independent, eigenvectors (thus defining a principal reference system).

Proof. The sufficiency is trivial. In fact, let us assume that the two tensors share a principal reference system. Represented in this system, the two tensors evidently commute. Assume now that the two tensors commute. Represented in the principal reference system of $\mathbf{A}$ these two tensors write

$$\mathbf{A} = \sum_{i=1}^{3} \alpha_i \mathbf{a}_i \otimes \mathbf{a}_i \,, \quad \mathbf{B} = \sum_{i,j=1}^{3} \beta_{ij} \mathbf{a}_i \otimes \mathbf{a}_j \,, \tag{A.1}$$

where $\mathbf{a}_i$, $i \in [1,3]$ are the unit eigenvectors of $\mathbf{A}$ and α_i the corresponding eigenvalues, β_{ij} are the components of $\mathbf{B}$ onto $\mathbf{a}_i$, which, due to symmetry, satisfy $\beta_{ij} = \beta_{ji}$.

Imposing the condition $\mathbf{AB} = \mathbf{BA}$ gives:

$$(\alpha_i - \alpha_j)\beta_{ij} = 0 \,, \quad (i \neq j \in [1,3]). \tag{A.2}$$

Condition (A.2) implies that either $\mathbf{a}_i$ and $\mathbf{a}_j$ are eigenvectors of $\mathbf{B}$, so that $\beta_{ij} = 0$, or the characteristic space corresponding to $\mathbf{a}_i$ and $\mathbf{a}_j$ is a plane, so that $\alpha_i = \alpha_j$. In the latter case, it is always possible to choose in this plane a reference system which is principal for $\mathbf{B}$.

□

Remark. 1 The above proof is restricted to dimension 3. Its generalization to dimension n is straightforward.

Remark. 2 It is important to realize that the above proposition *does not imply* that two coaxial tensors share *all* eigenvectors. Take for instance the identity tensor. This is

coaxial to every tensor, but shares only three eigenvectors with any symmetric second-order tensor with distinct eigenvalues.

APPENDIX B. On the metric induced by semi-positive definite tensors.

Let us consider the positive scalar valued function f defined over the space of nth-order tensors as

$$f(\mathbf{A}) = \sqrt{\mathbf{A} \cdot \mathbf{H}[\mathbf{A}]} \geq 0, \tag{B.1}$$

where $\mathbf{A}$ is a generic nth-order tensor and $\mathbf{H}$ is any semi-positive definite $(2 \times n)$th-order tensor (symmetries are not required). In particular, the cases in which $\mathbf{H}$ is a fourth-order tensor ($n = 2$) or second-order tensor ($n = 1$) are particularly relevant here. We begin with a proof of the Cauchy-Schwarz inequality. Take any $\lambda \in \Re$ and consider, for every nth-order tensors $\mathbf{A}$ and $\mathbf{B}$

$$0 \leq [f(\lambda\mathbf{A} + \mathbf{B})]^2 = \lambda^2 \mathbf{A} \cdot \mathbf{H}[\mathbf{A}] + \lambda(\mathbf{A} \cdot \mathbf{H}[\mathbf{B}] + \mathbf{B} \cdot \mathbf{H}[\mathbf{A}]) + \mathbf{B} \cdot \mathbf{H}[\mathbf{B}]. \tag{B.2}$$

It may be observed from (B.2) that the discriminant of the second-order polynomial in λ must be negative or null. This condition yields the Cauchy-Schwarz inequality

$$\frac{|\mathbf{A} \cdot \mathbf{H}[\mathbf{B}] + \mathbf{B} \cdot \mathbf{H}[\mathbf{A}]|}{2} \leq \sqrt{\mathbf{A} \cdot \mathbf{H}[\mathbf{A}]}\sqrt{\mathbf{B} \cdot \mathbf{H}[\mathbf{B}]}. \tag{B.3}$$

Taking now $\lambda = 1$ in (B.2) and using (B.3), the triangle inequality is readily obtained:

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{H}[\mathbf{A} + \mathbf{B}] \leq \mathbf{A} \cdot \mathbf{H}[\mathbf{A}] + |\mathbf{A} \cdot \mathbf{H}[\mathbf{B}] + \mathbf{B} \cdot \mathbf{H}[\mathbf{A}]| + \mathbf{B} \cdot \mathbf{H}[\mathbf{B}] \tag{B.4}$$

$$\leq \left(\sqrt{\mathbf{A} \cdot \mathbf{H}[\mathbf{A}]} + \sqrt{\mathbf{B} \cdot \mathbf{H}[\mathbf{B}]}\right)^2. \tag{B.5}$$

As a consequence of (B.4), we note that

$$f(\mathbf{A}) = f(\mathbf{A} \pm \mathbf{B} \mp \mathbf{B}) \leq f(\mathbf{A} \pm \mathbf{B}) + f(\mathbf{B}), \quad f(\mathbf{B}) = f(\mathbf{B} \pm \mathbf{A} \mp \mathbf{A}) \leq f(\mathbf{A} \pm \mathbf{B}) + f(\mathbf{A}), \tag{B.6}$$

which imply

$$\sqrt{(\mathbf{A} \pm \mathbf{B}) \cdot \mathbf{H}[\mathbf{A} \pm \mathbf{B}]} \geq |\sqrt{\mathbf{A} \cdot \mathbf{H}[\mathbf{A}]} - \sqrt{\mathbf{B} \cdot \mathbf{H}[\mathbf{B}]}|. \tag{B.7}$$

Finally, we note that for every $\lambda \in \Re$

$$f(\lambda\mathbf{A}) = |\lambda|f(\mathbf{A}), \tag{B.8}$$

therefore, function f defines a seminorm on the space of nth order-tensors (a norm when $\mathbf{H}$ is positive definite).