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ON UNIQUENESS AND STRAIN LOCALIZATION IN PLANE STRAIN AND PLANE STRESS ELASTOPLASTICITY

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Introduction

The interest in non-uniqueness of the rate problem in small strain elastoplasticity [1–6] has been renewed by recent attempts to numerically simulate failure in brittle materials such as concrete, rock and soil [4,7-9]. For these materials, diffused and localized bifurcation modes are commonly attribuited to strain softening and/or plastic strain rate non-normality developing without geometrical non-linearities, i.e. at small displacement gradients. It is believed that constitutive laws for these materials can only be used when the deformation of a representative element is homogeneous and unique. However, this is not the case, e.g. when localization of deformation occurs. After the localization, the behavior should be characterized in different terms, such as fracture propagation in rock and concrete or force – displacement relationship at the shear band in soils [10]. Hence, it becomes important to define the threshold between the homogeneous deformations range and the non-homogeneous range.

Work in this area concentrated on deriving criteria for loss of uniqueness of velocities. The criteria may be either overall, i.e. with respect to the boundary value problem [11-14] or local, i.e. with respect to behavior of an infinitesimal element [15-18]. Two local criteria will be discussed here: that of positiveness of the second work density (see e.g. [19-21]), and that of localization of deformation into a planar band [10,22-25]. In general these criteria have different meanings. The criterion of positiveness of the second order work density is, in view of its relation to Hill's overall criterion [11], a sufficient condition for uniqueness. The other criterion [23] is connected to a particular deformation mechanism and thus it is a criterion for bifurcation by strain localization into the specific form of a planar band. The above means that unless the two criteria coincide, other modes of bifurcation may appear in a straining process, after the second order work becomes zero, but before the localization criterion is satisfied. Such a circumstance for plane strain has been indicated at large displacement gradients in [26] in the context of properties of the governing differential equations in their elliptic, parabolic and hyperbolic regimes. In particular for finite strain uniaxial compression and extension of a rectangualr block in plane isochoric motion and obeying a particular constitutive law it was shown [27] that diffuse necking can precede shear band formation.

In this paper associative plasticity with smooth yield surface allowing for hardening and/or softening is addressed under the assumption of small displacement gradients. Moreover, the restriction is imposed on current stress magnitude which is required to be significantly less than the current elasticity modulus. The latter restriction is usually satisfied in moderately brittle materials, in which the elasticity does not drop significantly due to damage. On the other hand, such a drop necessarily leads to a non-associativity due to elastic-plastic coupling [14,20] and is dealt with separately [28]. The above approximations are of undubious practical significance for the analysis of bifurcation in brittle and frictional materials, such as rock, concrete and soil and are widely accepted in this context [1, 2, 4, 7–9, 29].

By specializing the known criterion for localization [23] to the conditions of plane strain and plane stress, explicit expressions for the critical hardening modulus for localization and for the inclination of the band are found. This is a numerically much more amenable form of the coupled condition for localization given in [24]. It is also concluded that in plane stress, the critical modulus is always null. On the other hand, developing the criterion for the positiveness of the second order work density [20] for plane strain and plane stress, it is seen that this criterion coincides in the above cases with the localization criterion [24]. Consequences of this coincidence are then discussed in terms of uniqueness of the rate problem. Non-associative plasticity is dealt with elsewhere [28].

Outline of the approaches to bifurcation

Rate independent behavior of the isotropic, homogeneous material which is discussed here is characterized by a linear elastic deformation, a smooth yield surface, the associative plastic flow rule, and a hardening/softening rule. The stress-strain relationship in terms of rates is:

$$\dot{\sigma} = E : \dot{\varepsilon} - \Lambda(\dot{\varepsilon})\, E : Q\,, \tag{1}$$

subjected to the conditions:

$$\Lambda(\dot{\varepsilon}) \geq 0, \quad \dot{f} \leq 0\,, \quad \dot{f}\,\Lambda(\dot{\varepsilon}) = 0\,, \tag{2}$$

where σ and ε are the second order tensor of stress and strain respectively, a dot denotes a rate, E is the fourth order isotropic elastic tensor, $f(\sigma,k) = 0$ describes a yield surface related to stress and the plastic hardening parameter k (k depends on the plastic strain history), Q is the second order tensor of the yield surface gradient, while the plastic multiplier $\Lambda(\dot{\varepsilon})$ is defined through:

$$\Lambda(\dot{\varepsilon}) = \frac{Q : E : \dot{\varepsilon}}{H + Q : E : Q}\,, \qquad H = -\frac{\partial f}{\partial \varepsilon^p} : Q\,, \tag{3}$$

where H is the plastic hardening modulus.

In what follows the elastoplastic rate stiffness tensor D will be used, obtained by combining (1) and (3):

$$\dot{\sigma} = D : \dot{\varepsilon}, \qquad \text{subjected to (2)}. \tag{4}$$

A sufficient criterion for uniqueness of incremental response of an elastoplastic body is positiveness of the overall second order work [11,12]:

$$\frac{1}{2}\int_V \Delta\dot{\sigma} : \Delta\dot{\varepsilon}\, dv > 0, \tag{5}$$

provided that $\Delta\dot{\sigma}$ and $\Delta\dot{\varepsilon}$ satisfy null traction rate and velocity conditions at specified portions of the boundary of the body of volume V. Condition (5) has been shown [10] to be equivalent to strain localization in a restricted class of boundary value problems which are kinematically constrained on the whole boundary.

The condition (5) is fulfilled if the second order work density is positive at every point of the body [11]. This corresponds to the requirement of the positive definiteness of the elastoplastic tensor D.

Loss of positive definiteness of the elastoplastic tensor D may be characterized by a critical hardening modulus which has been shown [19-21] to be zero for associative plasticity. The deformation at which zero second order work occurs is:

$$\dot{\varepsilon} = \alpha Q, \qquad \forall\, \alpha \in \mathbf{R} \tag{6}$$

Strain localization into a planar band takes place when a strain rate discontinuity normally to the plane of the band occurs so that Maxwell compatibility conditions are satisfied, [16]:

$$[\dot{\varepsilon}] = \frac{1}{2}(g \otimes n + n \otimes g) \tag{7}$$

where [ ] indicates the discontinuity, $\otimes$ denotes a tensorial product, n is the unit vector normal to the band and g is the vector which defines the discontinuity in the velocity derivatives. g normal to n describes a non dilatant shear band; if g is parallel to n, the localization occurs in the form of a split. The eigenvalues of the strain rate jump tensor are:

$$[\dot{\varepsilon}_{\mathrm{I}}] = \frac{1}{2}(|g| + gn), \quad [\dot{\varepsilon}_{\mathrm{II}}] = 0, \quad [\dot{\varepsilon}_{\mathrm{III}}] = \frac{1}{2}(-|g| + gn), \tag{8}$$

where a scalar product of two vectors x and y is denoted by xy.

The critical hardening modulus, corresponding to strain localization into a planar band, was found [24] to correspond to the solution of the constrained maximization problem:

$$H^{l}_{cr} = \max_{n} \{2G(2nQQn - (nQn)^2 - Q : Q - \frac{\nu}{1-\nu}(nQn - trQ)^2)\}, \tag{9}$$

subjected to $|n| = 1$,
where G is the elastic shear modulus, ν is the Poisson's ratio and tr denotes the trace of a tensor.

The velocity discontinuity vector g, for the n which maximizes (9), is:

$$g = 2nQ - \frac{1}{1-\nu}(nQn)n + \frac{\nu}{1-\nu}(trQ)n\,. \tag{10}$$

The critical hardening modulus as expressed through eq (9) is coupled with the vector n, and numerical procedures were developed (see e.g. [4,8]) to obtain its value.

Loss of positiveness of second order work density in plane strain and in plane stress

Plane Strain

From eq (6) one can see that in the general case, until at least one of the eigenvalues of the yield surface gradient Q is zero, the deformation rate for which zero second order work density is attained, is three-dimensional. In plane strain, the strain rate are:

$$\dot{\varepsilon}_{11} \neq 0, \quad \dot{\varepsilon}_{12} \neq 0, \quad \dot{\varepsilon}_{22} \neq 0, \quad \dot{\varepsilon}_{13} = \dot{\varepsilon}_{23} = \dot{\varepsilon}_{33} = 0, \tag{11}$$

and the second order work may be expressed as:

$$W = \frac{1}{2}\dot{\varepsilon} : D' : \dot{\varepsilon}, \tag{12}$$

where D' is the rate elastoplasticity tensor in plane strain:

$$D' = E' - \frac{N \otimes N}{H - H_0}, \tag{13}$$

and E' is the rate elasticity tensor in plane strain. In eq (13) the scalar H_0 is defined as:

$$H_0 = -Q_{ij}\, E_{ijkh}\, Q_{kh} \quad (i, j, k, h = 1, 2, 3) \tag{14}$$

and the second order tensor $N \in \mathbf{R}^2 \otimes \mathbf{R}^2$ as:

$$N_{kh} = Q_{ij}\, E_{ijkh} \qquad (i,j = 1,2,3; \quad k,h = 1,2) \tag{15}$$

For every purely elastic, incremental process ($\Lambda(\dot{\varepsilon}) = 0$) second order work density is always positive because D' reduces to E', which is positive definite. For a plastic process, loss of positive definiteness of plane strain tensor D' may be found through a strictly convex minimization of the second order work density (following [20]):

$$\min_{\varepsilon} \frac{1}{2} \{\dot{\varepsilon}_{ij} E'_{ijkh} \dot{\varepsilon}_{kh} - \dot{\varepsilon}_{ij} N_{ij}\}, \quad (i, j, h, k = 1, 2) \tag{16}$$

subjected to:

$$\dot{\varepsilon}_{ij} N_{ij} = H - H_0 . \quad (i,j = 1,2)$$

In (16) it has been assumed that $\Lambda(\dot{\varepsilon}) = 1$ because only the sign of second order work matters for positive definiteness of D'.

The optimal vector for the minimization problem (16) turns out to be:

$$\dot{\varepsilon}^c = \left(\frac{1}{2} - \omega\right) C' : N, \quad \omega = \left(\frac{1}{2} N : C' : N - H + H_0\right) (N : C' : N)^{-1}, \tag{17}$$

where ω is a Lagrangean multiplier and C' is the plane strain elastic compliance tensor, i.e.:

$$C'_{ijkh} = -\frac{\nu}{2G} \delta_{ij} \delta_{kh} + \frac{1}{4G} (\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}), \quad (j,i,k,h = 1,2)$$

and δ_{ij} is the Kronecker delta.

The critical hardening modulus H^u_{cr} corresponding to loss of positive definiteness of the plane strain tensor D', is obtained by imposing that the density of second order work produced by deformation (17) be zero:

$$H^u{}_{cr} = 2G \{-2 Q_{i3} Q_{i3} + (1 - \nu) Q^2_{33}\}. \quad (i = 1,2,3) \tag{18}$$

Hence, the deformation (17) corresponding to zero second order work is:

$$\dot{\varepsilon} = C' : N \quad \text{or} \quad \dot{\varepsilon}_{ij} = \nu Q_{33} \delta_{ij} + Q_{ij} \quad (i,j = 1,2). \tag{19}$$

Plane Stress

Deformation in plane stress is not constrained. Thus, the loss of positiveness of the second order work occurs, as in the general 3-D case, when the hardening modulus is equal to zero. Also the corresponding strain is that in (6), with the difference that the vector Q has, in general, only two non-zero components in the space of principal stresses.

Localization criteria in plane strain and plane stress

Plane Strain

In plane strain, as opposed to the general case (eqs 7-10), the shear band position is constrained, viz. it must develop perpendicular to the plane of deformation. Hence, the expression for the critical modulus (9) still holds, if vectors n and g belong to the plane of deformation.

By introducing a Lagrangian multiplier β, the constrained maximization problem (9) is reduced to:

$$L(n,\beta) = 2G\,\{2nQQn - (nQn)^2 - Q:Q - \frac{\nu}{1-\nu}(nQn - trQ)^2 - \beta\,(nn-1)\} \tag{20}$$

The maxima are characterized by:

$$nQQ - \frac{1}{1-\nu}(nQn)\,nQ + \frac{\nu}{1-\nu}(tr\,Q)\,n\,Q = \frac{\beta}{4G}\,, \tag{21}$$

By referring eq (21) to the principal axes of stress and keeping in mind that for plane strain condition at least one of the components of n must vanish, the following two equations are obtained:

$$Q_i^2 - \frac{1}{1-\nu}\{n_i^2\,(Q_i - Q_j) + Q_j\}\,Q_i + \frac{\nu}{1-\nu}(Q_i + Q_j + Q_{\mathrm{III}})\,Q_i = \frac{\beta}{4G}\,, \tag{22}$$

where repeated indices are not summed, $i \neq j$ and the range of indices i and j is I to II, which denote components along principal direction of stress in the deformation plane.

Examining hardening modulus for the cases where $n_i \neq n_j \neq 0$, $n_{\mathrm{III}} = 0$ or $n_i = 1$ $n_j = n_{\mathrm{III}} = 0$, it may be seen that the maximum is obtained for:

$$n_i^2 = \frac{Q_i + \nu Q_{\mathrm{III}}}{Q_i - Q_j}\,, \qquad i \neq j\,, \qquad i,j = \mathrm{I},\mathrm{II} \tag{23}$$

The corresponding critical hardening modulus is:

$$H_{cr}^l = -\,2G(1+\nu)\,Q_{II}^2 \tag{24}$$

Finally, vector g is found to be:

$$g_i = (Q_i - Q_j)\,n_i\,, \qquad g_{\mathrm{III}} = 0\,. \quad \text{(indices not summed)} \tag{25}$$

It should be noted that in this solution de-coupled expressions for the critical modulus H_{cr}^l and the vector n are obtained. This allows us to avoid complex numerical procedures necessary when the general coupled form is used [4,8]. The same result was obtained in [23] for Huber-von Mises and Drucker-Prager yield surfaces. It is worth noting that the critical hardening modulus for localization is always non-positive, as easily seen from (24). It is also seen that if eq (18) is expressed in the principal stress reference system, eq (24) becomes identical to (18) . Also if expressions (23) and (25) are substituted into eqs (8), the deformation rate in the band is found identical to that in (19). This indicates that for small displacement gradients, in plane strain for associative plasticity, loss of positiveness of the second order work density is equivalent to strain localization into planar band.

Remark: perfect plasticity.

i. As well-known, for a generic boundary value problem, the strain rate, at collapse, is entirely plastic. Thus Q_{III} must vanish at collapse and therefore from eq (24), $H_{l;cr} = 0$ is found. Hence, in the case of perfect plasticity in plane strain, collapse may occur with shear band formation.

ii. In the case of zero volumetric plastic strain, the additional condition $Q_i = -Q_j$ must hold at collapse. From (23) it is seen that the shear band is inclined at 45° with respect to the principal maximum stress direction.

Plane Stress

In plane stress the eq (8b) becomes immaterial. The critical hardening modulus is obtained by the maximization problem:

$$H^{l}_{cr} = \max_{n} 2G\,\{2nQQn - (nQn)^2 - Q : Q - \nu\,(nQn - trQ)^2\}, \tag{26}$$

subjected to $|n| = 1$, in which however n has two components and Q has only two eigenvectors.

The vector g is now:

$$g = 2nQ - (1 + \nu)\,(nQn)\,n - \nu\,(trQ)\,n\,, \tag{27}$$

Following the same procedure as in plane strain, the direction of the normal to the band is given by:

$$n_i^2 = \frac{Q_i}{Q_i - Q_j}\,, \tag{28}$$

where i = I and j = II or i = II and j = I indicate principal components of Q. The band inclination given by (28) corresponds to a null critical hardening modulus, $H_{l;cr} = 0$. Vector g finally results in:

$$g_i = (Q_i - Q_j)\,n_i\,, \quad \text{(indices not summed)} \tag{29}$$

Thus in plane stress loss of positiveness of the second order work density and strain localization coincide, as in plane strain.

Remark: perfect plasticity

i. For Huber-von Mises yield criterion, the shear band, in the case of thin sheets extension, results from eq (28) to be inclined at 54.73° to the direction of extension. This coincides with a well-known result (see, e.g. [16]).

Discussion

Associative plasticity, under the hypothesis of small displacement gradients has been considered. The following results concerning plane stress and plane strain conditions have been reached:

i. Starting from the condition for localization given in [24] and using the procedure similar to that in [23] one finds an explicit solution for the critical hardening modulus for localization and the corresponding inclination of the band. The result obtained here is an extension of that in [23] to any smooth yield surface.

ii. For the plane stress the critical moduls for localization is to be equal to zero. For plane strain the result of [10], that the critical hardening modulus cannot be positive, was confirmed.

iii. The criterion for positiveness of second order work was obtained through a minimization procedure constrained by the plane strain condition. Positiveness is lost at the critical hardening modulus identical to that for strain localization. Moreover, the strain rate at which the second order work vanishes is the same as that inside the shear band. A similar result has been reached for plane stress.

From the previous results it follows that:

α. Loss of positiveness of second order work and localization (and thus loss of ellipticity and vanishing speed of acceleration waves) are equivalent .

β. Non-uniqueness cannot occur in a boundary value problem before the condition for localization into a planar band is met at least at one point of the body.

This last result was obtained [10] for the very restrictive case of all-around displacement boundary conditions. It is immediate in the case of plane isochoric motion. It must be emphasised that the above equivalence holds only under the assumption of small displacement gradients, as opposed to finite displacement gradients. In the latter case an infinite number of diffuse bifurcation modes can precede localization [26,27], as already indicated in the Introduction. At small strain and negligible rotations, localization and loss of positiveness of second order work may be regarded as forms of instability due to constitutive properties. The fact that for small displacement gradients, loss of positiveness of second order work coincides with the localization condition, has a pratical meaning for numerical modeling. It allows us to use constitutive laws up to the point of localization, without a need for search for other prior bifurcation modes.

γ. An idealization of a three-dimensional problem made in terms of plane strain condition, may not be conservative from the point of view of a bifurcation analysis. Strain localization in plane strain is predicted to occur after (in terms of the hardening modulus) uniqueness is lost in the three-dimensional deformation problem.

The relation between localization and loss of uniqueness in the case of three-dimensional deformation and nonassociative plasticity is discussed elsewhere [28].

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ADDENDUM TO "ON UNIQUENESS AND STRAIN LOCALIZATION IN PLANE STRAIN AND PLANE STRESS ELASTOPLASTICITY" *

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In the paper the implicit assumption was made that eqns (23) and (28) are admissible solutions for the inclination of the shear band in plane strain and plane stress respectively. This assumption is satisfied when:

$$\mathrm{sign}(Q_I + \nu\, Q_{III}) \neq \mathrm{sign}(Q_{II} + \nu\, Q_{III}), \qquad \text{for plane strain} \tag{A.1}$$

$$\mathrm{sign}(Q_I) \neq \mathrm{sign}(Q_{II}), \qquad \text{for plane stress.} \tag{A.2}$$

Conditions (A.1) and (A.2) are equivalent to the requirement that the deformation rate components that make the second order work equal to zero have different signs. Conditions (A.1) and (A.2) are, for istance, always fulfilled, in the case of Huber-von Mises yield function in the presence of isotropic hardening, assuming $\sigma_I \geq \sigma_{III} \geq \sigma_{II}$ in plane strain and excluding biaxial tension or compression in plane stress. When second order work becomes zero but conditions (A.1) and (A.2) are not fulfilled, then the critical hardening modulus for localization corresponds to a band perpendicular to the axis I or II, for $Q_I \neq Q_{II}$ only. Viceversa $Q_I = Q_{II}$ corresponds to an infinite number of shear bands of indeterminate inclination. In both cases the critical hardening modulus becomes, in plane strain:

$$H^1_{cr} = \max_{i = I,II} \; - 2\,G\,\{(Q_i + \nu Q_{III})^2 / (1 - \nu) + (1 + \nu)\, Q^2_{III}\}, \tag{A.3}$$

corresponding to $n_i = 0$ if $Q_I \neq Q_{II}$ or to $n_i \in [0,1]$ if $Q_I = Q_{II}$. Analogously, in the case of plane stress:

$$H^1_{cr} = \max_{i = I,II} \; - 2\,G(1 + \nu)\, Q^2_i, \tag{A.4}$$

corresponding to $n_i = 0$ if $Q_I \neq Q_{II}$ or to $n_i \in [0,1]$ if $Q_I = Q_{II}$.

Note that moduli (A.3) and (A.4) are always smaller than the moduli corresponding to the vanishing of the second order work. Thus if localization does not coincide with the vanishing of the second order work, then there are two possibilities only (excluding reversals of the sign of the second order work in a loading process):

- the band forms perpendicular to the axis I or II, if $Q_I \neq Q_{II}$,
- the number of the possible bands becomes infinite, if $Q_I = Q_{II}$.

The solutions of the constrained maximization problem (9) [or (26)] include the possibility that localization occurs at the hardening modulus (A.3) [or (A.4)]. This is in fact the case when condition (A.1) [or (A.2)] is not fulfilled. Consequently, in a loading process, the zero second order work corresponds to localization, only if condition (A.1) [or (A.2)] is fulfilled. Thus, conclusion (iii.) in the paper is less general than it may appear. In fact the hypothesis that condition (A.1) [or (A.2)] be a-priori verified at the vanishing of the second order work is implicitely assumed.

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