

THE QUASI-STATIC FINITE CAVITY EXPANSION IN A NON-STANDARD ELASTO-PLASTIC MEDIUM

D. BIGONI and F. LAUDIERO

Istituto di Scienza delle Costruzioni, University of Bologna, Italy

(Received 4 August 1988; and in revised form 21 April 1989)

Abstract—A unified approach is presented for the analysis of the finite static expansion of a spherical or cylindrical cavity in an indefinite elastic-perfectly plastic medium. Mohr-Coulomb yield criterion is adopted with an arbitrary degree of non-associativity of the volumetric dilatational component of the plastic flow. This idealization makes the present analysis particularly appropriate for rock-like materials. The general solution obtained requires a numerical integration over the plastic zone. Some numerical examples, referring to both the spherical and the cylindrical cavity emphasize the determining role of the elastic deformations in the plastic region.

NOTATION

- a radius of the cavity at time t
- a_0 radius of the cavity at time $t = 0$
- b outer radius of the plastic zone
- c parameter identifying spherical ($c = 2$) or cylindrical ($c = 1$) cavity
- E Young modulus of elasticity
- k cohesion
- \mathbf{N} matrix collecting the direction cosines of the outward normals to the yield surface
- \mathbf{N}_D matrix giving the deviatoric part of the plastic flow
- \mathbf{N}_V matrix giving the volumetric part of the plastic flow (associated flow rule only)
- \mathbf{P} matrix giving the direction of the plastic strain vector (non-associated flow rule)
- \mathbf{P}_V matrix giving the volumetric part of the plastic flow (non-associated flow rule)
- p internal pressure
- p_0 hydrostatic pressure acting throughout the medium
- p_1 first yielding internal pressure
- r material coordinate
- r_0 r at time $t = 0$
- t time
- $\mathbf{U}_{m \times n}$ $m \times n$ matrix having unit entries
- u radial displacement
- u_1 first yielding displacement at the inner surface
- Y unconfined uniaxial compression strength
- $1 + \alpha$ unconfined uniaxial compression over tension strength
- β parameter ruling the degree of non-associativity
- Γ non-dimensional coefficient, see equation (39)
- ϵ strain vector
- $\dot{\epsilon}^P$ plastic strain rate vector
- $\dot{\epsilon}_{vol}^P$ plastic volumetric rate
- η non-dimensional coefficient, see equation (33)
- Θ parameter governing the volumetric part of the plastic flow
- λ activation intensity vector of the yield surface bounding planes
- ν Poisson ratio
- σ stress vector
- ϕ angle of shearing resistance
- ξ, ψ auxiliary variables, see equations (37)
- Ω non-dimensional parameter, see equation (23)
- ω non-dimensional coefficient, see equation (34)
- $()^T$ means transpose
- $(\dot{ })$ means rate

1. INTRODUCTION

The problem of the expansion of a spherical or a cylindrical cavity in an elasto-plastic infinite medium has been extensively studied over the last forty years. There are several

engineering problems concerning the loading, by internal pressure, of a thick-walled structure approaching to an infinite medium; e.g. problems of thick spherical or cylindrical containers subjected to the procedure known as auto-frettage as well as problems of deep-punching or explosions in metals, soils or rocks [1].

The thick-walled tube under internal pressure has been firstly solved by Nádai [2] assuming small strains and von Mises yield condition and neglecting the elastic volumetric change in the plastic zone. The elasto-plastic finite expansion of a spherical and a cylindrical cavity in an infinite medium was later solved [3] assuming Tresca yield condition and still neglecting the elastic deformation in the plastic zone. The latter assumption was subsequently removed to analyse the expansion of a cylindrical cavity, assuming small strains and Tresca yield condition [4]. In this paper the usual assumption that the longitudinal stress were the average of the principal stress components is firstly substituted by a more relaxed condition, i.e. the longitudinal stress is the intermediate principal stress component. Hill [5] gave a general solution of the finite expansion of the spherical cavity and a good approximation for the cylindrical cavity assuming Tresca yield condition. Chadwick [6] gave the analogous solution for the spherical cavity adopting Mohr–Coulomb yield criterion. It is worth underlining the two different approaches of the two authors since Hill integrates the displacement velocities directly whereas Chadwick integrates the strain increments introducing a finite logarithmic strain. Later Chadwick [7] gave a closed-form approximate solution for the expansion of a spherical cavity using Mohr–Coulomb yield criterion and making an approximate assumption about the elastic deformation in the plastic zone. He also studied the closely related problem of a thick-walled spherical shell loaded by an outer uniform pressure assuming elastic/work-hardening plastic behaviour [8]. In the dynamic range, pore-collapse analysis for an incompressible elasto-plastic material has also been developed [9].

Salençon [10] firstly analysed the expansion of a spherical or cylindrical cavity in an infinite elasto-plastic non standard medium obeying Mohr–Coulomb yield criterion. For the case of the cylindrical cavity the longitudinal plastic and elastic strain components were assumed to vanish rather than the sum of the two, which makes both the deviatoric and the volumetric plastic flow non associated. Even though such a general model is very appealing, it must be noted that the consequences of a non-associated deviatoric plastic flow are not discussed. Materials exhibiting a plastic strain rate vector deviating from the outward normal to the yield surface in a direction independent of that of the stress rate vector have been examined by several researches in the context of limit analysis [11–13] as well as in the study of incremental stress–strain relations [14–17]. Lack of normality may generate non-uniqueness of incremental solution and material instability in Hill's sense [18] even in the presence of work-hardening. Conditions to be imposed upon the constitutive laws in order to ensure response uniqueness and stability have been determined in [14–17, 19].

The expansion of a spherical or a cylindrical cavity in an infinite medium has been reconsidered by many authors in the context of soil mechanics. In [20] a non-associated flow rule is adopted in which the average plastic (negative) volumetric change is to be experimentally determined.

In [21] the decrease of the angle of shearing resistance with the mean normal stress is considered. In [22] a strain softening model is proposed for a purely cohesive medium. In [23, 24] the expansion of a cylindrical cavity in a dilatant soil is analysed. The same problem is studied in [25] adopting a non-linear stress–strain relation in the elastic zone. All of these authors neglect the elastic deformations in the plastic zone. This restriction is removed in [26] where a model quite similar to that of Salençon [10] is adopted. In this paper, the limit pressure for the case of finite strains is calculated for some values of the relevant parameters even though full solutions are not obtained.

In the present paper a unified approach is proposed for the analysis of the quasi-static finite expansion of a spherical or cylindrical cavity in an indefinite elastic perfectly plastic medium. A Mohr–Coulomb yield condition with non-associated flow rule is assumed (non standard material) which makes it possible to consider plastic volumetric changes less than those considered by the associated law. The above idealization is significant for many media

of engineering importance, particularly rock-like materials whose plastic volumetric changes are generally over-estimated [27, 28].

The non-associativity of the flow-rule has been given experimental evidence in [29, 30] and has been restricted to the volumetric plastic flow in [31–35]. In the assumed flow rule, the deviatoric plastic flow is associative, whereas the volumetric, non-associative, plastic flow is ruled by a parameter Θ . As a particular case, the associative flow rule is recovered. In the proposed formulation the elastic deformations in the plastic zone are fully considered.

As far as the cylindrical cavity is concerned, the longitudinal normal stress is assumed to coincide with the mean of the other two principal stresses. This generally accepted assumption [2, 3, 5, 20, 22, 36] is slightly more restrictive than that which postulates that the longitudinal principal stress is intermediate with respect to the other two [4, 37, 38].

A general solution is obtained which requires a numerical integration over the plastic zone; subsequently, a closed-form solution is derived by neglecting the elastic deformations in the plastic region. A comparison is made between the exact and the approximate solution in a quite wide range of the relevant parameters, showing the increasing role of the elastic deformations as the plastic zone spreads into the indefinite medium. Referring to the same material parameters, the increasing strength of the medium is underlined as the incremental plastic strain vector tends to be normal to the yield surface.

2. BASIC STATEMENTS

In the following, the problems of the expansion of the spherical and the cylindrical (of infinite length) cavities will be analysed simultaneously. For this purpose, in the governing equations, a parameter c will be introduced which takes two alternative values, i.e. $c = 1$ for a cylindrical cavity and $c = 2$ for a spherical cavity. An unbounded, homogeneous medium is considered made of an ideal linearly elastic–perfectly plastic, isotropic material. Mohr–Coulomb yield criterion is assumed with non-associated flow rule which makes it possible to select the appropriate plastic volumetric change depending on the specific material properties. A hydrostatic pressure p_0 acts through the medium whereas an additional increasing pressure p is superimposed inside the cavity which expands from an initial radius a_0 to a radius a . Consequently, a plastic region spreads into the medium whose boundary, for reasons of symmetry, is homothetic to the original shape of the cavity. The outer radius of the plastic zone, at any moment, is denoted by b . A spherical $O(r, \vartheta, \psi)$ and cylindrical $O(r, \vartheta, z)$ reference system measured about the centre of the cavity, are assumed each for the relevant problem.

Two general boundary conditions hold at any time, i.e.

$$\sigma_r(a) = -p - p_0 \quad (1)$$

$$\lim_{r \rightarrow \infty} \sigma_r(r) = -p_0.$$

Because of the symmetry, the equilibrium equations reduce to the single relation:

$$\frac{d\sigma_r}{dr} + c \frac{\sigma_r - \sigma_\vartheta}{r} = 0. \quad (2)$$

In the case of the spherical cavity, the symmetry implies $\sigma_\vartheta = \sigma_\psi$, whereas for the cylindrical cavity, the generally accepted assumption [2, 3, 5, 20, 22, 36] is introduced

$$\sigma_z = \frac{\sigma_r + \sigma_\vartheta}{2}, \quad (3)$$

instead of the more relaxed condition:

$$\sigma_r > \sigma_z > \sigma_\vartheta, \quad (4)$$

adopted in connection with Tresca yield criterion only [4, 36–38] so as to give null longitudinal plastic strains.

Mohr–Coulomb yield criterion defines an admissible stress range specified by the six simultaneous linear inequalities:

$$\mathbf{AN}^T \boldsymbol{\sigma} - Y \mathbf{U}_{3 \times 1} \leq \mathbf{0}, \quad (5)$$

where $\boldsymbol{\sigma}$ is the vector of the principal stress components, $\mathbf{U}_{m \times n}$ is a $m \times n$ matrix having unit entries and \mathbf{N} is a 3×6 matrix whose columns collect the direction cosines of the outward vectors normal to the six planes bounding the admissible stress region given by equation (5), i.e.

$$\mathbf{N} = \frac{1}{A} \begin{bmatrix} 1+\alpha & 1+\alpha & -1 & 0 & -1 & 0 \\ -1 & 0 & 1+\alpha & 1+\alpha & 0 & -1 \\ 0 & -1 & 0 & -1 & 1+\alpha & 1+\alpha \end{bmatrix}. \quad (6)$$

Moreover:

$$A = \sqrt{\alpha^2 + 2\alpha + 2}, \quad \alpha = \frac{2 \sin \phi}{1 - \sin \phi}, \quad Y = \frac{2k \cos \phi}{1 - \sin \phi}, \quad (7)$$

where Y is the uniaxial compressive strength, $1 + \alpha$ is the ratio of the uniaxial compressive to the tensile strength and finally k and ϕ represent the cohesion and the angle of shearing resistance of the material respectively.

According to Koiter [39] assumption of independent yielding modes and assuming a non-associated flow rule, the plastic strain rate vector may be written as follows:

$$\dot{\boldsymbol{\epsilon}}^p = \mathbf{P} \dot{\boldsymbol{\lambda}}, \quad (8)$$

where $\boldsymbol{\lambda}$ assembles the activation intensities of the six planes bounding the region given by equation (5) and \mathbf{P} is a 3×6 matrix whose columns yield the directions of the plastic strain rate vector. In the following \mathbf{P} is selected independent of the state variable $\boldsymbol{\sigma}$ and such that the deviatoric plastic flow only be associative [34, 35]. In fact, matrix \mathbf{N} can be split as follows:

$$\mathbf{N} = \mathbf{N}_D + \mathbf{N}_V \quad (9)$$

where

$$\mathbf{N}_V^T = \frac{1}{3} \mathbf{N}^T \mathbf{U}_{3 \times 3}, \quad (10)$$

and, by comparison with equation (6):

$$\mathbf{N}_V = \frac{\alpha}{3A} \mathbf{U}_{3 \times 6}. \quad (11)$$

Hence, \mathbf{P} will be written as:

$$\mathbf{P} = \mathbf{N}_D + \mathbf{P}_V, \quad (12)$$

with

$$\mathbf{P}_V = \frac{\Theta}{3A} \mathbf{U}_{3 \times 6}. \quad (13)$$

The parameter Θ rules the volumetric component of the plastic deformation, i.e. when $\Theta = \alpha$, the associative flow rule is recovered; when $\Theta = 0$, no volumetric deformation is developed; otherwise any other intermediate (non-associative) case is obtained.

When a compression test of a cylindrical specimen is considered and according to Haar–Karman hypothesis radial and circumferential stresses are assumed to be coincident [40], making use of equations (5), (6), (8), (9), (11), (12) and (13) and performing some calculations yield a relation between the plastic volumetric change $\dot{\epsilon}_{vol}^p$, the axial strain $\dot{\epsilon}_a^p$ and Θ :

$$\frac{\dot{\epsilon}_{vol}^p}{\dot{\epsilon}_a^p} = \frac{3\Theta}{\Theta - 3 - \alpha}. \quad (14)$$

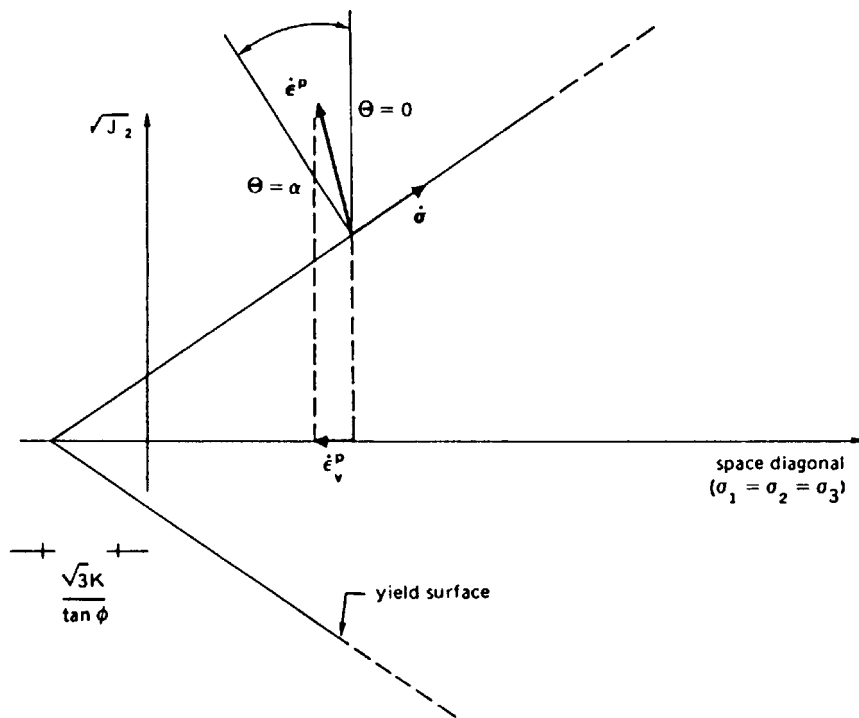


FIG. 1. Non-associated flow law; direction of $\dot{\epsilon}^p$ independent of $\dot{\sigma}$.

It can be easily checked, see Fig. 1, that for any case of practical interest, say $0 \leq \Theta \leq \alpha$, in a monotonically increasing loading path, the scalar product $\dot{\sigma} \dot{\epsilon}^p$ turns out to be always positive and, consequently, the second-order work density is positive as well. Hence, the sufficient condition for material stability proposed in [18] is satisfied for the considered loading path.

3. THE CALCULATION OF THE STRESSES

As the pressure p internal to the cavity increases from the zero value, the whole medium behaves at first in a linear range and the stress-strain-displacement relations are given by the well-known elastic solution:

$$\begin{aligned} \epsilon_r^{el} &= \frac{du}{dr} = \frac{1-\nu^2(2-c)}{E} \left\{ \sigma_r - \frac{c\nu}{1-\nu(2-c)} \sigma_\theta \right\} \\ \epsilon_\theta^{el} &= \frac{u}{r} = \frac{1-\nu^2(2-c)}{E} \left\{ -\frac{\nu}{1-\nu(2-c)} \sigma_r + [1-\nu(c-1)] \sigma_\theta \right\} \\ \epsilon_3^{el} &= (c-1)\epsilon_\theta, \end{aligned} \tag{15}$$

where u represents the radial displacement.

Substituting equation (15) into the equilibrium equation (2) and making use of the boundary conditions (1) yield:

$$\begin{aligned} \sigma_r &= -p_0 - p \left(\frac{a}{r} \right)^{c+1} \\ \sigma_\theta &= -p_0 + \frac{1}{c} p \left(\frac{a}{r} \right)^{c+1}, \end{aligned} \tag{16}$$

and

$$u(r) = \frac{1}{c} p \frac{1+\nu}{E} \left(\frac{a}{r} \right)^{c+1} r. \tag{17}$$

As the internal pressure increases further, first yielding occurs when the stress point is on the boundary of the admissible region (5), i.e. when stresses σ_r , σ_θ satisfy the equation:

$$(1 + \alpha)\sigma_\theta - \sigma_r = Y, \quad (18)$$

where $\sigma_r < \sigma_\theta = \sigma_\psi$ and $\sigma_r < \sigma_z < \sigma_\theta$ hold for the spherical and the cylindrical cavity respectively. Hence, the following limit value for the internal pressure is obtained:

$$p_1 = \frac{c(Y + \alpha p_0)}{c + 1 + \alpha}, \quad (19)$$

whereas equation (17) yields the corresponding value for the displacements at the inner surface:

$$u_1 = \frac{(1 + \nu)(Y + \alpha p_0)}{(c + 1 + \alpha)E} r \left(\frac{a}{r}\right)^{c+1}. \quad (20)$$

When $p > p_1$ a plastic zone, bounded by an inner radius a and an outer radius b , expands around the cavity. Substituting the yield condition (18) into the equilibrium equation (2) and integrating the resulting differential equation, yield:

$$\sigma_r = \frac{Y}{\alpha} - \frac{(c + 1)(1 + \alpha)(Y + \alpha p_0)}{\alpha(c + 1 + \alpha)} \left(\frac{b}{r}\right)^{\frac{c\alpha}{1 + \alpha}} \quad a \leq r \leq b \quad (21)$$

$$\sigma_\theta = \frac{Y}{\alpha} - \frac{(c + 1)(Y + \alpha p_0)}{\alpha(c + 1 + \alpha)} \left(\frac{b}{r}\right)^{\frac{c\alpha}{1 + \alpha}},$$

account taken for the continuity condition $\sigma_r(r = b) = p_1$. In the outer, elastic, zone ($r \geq b$) the stresses are obtained from equation (16) replacing a by b and p by p_1 .

Making use of (1)₁ and (21)₁, it is readily obtained:

$$\frac{b}{a} = \Omega^{\frac{\alpha+1}{c\alpha}}, \quad (22)$$

where:

$$\Omega = \frac{(Y + \alpha p + \alpha p_0)(c + 1 + \alpha)}{(c + 1)(\alpha + 1)(Y + \alpha p_0)}. \quad (23)$$

4. THE VOLUME PLASTIC RATE

For the spherical cavity, since symmetry requires $\dot{\epsilon}_3 = \dot{\epsilon}_\psi$, it will be assumed:

$$\dot{\epsilon}_1^p = \dot{\epsilon}_r^p, \quad \dot{\epsilon}_2^p = \dot{\epsilon}_3^p = \dot{\epsilon}_\psi^p. \quad (24)$$

When plastic flow occurs, the condition $\sigma_\theta = \sigma_\psi$ together with equation (24) require:

$$\dot{\lambda}_3 = \dot{\lambda}_5 \neq 0 \quad \text{and} \quad \dot{\lambda}_i = 0, \quad i \neq 3, 5. \quad (25)$$

For the cylindrical cavity, it will be assumed:

$$\dot{\epsilon}_1^p = \dot{\epsilon}_r^p, \quad \dot{\epsilon}_2^p = \dot{\epsilon}_z^p, \quad \dot{\epsilon}_3^p = \dot{\epsilon}_\theta^p. \quad (26)$$

Hence, taking into account equation (3), it is obtained for the plastic zone:

$$\dot{\lambda}_5 \neq 0 \quad \text{and} \quad \dot{\lambda}_i = 0, \quad i \neq 5. \quad (25'')$$

Making use of (25) and taking into account equation (8), the volume plastic rate turns out to be a function of one plastic multiplier only which, in turns, can be expressed in terms of $\dot{\epsilon}_r^p$. Hence

$$\dot{\epsilon}_{vol}^p = (1 - \beta)\dot{\epsilon}_r^p, \quad (27)$$

or more explicitly:

$$\beta \dot{\varepsilon}_r^p + c \dot{\varepsilon}_\theta^p + (2-c) \dot{\varepsilon}_z^p = 0, \quad (28)$$

where

$$\beta = 1 - \frac{3\Theta}{\Theta - 3 - \alpha}. \quad (29)$$

It can be noted that, for the case of the spherical cavity, equation (28) coincides with the analogous expression in [10, 26] and reduces to [3, 5, 20, 41] for $\beta = 1$ and to [6] for $\beta = 1 + \alpha$.

Similarly, for the case of the cylindrical cavity, equation (28) reduces to [10, 26] when $\dot{\varepsilon}_z^p = 0$ and to [2, 3, 4, 5, 20, 36–38, 41] when $\dot{\varepsilon}_z^p = 0$ and $\beta = 1$.

5. THE ELASTO-PLASTIC SOLUTION

By substituting the plastic strain rates in equation (28) as the differences between the total strain rates and the elastic strain increments obtained from equation (15), it is readily obtained:

$$\begin{aligned} \beta \dot{\varepsilon}_r + c \dot{\varepsilon}_\theta = \frac{1}{E} \left\{ \frac{1}{3-c} [(3-c)\beta - (8-3c)v - (2-c)(\beta v - 1)] \dot{\sigma}_r \right. \\ \left. + \frac{1}{3-c} (4-c)(1-v-\beta v) \dot{\sigma}_\theta \right\}, \end{aligned} \quad (30)$$

where, for the case of cylindrical cavity, equation (3) has been accounted for. Considering that at $t = 0$, $\varepsilon_r = \varepsilon_\theta = 0$ and $\sigma_r = \sigma_\theta = -p_0$, integrating equation (30) yields:

$$\begin{aligned} \beta \varepsilon_r + c \varepsilon_\theta = \frac{1}{E} \left\{ \frac{1}{3-c} [(3-c)\beta - (8-3c)v - (2-c)(\beta v - 1)] \sigma_r \right. \\ \left. + \frac{1}{3-c} (4-c)(1-v-\beta v) \sigma_\theta \right\} + \frac{1}{E} (1-2v)(\beta+2)p_0. \end{aligned} \quad (31)$$

Substituting equation (21) into equation (31) gives:

$$\varepsilon_r + \frac{c}{\beta} \varepsilon_\theta = \ln \eta - \omega \left(\frac{b}{r} \right)^{\frac{c\alpha}{1+\alpha}} \quad (32)$$

where

$$\eta = \exp \left[\frac{(1-2v)(2+\beta)(Y + \alpha p_0)}{\alpha \beta E} \right] \quad (33)$$

$$\begin{aligned} \omega = \frac{(2c-1)(Y + \alpha p_0)}{\alpha \beta E(\alpha + c + 1)} \{ (1-2v)(2+\beta) + \alpha [(3-c)\beta \\ - (8-3c)v - (2-c)(\beta v - 1)] \}. \end{aligned} \quad (34)$$

As the internal pressure increases, the total strains become large and equation (32) requires finite strain components to be adopted. Following Chadwick [6] a logarithmic strain will be assumed, that is:

$$\varepsilon_r = \ln \left(\frac{dr}{dr_0} \right), \quad \varepsilon_\theta = \ln \left(\frac{r}{r_0} \right), \quad (35)$$

where $a \leq r \leq b$ and r_0 is the original length of a given radius $r(t)$. Substituting (35) into (32) yields:

$$\ln \left[\left(\frac{r}{r_0} \right)^{c/\beta} \left(\frac{dr}{dr_0} \right) \right] = \ln \eta - \omega \left(\frac{b}{r} \right)^{\frac{c\alpha}{1+\alpha}}. \quad (36)$$

By means of the transformations:

$$\psi = \omega \left(\frac{b}{r} \right)^{\frac{c\alpha}{1+\alpha}}, \quad \xi = \left(\frac{r_0}{b} \right)^{\frac{\beta+c}{\beta}}, \quad (37)$$

equation (36) is easily integrated over the interval $[r, b]$, resulting in:

$$\begin{aligned} \Gamma & \left\{ \left[1 - \frac{(1+\nu)(Y+\alpha p_0)}{(1+c+\alpha)E} \right]^{\frac{\beta+c}{\beta}} - \left(\frac{r_0}{b} \right)^{\frac{\beta+c}{\beta}} \right\} \\ & = - \int_{\omega \left(\frac{b}{r} \right)^{\frac{c\alpha}{1+\alpha}}}^{\omega} e^{\psi} \psi^{-\frac{c(1+\alpha)+\beta[1+(c+1)\alpha]}{c\alpha\beta}} d\psi, \end{aligned} \quad (38)$$

where:

$$\Gamma = \eta \frac{c\alpha\beta}{(1+\alpha)(c+\beta)} \omega^{-\frac{(1+\alpha)(c+\beta)}{c\alpha\beta}}. \quad (39)$$

Assuming $r_0 = a_0$ and $r = a$, equations (38) and (22) yield:

$$\begin{aligned} \Gamma & \left\{ \left[1 - \frac{(1+\nu)(Y+\alpha p_0)}{(1+c+\alpha)E} \right]^{\frac{\beta+c}{\beta}} - \Omega^{-\frac{(1+\alpha)(\beta+c)}{c\alpha\beta}} \left(\frac{a_0}{a} \right)^{\frac{\beta+c}{\beta}} \right\} \\ & = \int_{\omega}^{\omega\Omega} e^{\psi} \psi^{-\frac{c(1+\alpha)+\beta[1+(c+1)\alpha]}{c\alpha\beta}} d\psi. \end{aligned} \quad (40)$$

For any given value of p , equation (23) yields the relevant value of Ω ; hence, by a numerical integration of the right-hand side of (40), the corresponding value of a_0/a is determined. Moreover, making use of equations (22), (36), (21), (35)₂ and (31), by a chain substitution, b, r, σ_r , and $\sigma_\theta, \varepsilon_\theta$ and ε_r , are obtained for the plastic zone. Finally, b being known, equations (15), through (20) give the relevant quantities in the elastic zone. The limit pressure will be further obtained when the second term in the left-hand side of (40) becomes close to zero. Equation (40) when $\beta = 1 + \alpha$ reduces to the solution given by Chadwick [6] for the spherical cavity.

When the elastic deformations in the plastic zone are neglected, the right-hand side of equation (30) goes to zero; hence, following the same procedure, an approximate evaluation of equation (40) is obtained:

$$\Omega^{\frac{(\alpha+1)(c+\beta)}{c\alpha\beta}} = \left[1 - \left(\frac{a_0}{a} \right)^{\frac{c+\beta}{\beta}} \right] \left\{ 1 - \left[1 - \frac{(1+\nu)(Y+\alpha p_0)}{(c+1+\alpha)E} \right]^{\frac{c+\beta}{\beta}} \right\}^{-1}. \quad (41)$$

Putting $a_0/a = 0$, equation (41) yields an evaluation of the limit pressure which reduces to the solution given by Vesic [20] for both cavities when β is set equal to one.

6. THE NUMERICAL SOLUTION

A numerical integration of the r.h.s. of equation (40) has been performed adopting Gauss procedure. The material parameters $E = 1000$, $\nu = 0.3$ have been adopted. The parameter β has been assumed to take the extreme values $\beta = \alpha + 1$ and $\beta = 1$ corresponding to the associative volumetric plastic flow and to null volumetric change respectively. For each of these values, the parameter α has been given the following discrete values (1.463911, 2.000000, 2.690168, 3.598903, 4.828417) corresponding to the following values of ϕ (25°, 30°, 35°, 40°, 45°). Finally:

$$p_0 + \frac{Y}{\alpha} = \frac{E}{1000} \quad (42)$$

has been adopted. Consequently, the obtained results hold for any Y and p_0 which satisfy equation (42). As a matter of fact, the principle of corresponding states [42] implies that any

solution obtained for a given couple of values of \bar{p}_0 and \bar{Y} holds for any p_0 and Y such that:

$$p_0 + \frac{Y}{\alpha} = \bar{p} \tag{43}$$

where

$$\bar{p} = \bar{p}_0 + \frac{\bar{Y}}{\alpha} \tag{44}$$

Figures 2, 3, 4 and 5 show the variations of the cavity radius versus the internal pressure for the values of the shearing resistance angle $\phi = 30^\circ$ and $\phi = 40^\circ$ including the case in which

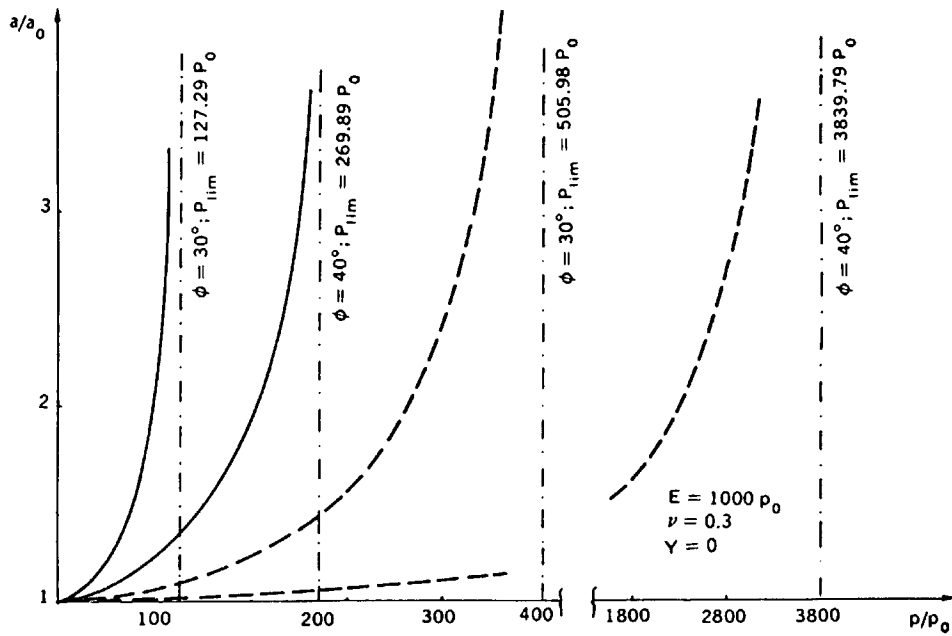


FIG. 2. Spherical cavity. Radius variation vs internal pressure; associated flow rule ($\beta = 1 + \alpha$);
 ---- no elastic deformations in the plastic zone.

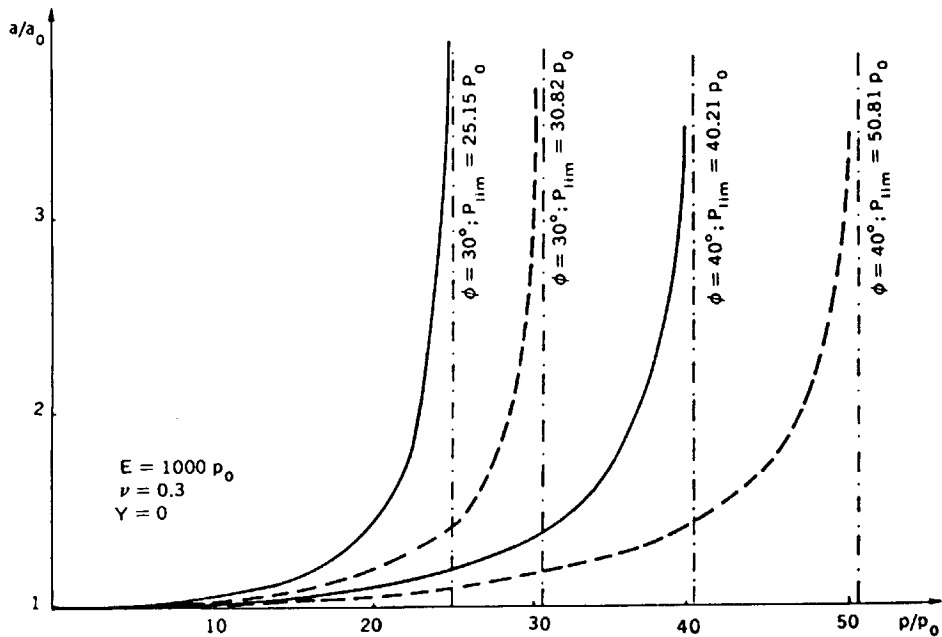


FIG. 3. Spherical cavity. Radius variation vs internal pressure; non-associated flow rule ($\beta = 1$);
 ---- no elastic deformations in the plastic zone.

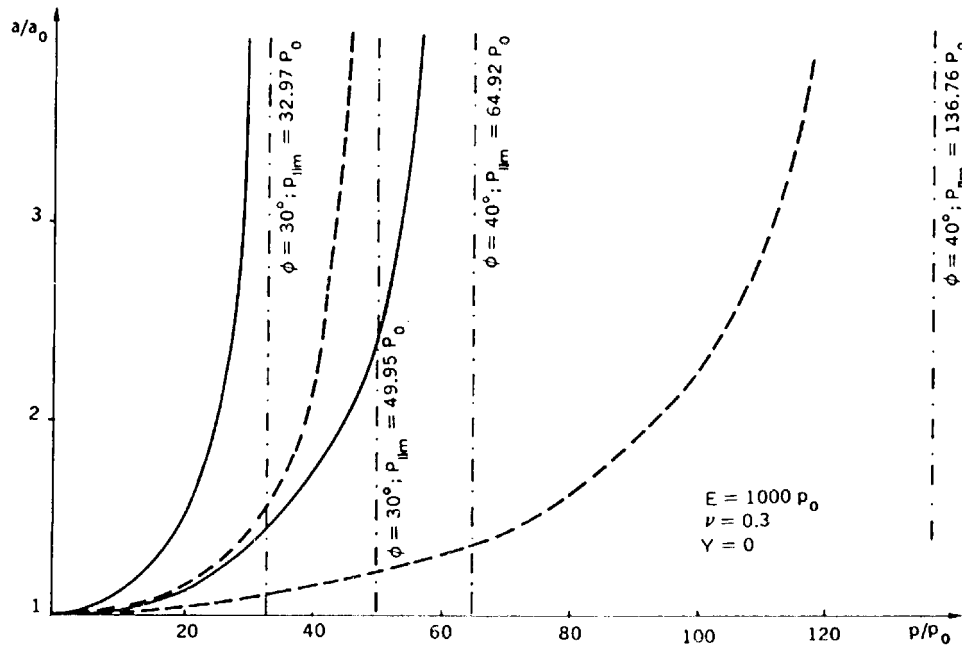


FIG. 4. Cylindrical cavity. Radius variation vs internal pressure; associated flow rule ($\beta = 1 + \alpha$); no elastic deformations in the plastic zone.

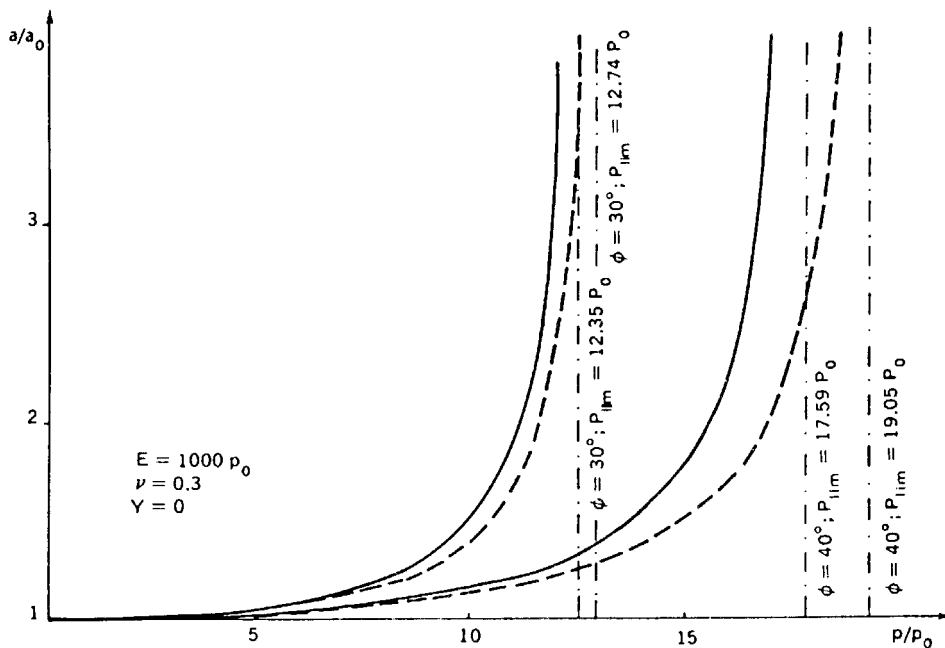


FIG. 5. Cylindrical cavity. Radius variation vs internal pressure; non-associated flow rule ($\beta = 1$); no elastic deformations in the plastic zone.

the elastic deformations in the plastic zone are neglected. Figures 2 and 3 refer to the spherical cavity for the cases of associated and non-associated flow rules respectively. Analogously, Figs 4 and 5 refer to the cylindrical cavity. Table 1 collects the limit pressure values for all the cases reported. Figure 6 shows the variation of the limit pressure with the angle of shearing resistance for both the cavities with the associated ($\beta = 1 + \alpha$) and non-associated ($\beta = 1$) flow rule.

It can be noted that neglecting the elastic deformations in the plastic region gives an error which increases with the dilatational component of the plastic flow, i.e. with both the

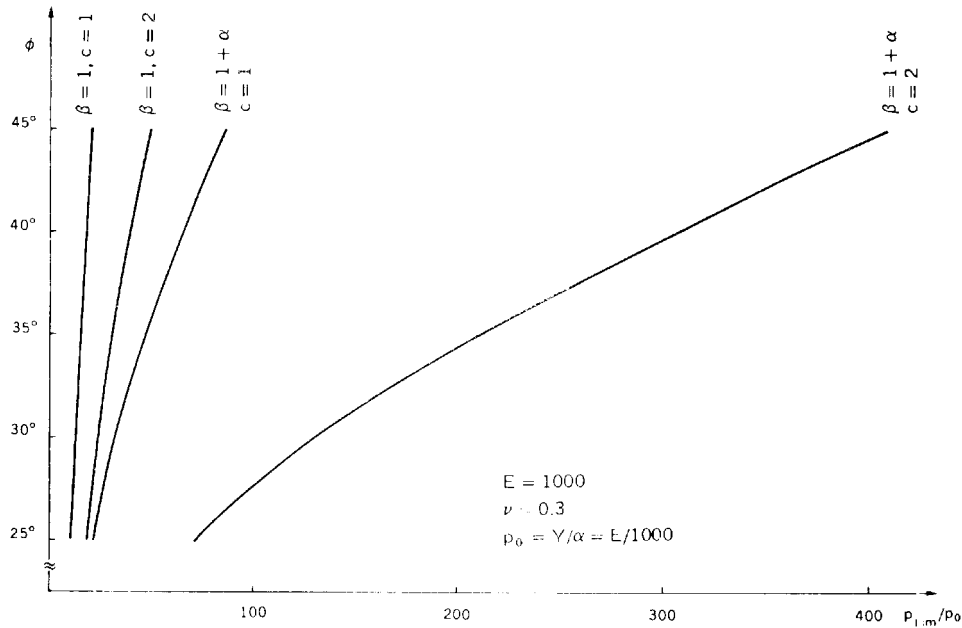


Fig. 6. Non-dimensional limit pressure vs angle of shearing resistance; associated flow rule ($\beta = 1 + \alpha$); non-associated flow rule ($\beta = 1$).

material parameters ϕ and β . As a consequence, especially for the spherical cavity, approximate solutions [7, 10] tend to give unacceptable errors as ϕ and β increase.

Finally it must be noted that even when the dilatational component of the plastic flow vanishes, neglecting the elastic deformations in the plastic zone [20] overestimates the limit pressure for the spherical cavity by some 20%. An analogous observation was made by Hill [5] with reference to metal obeying Tresca yield criterion.

TABLE I. NON-DIMENSIONAL LIMIT PRESSURES p_{lim}/\bar{p} ($E = 1000$; $\nu = 0.3$; $p_0 + Y/\alpha = E/1000$)

ϕ	Elastic deform. plastic zone	Associated flow-rule $\beta = 1 + \alpha$		Non-associated flow-rule $\beta = 1$	
		Spherical	Cylindrical	Spherical	Cylindrical
25	Yes	71.28	21.15	18.17	9.70
	No	181.08	28.40	22.17	9.77
30	Yes	127.29	32.97	25.15	12.35
	No	505.98	49.95	30.82	12.74
35	Yes	211.58	36.94	32.12	15.01
	No	1404.22	84.31	40.49	15.87
40	Yes	269.89	64.92	40.21	17.59
	No	3839.79	136.76	50.81	19.05
45	Yes	512.79	83.85	48.53	20.01
	No	10231.11	213.20	61.42	22.19

7. CONCLUSIONS

A general solution has been proposed for the analysis of the finite static expansion of both spherical and cylindrical cavities in an unbounded elastic–perfectly plastic medium obeying Mohr–Coulomb yield criterion with non-associated flow law. By means of numerical (Gaussian) integration the expansion of both the cavities has been analysed for a rather wide range of material parameters.

The numerical solutions developed have considered the two limit cases of the volumetric dilatation either with null value or corresponding to the associated flow law.

The elastic deformations in the plastic zone have been found to play an important role which increases with the dilatational component of the plastic flow.

The strong influence of the non-associativity degree of the plastic deformations has been stressed by the numerical results.

Acknowledgement –The financial support of (Italian) M.P.I. is gratefully acknowledged.

REFERENCES

1. H. G. HOPKINS, Dynamic expansion of spherical cavities in metals. In *Progress in Solid Mechanics* (Edited by I. N. SNEDDON and R. HILL), pp. 84–164. North Holland, Amsterdam (1960).
2. A. NADAI, *Plasticity*, pp. 182–200. McGraw-Hill, New York (1931).
3. R. F. BISHOP, R. HILL and N. F. MOTT, The theory of indentation and hardness tests. *Proc. Phys. Soc. Lond.* **57**, part 3, No. 321, 147 (1945).
4. R. HILL, E. H. LEE and S. J. TUPPER, The theory of combined plastic and elastic deformation with particular reference to a thick tube under internal pressure. *Proc. R. Soc. A* **188**, 278 (1947).
5. R. HILL, *The Mathematical Theory of Plasticity*, pp. 97–127. Clarendon Press, Oxford (1950).
6. P. CHADWICK, The quasi-static expansion of a spherical cavity in metals and ideal soils. *Q. Jl Mech. Appl. Math.* **XII**(1), 52 (1957).
7. P. CHADWICK, Propagation of spherical plastic–elastic disturbances from an expanded cavity. *Q. Jl Mech. Appl. Math.* **XV**(3), 349 (1962).
8. P. CHADWICK, Compression of a spherical shell of work-hardening material. *Int. J. Mech. Sci.* **5**, 165 (1963).
9. M. M. CARROLL and A. C. HOLT, Static and dynamic pore-collapse relations for ductile porous materials. *J. appl. Phys.* **43**(4), 1626 (1972).
10. J. SALENÇON, Expansion quasi-statique d'une cavité à symétrie sphérique ou cylindrique dans un milieu elastoplastique. *Annales de Ponts et Chaussées* **III**, 175 (1966).
11. D. RADENKOVIĆ, Théorie des charges limites, extension à la mécanique des sols. Séminaires de Plasticité, Ec. Polytech., Pub. Sc. et Tech. Minist. Air. n. N.T. 116 (1961).
12. G. DE JOSSELING DE JONG, Lower bound collapse theorem and lack of normality of strain rate to yield surface for soils. *Proc. I.U.T.A.M. Symp. on Rheology and Soil Mech.*, Grenoble, pp. 69–78 (1964).
13. A. C. PALMER, A limit theorem for materials with non-associated flow laws. *J. de Mécanique* **V**(2), 217 (1966).
14. Z. MROZ, Non-associated flow laws in plasticity. *J. de Mécanique* **II**(1), 23 (1963).
15. G. MAIER, A minimum principle for incremental elastoplasticity with nonassociated flow-laws. *J. Mech. Phys. Solids* **18**, 319 (1970).
16. G. MAIER and T. HUECKEL, Nonassociated and coupled flow rules of elasto-plasticity for rock-like materials. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **16**, 77 (1979).
17. T. HUECKEL and G. MAIER, Incremental boundary value problems in the presence of coupling of elastic and plastic deformations: a rock mechanics oriented theory. *Int. J. Solids Struct.* **13**, 1 (1977).
18. R. HILL, A general theory of uniqueness and stability in elastic-plastic solids. *J. Mech. Phys. Solids* **6**, 236 (1958).
19. J. MANDEL, Conditions de stabilité et postulat de Drucker. In *Rheology and Soil Mechanics, IUTAM*, Grenoble 1964 (Edited by J. KRAVCHENKO and P. M. SIRIEYS), Springer, Berlin (1966).
20. A. S. VESIC, Expansion of cavities in infinite soil mass. *J. Soil Mech. Found. Div. ASCE* **98** (SM3), 265 (1972).
21. M. M. BALIGH, Cavity expansion in sands with curved envelopes. *J. Geotech. Engng Div. ASCE* **102** (GT11), 1131 (1976).
22. J. H. PREVOST and K. HOEG, Analysis of pressuremeter in strain-softening soil. *J. Geotech. Engng Div. ASCE* **101**(GT8), 717 (1975).
23. J. M. HUGHES, C. P. WROTH and D. WINDLE, Pressuremeter tests in sands. *Geotechnique* **27**(4), 455 (1977).
24. A. P. S. SELVADURAI, Large strain and dilatancy effects in pressuremeter. *J. Geotech. Engng Div. ASCE* **110**(3), 421 (1984).
25. S. M. SAYED, Cylindrical cavity expansion in nonlinear dilatant media. *Mech. Res. Commun.* **14**(4), 219 (1987).
26. J. P. CARTER, J. R. BOOKER and S. K. YEUNG, Cavity expansion in cohesive frictional soils. *Geotechnique* **36**(3), 349 (1986).
27. F. P. SMITS, Private communication. Laboratory of soil mechanics, ISMES, Bergamo (Italy).
28. P. CHADWICK, A. D. COX and H. G. HOPKINS, Mechanics of deep underground explosions. *Phil. Trans. R. Soc. (Lond.)* **A256**, 235 (1964).

29. P. V. LADE and J. M. DUNCAN, Cubical triaxial tests on cohesionless soil. *J. Soil Mech. Found. Div. ASCE* **99** (SM10), 793 (1973).
30. P. V. LADE and J. M. DUNCAN, Elastoplastic stress-strain theory for cohesionless soil. *J. Geotech. Engng Div. ASCE* (GT10), 1037 (1975).
31. J. M. RUDNICKI and J. R. RICE, Conditions for the localization of deformation in pressure-sensitive dilatant materials. *J. Mech. Phys. Solids* **23**, 371 (1975).
32. J. R. RICE, The localization of plastic deformation. *Proc. 14th Int. Congr. Theor. and Appl. Mech.* (Edited by W. T. KOITER), pp. 207–220. Delft (1977).
33. J. W. RUDNICKI, The inception of faulting in a rock mass with a weakened zone. *J. geophys. Res.* **82**(5), 844 (1977).
34. B. LORET and J. H. PREVOST, Accurate numerical solutions for Drucker Prager elastic-plastic models. *Comp. Meth. Appl. Mech. Engng* **54**, 259 (1986).
35. J. H. PREVOST, Modeling the behavior of geomaterials. Lecture Notes, Lousanne, August (1987).
36. L. M. KACHANOV, *Foundations of the Theory of Plasticity*, pp. 118–121. North-Holland Publ. Comp., Amsterdam (1971).
37. H. GEIRINGER, Ideal plasticity. In *Handbuch der Physik* (Edited by S. FLÜGGE), pp. 506–515. Springer-Verlag, VI a/3, Berlin (1973).
38. W. T. KOITER, On partially plastic thick-walled tubes, Biezeno Anniversary Vol. in *Appl. Mech.*, N.V. De Technische Uitgeverij H. Stam, Haarlem, pp. 232–251 (1953).
39. W. T. KOITER, Stress-strain relations uniquenesses and variational theorems for elastic plastic materials with a singular yield surface. *Q. Appl. Math.* **3**, 350 (1953).
40. R. M. HAYTHORNTHWAITE, Mechanics of the triaxial test for soils. *J. Soil Mech. Found. Div. ASCE* **86** (SM5), 35 (1960).
41. A. MENDELSON, *Plasticity: theory and application*, the Macmillan Comp., pp. 135–163 (1968).
42. J. SALENÇON, *Applications of the Theory of Plasticity in Soil Mechanics*, p. 68. Wiley, New York (1974).