Elasticity with Hierarchical Disarrangements: a Field Theory that Admits Slips and Separations at Multiple Submacroscopic Levels

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Abstract

The complexity and variety of geometrical changes in physical systems at submacroscopic levels has led to various approaches to the broadening of the classical theory of finite elasticity. One approach, the field theory "elasticity with disarrangements", employed the multiscale geometry of structured deformations in order to incorporate the effects of disarrangements such as slips and separations at a single submacroscopic level on the macroscopic response of a continuous body. This article extends that field theory by enriching the underlying geometry so as to include the effects of disarrangements at more than one submacroscopic level. The resulting field theory broadens the scope of this approach, sharpens the description of the physical nature of dissipative mechanisms that can arise, and increases the variety of systems of contact forces that can serve as boundary loadings for a body that evolves via multiscale geometrical processes.

To Walter Noll,

whose writings set the foundations of continuum mechanics and whose commitment to colleagues, friends, and family endures in our memory.

1 Introduction

In the article [2] we provided a first step in a program to employ structured deformations of continua [1] in order to obtain a field theory capable of describing, in the context of dynamics and large isothermal deformations, the evolution of bodies that undergo smooth deformations at the macroscopic level, that can experience piecewise smooth deformations at submacroscopic levels, and that can not only store energy but can also dissipate energy during such multilevel geometrical changes. The field theory [2] that we formulated incorporated the effects at the macrolevel of smooth deformations and of non-smooth deformations (disarrangements) at one submacroscopic level by providing field relations that govern the (time-dependent) macroscopic deformation g of the body as well as the tensor field G of deformations without disarrangements at the submacroscopic level. Among the field relations in [2] that govern the structured motions (g, G) of the body is the accommodation inequality that includes the relation $0 < \det G \leq \det \nabla g$, i.e., the volume changes caused by smooth deformations at the submacroscopic level cannot exceed the volume changes at the macroscopic level. The accommodation inequality is the basis for the Approximation Theorem: at each time there exist piecewise smooth, injective mappings f_n on the body that tend to g and whose gradients ∇f_n tend to G as n tends to ∞. The passage to ∞ can be interpreted as the process of "zooming out" from the generally non-smooth geometrical changes at the submacroscopic level to the generally smoother geometrical fields g and G. In other words, g and G provide at the macrolevel some (but generally not all) of the geometrical information available at the submacroscopic level. The additional information carried in the field G, the deformation without disarrangements, is complemented by the information carried in the field $M = \nabla g - G$, which can be shown to capture at the macrolevel the geometrical effects of slips and separations in the approximating deformations f_n in the limit as n tends to ∞.

The main obstacle in formulating a field theory in terms of g and G is to provide relations between these fields in addition to the algebraically obvious additive decomposition

$$\nabla g = G + M.$$

In [2] we provided additional relations among these fields by obtaining for the stress S in the macroscopic reference configuration (the Piola-Kirchhoff stress) both an additive decomposition $\det((\nabla g)^{-1}G)S = S_\backslash + S_d$ and a multiplicative decomposition $\det((\nabla g)^{-1}G)S = S_\backslash G^T(\nabla g)^{-T}$. The theory of structured deformations permits us to identify the field $S_\backslash$ as the stress without disarrangements and the field S_d as the stress due to disarrangements, and the two decompositions of S show that, given the structured deformation (g, G), the two fields $S_\backslash$ and S_d cannot be prescribed independently. Thus, there is a "consistency relation"

$$S_\backslash G^T(\nabla g)^{-T} = S_\backslash + S_d \tag{1}$$

that the two refined stress fields must satisfy. This consistency relation is universal, i.e., it is valid for every body undergoing structured deformations in the presence of a given system of contact forces that determine S, without regard to the material comprising that body [42]. The constitutive assumptions in [2] then allow us through the response function $(G, M) \longmapsto \Psi(G, M)$ for the free energy density ψ to interpret the refined stress $S_\backslash$ already defined in terms of S and (g, G), as a "driving stress" corresponding to the geometrical field G and to interpret the refined stress S_d as a "driving stress" corresponding to the geometrical field M. These constitutive relations express the dependence of both $S_\backslash$ and S_d upon G and M, and the consistency relation then provides the desired further restriction on the three fields ∇g, G, and M. Moreover, the additive decompositions both of S and of ∇g as well as our constitutive assumptions allow us to calculate the internal dissipation Γ through the formula

$$\begin{aligned}\Gamma &= S \cdot \nabla \dot{g} - \dot{\psi} = (S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}) / \det((\nabla g)^{-1} G) \\ &= D_G \Psi \cdot \dot{M} + D_M \Psi \cdot \dot{G}.\end{aligned}$$

These formulas give us a precise physical as well as mathematical insight into the sources of dissipation in this multilevel setting and led to a necessary departure from the standard approach [3] for assuring compatibility of constitutive relations with the Second Law.

Our field theory [2] and its refined geometry have been applied (i) to distinguish among the geometrical phenomena of contact, of intermingling and of mixing of elastic constituents and to identify the additional forces and moments exerted by a mixture of elastic bodies on each constituent [4], (ii) to define and apply a notion of submacroscopically stable equilibrium of elastic bodies that selects among the many phases available to the body those that dissipate energy faster than they store energy in quasistatic, purely submacroscopic processes [5], and (iii) to define and identify a related notion of stable disarrangement phases of elastic bodies that, in the context of aggregates of small elastic bodies, provides a setting for the emergence of no-tension materials with nonlinear response in compression [6].

These applications indicate the variety of possibilities afforded by structured deformations for enriching through effects at a single submacroscopic level the purely macroscopic field theory of non-linear elasticity. Nevertheless, many (i) *natural* and (ii) *man-made* physical systems have a rich enough geometrical structure to permit the identification of hierarchies consisting of more than one physically meaningful submacroscopic level. In each of the above categories both (a) *soft* and (b) *hard* materials can be singled out.

Natural soft tissues exhibiting hierarchies include muscles, cartilage, cornea, tendons (see, e.g., [7, 8, 9, 10]), while natural hard tissues with hierarchical structures include bone, nacre, enamel, etc. (see, e.g., [7, 11, 12, 16, 13]). In [9] and[13] one finds evidence that submacroscopic disarrangements in hierarchical materials can occur at various levels of the hierarchy and can be influenced by the organization of the hierarchy. This evidence is provided with particular regard to muscles and enamel.

Furthermore, in [8, 16] it is suggested that toughening mechanisms, characterized by distributions of submacroscopic separations at the various levels of the hierarchies, do closely follow the internal arrangements of such materials. For such mechanisms a statistical model based on bundles of fibers has been extensively used in this context [17, 18, 19]. Among natural materials exhibiting hierarchical structures and analogous modes of disarrangements, one finds wood, bamboo [14, 16, 15], and plants in general, forming a span of hard-to-soft materials.

Man-made materials with similar features are primarily bioinspired composites [16], nowadays available thanks to the evolving advancements in additive manufacturing [20, 21].

Our goal here is to take a further step in our program by broadening the

Figure 1: An example of three-level hierarchies: large stack (macroscale) of bundled papers (intermediate level). The finest scale is represented by each paper sheet (reproduced with permission of the photographer Mr. M. Schmorgan).

field theory [2] to include the effects of geometrical changes at more than one submacroscopic level. To simplify the explanations and to provide focus, we restrict our attention to the case of two submacroscopic levels: submacroscopic level 2 , the finer submacroscopic level that represents in this case the most magnified view of geometrical changes, and submacroscopic level 1, a level intermediate to submacroscopic level 2 and to the macroscopic level (level 0). Thus, we provide not only the macroscopic deformation field g at each time, but also two tensor fields G_1 and G_2 on the body that ultimately provide the effects at the macrolevel of geometrical changes without disarrangements at each of the submacrocscopic levels.

In Section 2.1 we describe the refined geometry of three-level structured deformations (g, G_1, G_2) based partly on discussions with Gianpietro Del Piero in the late $1990'$s. The three-level analogues (3) and Theorem 1 in the Appendix of the accommodation inequality and the Approximation Theorem identify G_2 as a gradient followed by two limit operations that correspond to zooming out through both submacroscopic levels . As such, G_2 incorporates at the macrolevel nothing of the slips and separations at either submacroscopic level. By contrast, G_1 is identified as a limit (zooming out from submacroscopic level 2 to level 1) followed by a gradient followed by a second limit (zooming out from submacroscopic level 1 to the macrolevel), and so incorporates the effects both of smooth and of non-smooth changes at level 2, but only the effects of

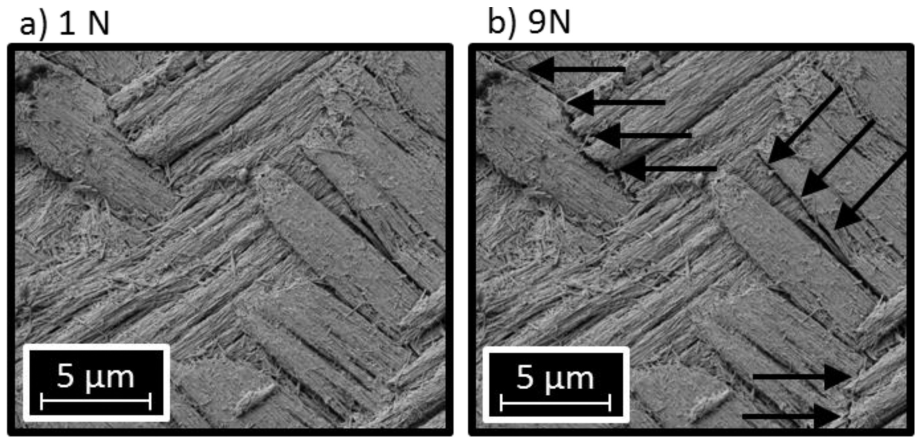

Figure 2: Disarrangements at different levels of hiearchy: SEM-image of an *enamel specimen* a) after pre-loading, showing very little submacroscopic cracks opening, b) at a load level of 9N, displaying more opening (reproduced from [13] with permission of Prof.Dr.rer.nat. G.A. Schneider).

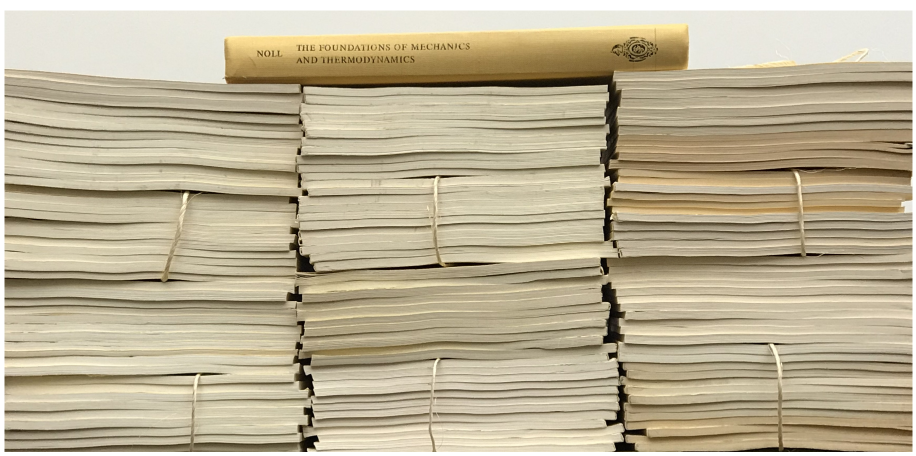

Figure 3: Zoom-in across the scales: the presence of disarrangement at two different levels is evident because of the separation between (i) bundled papers and (ii) sheets.

smooth deformations at level 1. The three-level hierarchy is completed with the field $G_0 := \nabla g$, consisting first of two limits that span both the submacroscopic levels, followed by the gradient operation. Consequently, G_0 incorporates both types of geometrical changes at both submacroscopic levels.

The analysis in Section 2.2 includes a discussion of the algebraically trivial additive decomposition $G_0 = G_2 + (G_1 - G_2) + (G_0 - G_1)$, in which each of the differences $G_{i-1} - G_i$ on the right-hand side represents the effect of interchanging the order of a "limit operator" with the gradient operator and, consequently, incorporates only the effects of disarrangements at level i. Our definition $M_i = G_{i-1} - G_i$ of the disarrangement tensor at level i recasts the last additive decomposition in the form

$$\nabla g = G_0 = G_2 + M_2 + M_1 \tag{2}$$

and so resolves the difference $G_0 - G_2$, spanning both submacroscopic levels, separately into the effects M_2 of disarrangements at level two only, added to the effects M_1 of disarrangements at level one only. This is the source of the term "hierarchical disarrangements" and motivates the preferred choice of refined geometrical variables G_2, M_2, M_1 in most of our treatment. We show also in Section 2.2 how factoring the invertible tensor field G_2 from either the left of the sum $G_2+M_2+M_1$ or from the right yields two multiplicative decompositions of $\nabla g = G_0$ in both of which each factor has a definite interpretation supported by limit formulas derived from the Approximation Theorem. Moreover, a notion of composition of three-level structured deformations permits us to factor via composition an arbitrary three-level structured deformation into a succession of three three-level structured deformations, each of which describes geometrical changes occuring at exactly one of the three levels under consideration.

For each three-level structured deformation and for each tensor field N ("tensor flux") on the macroscopically deformed body, we introduce in Section 2.3, for $i = 0, 1, 2, \ldots$ various refined versions N_i, $N_{i-1.\backslash}$ and $N_{i-1.d}$ of the field N that depend upon the submacroscopic or macroscopic level i and depend upon whether disarrangements at that level are to be incorporated. Our development focuses on the example $N = T$, the Cauchy stress (stress in the deformed configuration), in which case T_0 is the (first) Piola-Kirchhoff stress (stress in the reference configuration). However, the treatment here could also apply to other tensor fluxes such as the momentum flux. The interpretations of the refined versions N_i, $N_{i-1.\backslash}$ and $N_{i-1.d}$ are justified on the basis of the Approximation Theorem, and we provide additive decompositions that parallel the decomposition of ∇g in (2) as well as multiplicative decompositions. The availability of universal additive and of multiplicative decompositions of tensor fluxes at each level permits us in Section 2.4 to derive a consistency relation for the tensors $N_{i-1.\backslash}$ and $N_{i-1.d}$ at each submacroscopic level analogous to (1) for the case of the stress and for only one submacroscopic level.

In Section 3 the notion of a structured motion as a time-parameterized family of three level structured deformations is introduced. In this context, one can use the three-term additive decomposition (2) of G_0 and the corresponding decomposition of tensor fluxes N to compute a nine-term additive decomposition of the inner product $N_0 \cdot \dot{G}_0$, where $\dot{G}_0$ is the time derivative of G_0. When N is taken to be the Cauchy stress field T defined on the deformed configuration of the body at each time, then $T_0 \cdot \dot{G}_0$ is the stress power per unit volume in the reference configuration of the body. Three of the nine terms in the expression for the stress power contain a "matching" pair of factors, indicating that the location of the corresponding tractions are in proximity to the resulting geometrical changes both in the sense of spatial location and in the sense of hierarchical level. The remaining six terms of $T_0 \cdot \dot{G}_0$ each contains a mismatched pair of factors: *either* the two hierarchical levels of the refined stress and of the rate of deformation measure do not match *or* the disarrangement status "without disarrangements" versus "due to disarrangements" of the two factors do not match. This remote character of the stress and deformation rates in the mismatch terms lead us in subsequent sections to assign a special thermodynamical status to these six terms.

Section 4 contains the conceptual background for the specific constitutive assumptions that we make in Section 5. This background is needed, since it provides a procedure for assuring that constitutive assumptions are in accord with the Second Law of Thermodynamics that is less restrictive than the now standard procedure of Coleman and Noll [3]. The key observation already employed in [2] that we use here in Section 4 is the following: constitutive assumptions identify a constitutive class, i.e., a subset $\mathcal{C}$ of the collection of all dynamical processes available to a body, and the requirement that the Second Law of Thermodynamics be satisfied also identifies a set $\mathcal{T}$ of dynamical processes; compatibility of the constitutive assumptions with the Second Law is simply the requirement that the first set be included in the second: $\mathcal{C} \subset \mathcal{T}$. For a given free energy response function, our approach provides in Section 5 constitutive relations defined in terms of that response function and its partial derivatives, including an inequality involving the fields G_2, M_2, M_1 and their time derivatives. These relations define a constitutive class that we denote by $\mathcal{E}_{hd}$, and we verify the inclusion $\mathcal{E}_{hd} \subset \mathcal{T}$. An advantage of our particular choice $\mathcal{E}_{hd}$ is the particular physical interpretation that it affords for the internal dissipation $\Gamma = T_0 \cdot \dot{G}_0 - \dot{\psi}$ in dynamical processes: Γ turns out to be the sum of the six "mismatch" terms in the expression for the stress power $T_0 \cdot \dot{G}_0$, and internal dissipation is thereby identified with lack of proximity of applied tractions to the sites of occurrence of the geometrical changes that they work against. Here, "proximity" means that the tractions and the rates of deformation must correspond to the same (submacroscopic or macroscopic) level and to the same disarrangement status ("without disarrangements" or "due to disarrangements").

In Section 5.1 we detail the choice of $\mathcal{E}_{hd}$, the constitutive class mentioned in the previous paragraph. One of our constitutive equations requires that the free energy density at any point and time be determined through a given response function Ψ by the values at that point and time of G_2, M_2, and M_1. Three additional constitutive equations relate the values of the refined stress measures $T_{1.\backslash}$, $T_{1.d}$, and $T_{0.d}$ at each point and time to the values of G_2, M_2, and M_1 by means of the partial derivatives $D_{G_2}\Psi$, $D_{M_2}\Psi$, and $D_{M_1}\Psi$ of Ψ, respectively. Doing so permits us to interpret each of these stress measures as a "driving force (per unit area)" corresponding to a refined geometrical variable. The final constitutive assumption is the requirement that the power expended by the "mismatch" terms in the stress power $T_0 \cdot \dot{G}_0$ be non-negative. This inequality is a restriction on dynamical processes that, together with the other constitutive assumptions, assures that the constitutive class $\mathcal{E}_{hd}$ is compatible with the Second Law. In Section 5.2 we rewrite with $N = T$ the two consistency relations by substituting into them the constitutively assigned expressions for the refined stress measures in terms of the derivatives of the free energy response Ψ. We so obtain two tensorial consistency equations that directly restrict the three fields G_2, M_2, and M_1 and that eventually provide closure to the field theory under consideration. We also record the "stress relation" that gives the Piola-Kirchhoff stress $S = T_0$ in terms of G_2, M_2, and M_1 as the sum of the three partial derivatives of Ψ. The stress relation then permits one to write the balance of linear momentum equation as a restriction on the fields g, $G_1 = G_2 + M_2$, and G_2. Additional stress relations for the referential stresses T_1 and T_2 at submacroscopic levels 1 and 2 also are obtained in terms of the partial derivatives of Ψ.

We are aware that the accomodation inequality $0 < \det G \leq \det F$ could be treated as an internal constraint and, hence, appropriate reaction stresses would arise. This would lead to an enrichment of the refined measures for the stress and, ultimately, of the stress measures themselves (see [35] for details in the context of ordinary disarrangements and of "gradient disarrangements").

The refined geometry of three-level structured deformations permits the determination of transformation rules under change of observer for the new tensor fields that enter into our constitutive relations, and we note in Section 5.3 that G_1, G_2, M_2, M_1, T_i for $i = 0, 1, 2$, as well as $T_{i-1.\backslash}$ and $T_{i-1.d}$ for $i = 1, 2$ all transform in the same manner under change of observer as does $G_0 = \nabla g$. Frame-indifference in the present context is the statement that the constitutive class $\mathcal{E}_{hd}$ chosen in Section 5.1 be closed under change of observer, i.e., a dynamical process is in $\mathcal{E}_{hd}$ if and only if all the dynamical processes obtained from it by a change of observer also are in $\mathcal{E}_{hd}$. It is interesting that we are able to show in Section 5.3 that frame-indifference here is equivalent to the usual invariance property of the free energy response Ψ together with the symmetry of the Cauchy stress T. Consequently, balance of angular momentum is guaranteed on dynamical processes in $\mathcal{E}_{hd}$, once it is required that $\mathcal{E}_{hd}$ be closed under change of observer.

In Section 5.4 we extend to elasticity with hierarchical disarrangements the notion of "stable disarrangement phase" already defined and studied ([5],[6],[43]) in the context of elasticity with disarrangements at only one submacroscopic level [2]. In the present context, for a given tensor $F = G_0$, the decomposition $F = G_2 + M_2 + M_1$, the consistency relations for stress at submacroscopic levels 1 and 2 and a symmetry condition arising from frame indifference, along with the accommodation inequalities, provide four tensorial equations and three inequalities that restrict the tensor variables G_2, M_2, and M_1. A triple G_2, M_2, M_1 that satisfies these equations and inequalities is called a disarrangement phase corresponding to F. We pointed out in the context of elasticity with disarrangements ([5],[6],[43]) that multiple disarrangement phases corresponding to a given F are to be expected, and the same assertion applies in the present, richer context. Accordingly, we follow [6] and define a stable disarrangement phase G_2, M_2, M_1 corresponding to F to be a disarrangement phase corresponding to

F with the lowest free energy among all disarrangement phases corresponding to F. Because of the constraints imposed by (i) the consistency relations and frame indifference, (ii) the decomposition for F, and (iii) the accommodation inequalities, a stable disarrangement phase need not be a stationary point of the free energy and, therefore, the stress need not vanish at a stable disarrangement phase and (as examples in the two-level case show) need not be hydrostatic.

We collect together in Section 5.5 all of the geometrical and constitutive information from previous sections and record the system of field relations for elasticity with hierarchical disarrangements. The field relations amount to thirty-three scalar equations and four inequalities to be satisfied by the thirty components of g, G_2, M_2, and M_1. (Three of the scalar equations express the symmetry of the Cauchy stress and can be omitted in cases where a particular free energy response function automatically provides the required symmetry.) We show how under particular circumstances these field relations in the context of two submacroscopic levels reduce to those for elasticity with disarrangments, in which only one submacroscopic level is admitted, or, under other particular circumstances reduce to the field relations of finite elasticity, in which no submacroscopic levels are admitted.

In Section 6 the case where the response function Ψ does not depend upon the disarrangement tensors M_1 and M_2 is considered. This special circumstance bars the storage of energy in the body through non-smooth geometrical changes, and, following terminology introduced in the two-level case [23], we call this the case of purely dissipative disarrangements. The field relations simplify considerably in the case of purely dissipative disarrangements, and we point out that among the field relations, all but the accommodation inequalities are invariant under interchange of the fields M_1 and M_2. Examples are provided in Section 6 that illustrate how the accommodation inequalities serve in the case of purely dissipative disarrangements to determine the particular level at which given disarrangements may arise.

The refined geometry of hierarchical structured deformations provides, in addition to the Piola-Kirchhoff stress $T_0 = S$, the submacroscopic reference stresses T_1 and T_2 introduced in Section 2.3. Each incorporates to a different extent the disarrangements occuring at submacroscopic levels one and two. In Section 7 we use the reference stresses T_0, T_1, and T_2 to refine and broaden the types of boundary conditions available for the fields g, G_1, G_2 that determine structured deformations satisfying the field relations in Section 5.5.

The paper concludes with an appendix in which a complete statement and proof of the Approximation Theorem for three-level structured deformations are provided.

2 Hierarchical structured deformations

2.1 Definition and Approximation Theorem

The geometrical basis for our present considerations is the multilevel geometry of hierarchical structured deformations that we describe here for the case of three levels: the macroscopic level along with two submacroscopic levels. The case of a multilevel geometry with only one submacroscopic level was developed in a series of papers on structured deformations ([1], [25], [29], [24], [31], [38], [39]) from which followed a number of applications (e.g., [30], [26], [44], [36], [37], [2], [4], [6], [43], [40], [41]) that include a field theory of elasticity with (submacroscopic) disarrangements [2] and a broadened field theory that includes the effects of gradient disarrangements [35]). The case with more than two submacroscopic levels can be inferred easily from the present treatment. For present purposes, we may define a *three-level structured deformation* to be a triple (g, G_1, G_2) in which the injective, piecewise continuously differentiable point-valued mapping $g : \mathcal{B} \longrightarrow \mathcal{E}$ from the body $\mathcal{B}$ into Euclidean space $\mathcal{E}$ is called the *macroscopic deformation*, and the piecewise continuous tensor-valued mappings G_1 and G_2 from $\mathcal{B}$ into $Lin\mathcal{V}$ are called the *deformation without disarrangements at submacroscopic level one* and *the deformation without disarrangements at submacroscopic level two*, respectively. The triple (g, G_1, G_2) is required to satisfy the *accommodation inequalities* throughout $\mathcal{B}$ (excepting the points of discontinuity of ∇g, G_1 and G_2):

$$c < \det G_2(X) \leq \det G_1(X) \leq \det \nabla g(X) \tag{3}$$

with c a positive constant that may depend upon the given triple but not upon the points X in $\mathcal{B}$. (The notions of piecewise continuous differentiability and piecewise continuity are made precise in [1] through the notion of "simple deformations" and "piecewise-fit regions"; the set κ that appears in the definitions of simple and structured deformations in [1] is here taken to be the empty set). Here, ∇ denotes the classical derivative of a smooth mapping or of the restriction of a piecewise smooth mapping to points of differentiability. When the body has the regularity of a piecewise-fit region [1], the Approximation Theorem for three-level structured deformations provides a double sequence $(n_1, n_2) \longmapsto f_{n_1 n_2}$ of injective, piecewise continuously differentiable mappings from $\mathcal{B}$ into $\mathcal{E}$ satisfying

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} f_{n_1 n_2} = g, \tag{4}$$

so that the deformation gradient ∇g is given by

$$G_0 := F := \nabla g = \nabla \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} f_{n_1 n_2}, \tag{5}$$

as well as satisfying

$$G_1 = \lim_{n_1 \to \infty} \nabla \lim_{n_2 \to \infty} f_{n_1 n_2} \tag{6}$$

$$G_2 = \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \nabla f_{n_1 n_2}. \tag{7}$$

Here, each of the limits $\lim_{n_1\to\infty}$, $\lim_{n_2\to\infty}$ separately may be taken in the sense of L^∞-convergence on the body, i.e., in the sense of essential uniform convergence, although other settings may require different choices of regularity of the fields g, G_1, and G_2 and possibly weaker notions of convergence (see [29], [30], [31], [38] for the case of two-level structured deformations). A complete statement of the Approximation Theorem for three-level structured deformations and a proof are provided in the Appendix.

It is clear from (5) - (7) that G_0, G_1, and G_2 capture the effects of applying the gradient operator ∇ following, between, or prior to applying the limit operations $\lim_{n_1\to\infty}$ and $\lim_{n_2\to\infty}$, and we explore now in more depth the significance of these relations. We interpret each injective mapping $f_{n_1 n_2}$ in (4) - (7) to be a description of the geometrical changes occurring at the finest level (highest magnification), i.e., at the submacroscopic level 2. The non-smooth changes (disarrangements) may consist of slips and separations of small pieces of the body that, individually, deform smoothly. The classical gradient $\nabla f_{n_1 n_2}$ then reflects only the smooth changes occurring at level 2 in each of the pieces. The limits $\lim_{n_2\to\infty} f_{n_1 n_2}$ and $\lim_{n_2\to\infty} \nabla f_{n_1 n_2}$ reflect the process of "zooming out" from submacroscopic level 2 to submacroscopic level 1: $\lim_{n_2\to\infty} f_{n_1 n_2}$ reflects the effects at level 1 of both disarrangements occuring at level 2 and of smooth changes occurring at level 2; on the contrary, $\lim_{n_2\to\infty} \nabla f_{n_1 n_2}$ reflects the effects at level 1 only of deformations without disarrangements occurring at level 2. The process $\lim_{n_1\to\infty}$ of zooming out from level 1 to the macrolevel, level 0, along with the relation (7), then justifies the terminology already introduced: "deformation without disarrangements at level 2" for the tensor field G_2. In the same spirit, for each n_1 the field $\nabla \lim_{n_2\to\infty} f_{n_1 n_2}$ only reflects the smooth part of the deformations $\lim_{n_2\to\infty} f_{n_1 n_2}$ at level 1, so that the iterated limit $\lim_{n_1\to\infty} \nabla \lim_{n_2\to\infty} f_{n_1 n_2}$ reflects the effects at the macrolevel only of the smooth part of the deformations $\lim_{n_2\to\infty} f_{n_1 n_2}$ at level 1. The relation (6) then justifies the terminology already introduced: "deformation without disarrangements at level 1" for the tensor field G_1.

As an example we consider the *three-level shear,* an adaptation to the present context of the two-level shear for two-level structured deformations [1], [26]. Let u, v be orthogonal unit vectors, let μ_0, μ_1, μ_2 be scalars, and define for each X in $\mathcal{B}$

$$g(X) \quad = \quad X + \mu_0\,(v\cdot(X-o))\,u \tag{8}$$

$$G_1(X) \quad = \quad I + \mu_1 u\otimes v, \quad G_2(X) = I + \mu_2 u \otimes v \tag{9}$$

with o a preassigned point in $\mathcal{B}$, the unit cube with one vertex at o and two edges parallel to u and v. The macroscopic deformation g is a simple shear of amount μ_0 in the direction u whose shearing plane has normal v, while the angle of shearing is $\tan^{-1}\mu_0$. Because $G_0(X) = \nabla g(X) = I + \mu_0\, u\otimes v$, we have $\det G_i = 1$ for $i = 0, 1, 2$, so that the accommodation inequalities (3) are satisfied with $c = 1/2$ and with equality in the last two.

An approximating sequence $(n_1, n_2) \longmapsto f_{n_1 n_2}$ for the three-level shear that satisfies (4) - (7) in the Approximation Theorem is described geometrically as follows. The mapping $f_{n_1 n_2}$ first slices $\mathcal{B}$ into n_1 pieces ("cards") by means of $n_1 - 1$ parallel planes with normal the unit vector v that are separated by amount $1/n_1$. It leaves the bottom card fixed and translates each higher card in the deck, relative to the one below it, by the amount $(\mu_0 - \mu_1)/n_1$ in the direction u, and successively shears the i^{th} translated card via a simple shear of amount μ_1, again in the direction u with normal orientation v. The resulting deck of n_1 translated and sheared cards approximates the range $g(\mathcal{B})$ of the original mapping g, and this sequence of injective, piecewise smooth approximations indexed on n_1 would converge to g uniformly, while its gradients would converge uniformly to G_1, if we at this point would let n_1 tend to infinity. In fact, the description up to this point is precisely that of a two-level shear and its approximates at submacroscopic level 1 [1]. The further action of $f_{n_1 n_2}$ involves slicing each of the n_1 translated and sheared level 1 cards into n_2 level 2 cards by means of $n_2 - 1$ planes all with normal v as before. Within the i^{th} level 1 card, the j^{th} level 2 card is translated relative to the card below it by amount $(\mu_1 - \mu_2)/n_2$ in the direction u and sheared via a simple shear of amount μ_2 with direction u and normal orientation v. The resulting sequence of multilevel shears $f_{n_1 n_2}$ is easily shown to satisfy (4) - (7) for the specific three-level structured deformation defined in (8) and (9).

It is worth pointing out in the example of three-level shears that the differences

$$G_0 - G_1 = (\mu_0 - \mu_1) u \otimes v \quad \text{and} \quad G_1 - G_2 = (\mu_1 - \mu_2) u \otimes v$$

involve the differences $\mu_0 - \mu_1$ and $\mu_1 - \mu_2$ that appear in the numerators of the expressions for the tangential discontinuities $(\mu_0 - \mu_1)/n_1$ and of $(\mu_1 - \mu_2)/n_2$ caused by the approximating sequence of piecewise smooth, injective deformations $f_{n_1 n_2}$. This suggests that $G_0 - G_1$ captures the deformation due to disarrangements at submacroscopic level 1, while $G_1 - G_2$ captures the deformation due to disarrangements at submacroscopic level 2. An analysis analogous to that given in [25] for general two-level structured deformations shows that for arbitrary three-level structured deformations this interpretation of the differences $G_{i-1} - G_i$ for $i = 1, 2$ is justified. In fact, for each three-level structured deformation (g, G_1, G_2) and for each sequence $f_{n_1 n_2}$ satisfying (4) - (7) in the Approximation Theorem, we have for every point X and $r > 0$:

$$\int_{B_r(X)} (G_1 - G_2)(y) dV_y = \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \int_{J(f_{n_1 n_2}) \cap B_r(X)} [f_{n_1 n_2}](y) \otimes \nu_y \, dA_y \quad (10)$$

and

$$\int_{B_r(X)} (G_0 - G_1)(y) dV_y = \lim_{n_1 \to \infty} \int_{J(\lim_{n_2 \to \infty} f_{n_1 n_2}) \cap B_r(X)} [\lim_{n_2 \to \infty} f_{n_1 n_2}](y) \otimes \nu_y \, dA_y. \quad (11)$$

In these relations, $B_r(X)$ denotes the ball of radius r centered at the point X in $\mathcal{B}$, while $J(h)$ denotes the jump set of a function h, ν_y the normal to $J(h)$ at the point y, and $[h](y)$ its jump at y. While (6) and (7) show that G_1 and G_2

capture the effects at the macrolevel of smooth submacroscopic deformations at levels 1 and 2, respectively, the relations (10) and (11) show that the tensor

$$M_1 := G_0 - G_1 \tag{12}$$

captures the effects at the macrolevel of disarrangements (discontinuous deformations) at submacroscopic level 1 and that

$$M_2 := G_1 - G_2 \tag{13}$$

captures the effects at the macrolevel of disarrangements at submacroscopic level 2. We call M_1 and M_2 the *disarrangement tensors* for the three-level structured deformation (g, G_1, G_2). Their role from (10) and (11) in capturing separately the non-smooth deformations that occur at different levels motivates the term "hierarchical disarrangements" in the title of this paper. It is of mathematical interest to note from (5) - (7), (12) and (13) that each of the disarrangement tensors M_1 and M_2 captures quantitatively the effect of interchanging once the operation "∇", the classical gradient operator, and the operation "lim" that here describes zooming out one level.

2.2 Additive decompositions, multiplicative decompositions, and factorizations

We record using (5), (10), and (11) the algebraically simple but geometrically significant relation

$$\nabla g = G_0 = G_2 + M_2 + M_1 \tag{14}$$

that provides an additive decomposition of the macroscopic deformation gradient as a sum of the disarrangement tensors M_1 and M_2 at levels one and two, respectively, and of the tensor without disarrangements at level two G_2. Thus, all of the local geometrical changes captured by $G_0 = \nabla g$ are described in a simple way in terms of the deformation without disarrangements G_2 at level two and the deformations due to disarrangements M_1 and M_2 at levels one and two. Similarly, we may write for $i = 1, 2$:

$$G_{i-1} = G_i + M_i \tag{15}$$

to obtain additive decompositions for G_0 and for G_1 into a part without disarrangements and a part due to disarrangements. While each of these decompositions is algebraically trivial, the identification relations (6), (7), (10), and (11) fully justify the interpretation and the terminology that we provide for each term in the sums on the right-hand sides of (14) and (15).

We emphasize that the additive decompositions (14) and (15) are valid whatever the size of the underying deformations. The additive decomposition (14) for ∇g provides immediately two multiplicative decompositions:

$$\nabla g = G_2(G_2^{-1}M_1 + G_2^{-1}M_2 + I) =: G_2 M_r \tag{16}$$

$$\nabla g = (M_1 G_2^{-1} + M_2 G_2^{-1} + I)G_2 =: M_l G_2. \tag{17}$$

In many theories of inelastic behavior of materials, particularly in theories of plasticity, analogous decompositions are introduced either as constitutive assumptions or as geometrical assumptions that rest on the existence of intermediate configurations that are not attainable in general by a three-dimensional body. Here, the multiplicative decompositions are derived from additive decompositions, and the factors in each multiplicative decomposition have definite interpretations obtained from the identification relations (6), (7), (10), and (11). Moreover, each of the multiplicative factors M_r and M_l itself has an additive decomposition whose terms reflect the presence of disarrangements at two submacroscopic levels:

$$M_r = G_2^{-1}M_1 + G_2^{-1}M_2 + I \tag{18}$$
$$M_l = M_1G_2^{-1} + M_2G_2^{-1} + I. \tag{19}$$

(See [1], [32] for detailed discussions of the corresponding additive and multiplicative decompositions in the case of two-level structured deformations.)

In spite of the usefulness of multiplicative decompositions of the form (16) and (17), these decompositions do not reflect factorizations of the deformations of the body as a whole. We record here a definition of *composition of three-level structured deformations* (see [1] for the two-level counterpart) that does reflect geometrical changes of the body as a whole:

$$(g, G_1, G_2) \circ (\tilde{g}, \tilde{G}_1, \tilde{G}_2) = (g \circ \tilde{g}, (G_1 \circ \tilde{g})\tilde{G}_1, (G_2 \circ \tilde{g})\tilde{G}_2). \tag{20}$$

In this definition we use the same symbol "$\circ$" on the left to denote "composition of structured deformations" as we do on the right for "composition of tensor fields and point mappings", but no confusion should arise from this practice. This definition leads immediately to the following factorization of an arbitrary three-level structured deformation:

$$(g, G_1, G_2) = (g, G_0, G_0) \circ (\iota, G_0^{-1}G_1, G_0^{-1}G_1) \circ (\iota, I, G_1^{-1}G_2) \tag{21}$$

where ι denotes the identity mapping $X \longmapsto \iota(X) = X$, and where I is the constant tensor field whose only value is the identity tensor $u \longmapsto Iu = u$. The factorization is easily verified by evaluating the (associative) operation $\circ$ twice on the right-hand side of this relation, and we show below (22) that each factor on the right-hand side is, itself, a three-level structured deformation.

The following table records, for each of the three factors on the right-hand side of (21), the deformation without disarrangements at level two, "G_2", as well as the disarrangement tensors "M_1" and "M_2" at levels one and two:

factor in (21)	"G_2"	"M_1"	"M_2"
(g, G_0, G_0)	G_0	0	0
$(\iota, G_0^{-1}G_1, G_0^{-1}G_1)$	$G_0^{-1}G_1$	$I - G_0^{-1}G_1$	0
$(\iota, I, G_1^{-1}G_2)$	$G_1^{-1}G_2$	0	$I - G_1^{-1}G_2.$

The table shows that the first factor (g, G_0, G_0) causes no disarrangements at either submacroscopic level, and we call (g, G_0, G_0) a *classical deformation* or a *purely macroscopic deformation*, since in classical treatments of geometrical changes no disarrangements are included and only deformations at the macrolevel are considered. The second factor $(\iota, G_0^{-1}G_1, G_0^{-1}G_1)$ produces no geometrical changes at the macrolevel, since $i(X) = X$ for all $X \in \mathcal{B}$, and causes no disarrangements at submacroscopic level 2. The third factor $(\iota, I, G_1^{-1}G_2)$ produces no geometrical changes at either the macrolevel or at the submacroscopic level 1, since $\iota(X) = X$ for all $X \in \mathcal{B}$ and since $Iu = u$ for all $u \in \mathcal{V}$, and $(i, I, G_1^{-1}G_2)$ causes no disarrangements at submacroscopic level 1.

It is convenient to use the notation $K_i = G_{i-1}^{-1}G_i$ for $i = 1,2$ and to write the factorization (21) in the form

$$(g, G_1, G_2) = (g, G_0, G_0) \circ (\iota, K_1, K_1) \circ (\iota, I, K_2). \tag{22}$$

The accommodation inequalities (3) for the three-level structured deformation (g, G_1, G_2) tells us that at each point X both $\det K_1(X)$ and $\det K_2(X)$ lie in the interval $(0, 1]$, and we refer to $\det K_i$ as the volume fraction associated with the purely level-i factor, for $i = 1, 2$. Moreover, it is clear that each of the factors on the right-hand side of (22) is, itself, a three-level structured deformation, because each has the required regularity and satisfies the accommodation inequalities (3) in one of the following forms:

$$\begin{array}{c} 0 < c_0 < \det G_0(X) = \det G_0(X) = \det \nabla g(X) \quad \text{for} \quad (g, G_0, G_0) \\ 0 < c_1 < \det K_1(X) = \det K_1(X) \leq 1 = \det \nabla \iota(X) \quad \text{for} \quad (\iota, K_1, K_1) \\ 0 < c_2 < \det K_2(X) \leq 1 = \det I = \det \nabla \iota(X) \quad \text{for} \quad (\iota, I, K_2). \end{array}$$

In these relations, c_j for $j = 0, 1, 2,$ can be chosen using the fact that $\det G_j$ has a strictly positive lower bound and a finite upper bound on the closure of the body.

2.3 Decomposition of tensor fluxes

For a given three-level structured deformation (g, G_1, G_2) and for a smooth tensor field $N : g(\mathcal{B}) \longrightarrow Lin\mathcal{V}$, we call N a *tensor flux on the deformed configuration*, and we define for $i = 0, 1, 2$ the *reference flux at level* i to be the tensor field $N_i : \mathcal{B} \longrightarrow Lin\mathcal{V}$ given by

$$N_i = (N \circ g)G_i^*, \tag{23}$$

where, for each invertible tensor A, $A^* = (\det A)A^{-T}$. We also refer to N_0 as the macroscopic reference flux. For example, when $N = T$ is the Cauchy stress field on $g(\mathcal{B})$, then $T_0 = (T \circ g)G_0^* = (T \circ g)(\nabla g)^*$ is the Piola-Kirchhoff stress, usually denoted by S. The macroscopic reference stress T_0 as well as the submacroscopic reference stresses $T_i = (T \circ g)G_i^*$ for $i = 1, 2$ can be rewritten using (4), (6), and (7) as

$$\begin{array}{rcl} S &=& T_0 = (T \circ g)(\nabla \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} f_{n_1 n_2}))^* \\ T_1 &=& (T \circ g) \lim_{n_1 \to \infty} (\nabla \lim_{n_2 \to \infty} f_{n_1 n_2})^* \\ T_2 &=& (T \circ g) \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} (\nabla f_{n_1 n_2})^*. \end{array} \tag{24}$$

This is the primary example of tensor flux that we shall use here, and the reader may without loss read all of the following with N replaced by T. Later in this section we justify calling the submacroscopic reference stress T_1 the part of the macroscopic reference stress $T_0 = S$ without level one disarrangements, and we justify calling the submacroscopic reference stress T_2 the part of the submacroscopic reference stress T_1 without level two disarrangements (see relations (29)).

For a general flux N, the reference fluxes N_1 and N_2 at the submacroscopic levels one and two are intended to capture the idea that the geometry of three-level structured deformations allows us to identify refined measures of fluxes defined on the body $\mathcal{B}$ arising from the given flux N on the deformed body $g(\mathcal{B})$. In order to develop further this idea, we use the tensor fields K_i defined above (22) to write for $i = 1, 2$ the following product rule

$$\det K_i \ \operatorname{div} N_{i-1} = \operatorname{div}((\det K_i) N_{i-1}) - N_{i-1} \nabla \det K_i$$

for the divergence of the product $(\det K_i) N_{i-1}$. By adding and subtracting on the right-hand side the field $\operatorname{div}(N_{i-1} K_i^*)$, where $K_i^* = (\det K_i) K_i^{-T}$, we obtain for $i = 1, 2$

$$\det K_i \ \operatorname{div} N_{i-1} = \operatorname{div}(N_{i-1} K_i^*) + \operatorname{div}((\det K_i) N_{i-1} - N_{i-1} K_i^*) - N_{i-1} \nabla \det K_i. \tag{25}$$

Arguments given in [2] in the context of two-level structured deformations and summarized in [6] show, by applying the Approximation Theorem to the purely level-i factor in the factorization (22), (i, K_1, K_1) for $i = 1$ or (i, I, K_2) for $i = 2$, that the left-hand side $\det K_i \ \operatorname{div} N_{i-1}$ can be interpreted as a volume density of the flux of N_{i-1} taking into account both the sites of disarrangements and the preassigned surfaces across which the flux of N_{i-1} is measured. Moreover, the term $\operatorname{div}(N_{i-1} K_i^*)$ on the right-hand side can be interpreted as a volume density of the flux of N_{i-1}, taking into account only preassigned surfaces. Consequently, the remaining terms $\operatorname{div}((\det K_i) N_{i-1} - N_{i-1} K_i^*) - N_{i-1} \nabla \det K_i$ can be interpreted as the volume density of the flux of N_{i-1}, taking into account only the sites of disarrangements at level i, and we may call (25) the *decomposition at level i of the volume density of the flux of N_{i-1}*. It is shown further in [2] that the tensor field $N_{i-1} K_i^*$ accounts for all of the volume density of the flux without level i disarrangements, while the tensor field $(\det K_i) N_{i-1} - N_{i-1} K_i^*$ accounts for only the flux due to level i disarrangements.

The trivial algebraic identity for $i = 1, 2$

$$(\det K_i) N_{i-1} = N_{i-1} K_i^* + ((\det K_i) N_{i-1} - N_{i-1} K_i^*) \tag{26}$$

and the identification just above of each term on the right-hand side of (25) motivates the following terminology: $N_{i-1} K_i^*$ is *the part of N_{i-1} without level i disarrangements*, and $(\det K_i) N_{i-1} - N_{i-1} K_i^*$ is *the part of N_{i-1} due to level i disarrangements*. For conciseness, we use the following notation for the two parts of N_{i-1}:

$$N_{i-1.\backslash} := N_{i-1} K_i^* \quad \text{and} \quad N_{i-1.d} := (\det K_i) N_{i-1} - N_{i-1} K_i^*, \tag{27}$$

In the expression $N_{i-1}K_i^*$ the subscript $i-1$ on N indicates the level at which the reference flux is measured, while the subscript i on K_i and K_i^* indicates the level at which disarrangements are being considered.

We rewrite (26) in the shorter form for $i = 1, 2$:

$$(\det K_i)N_{i-1} = N_{i-1.\backslash} + N_{i-1.d}. \tag{28}$$

The appearance of $\det K_i$ multiplying N_{i-1} on the left-hand sides of (26) and (28) is natural, in light of the fact that the adjugate K_i^* of K_i also containes that factor. (One could divide the decomposition (26) through by $\det K_i$ and so obtain a decomposition of N_{i-1}, itself, but maintaining the presence of $\det K_i$ has advantages that are apparent in what follows.) One may view the role of $\det K_i$ in front of N_{i-1} in (28) as adjusting for the possible reduction in volume due to the "purely level-i factor" in the factorization (22). One also may regard the decomposition (28) as the counterpart for fluxes of the geometrical decomposition $G_{i-1} = G_i + M_i$ that results from the definition of M_i as the difference $G_{i-1} - G_i$.

We have argued for $i = 1, 2$ that $N_{i-1.\backslash} = N_{i-1}K_i^*$ represents the part of the reference flux N_{i-1} without disarrangements at level i, and it is interesting that this part of the reference flux N_{i-1} at level $i-1$ actually recovers all of the reference flux N_i at the next level, level i, i.e., we have at each of the two submacroscopic levels:

$$N_i = N_{i-1.\backslash} \quad \text{for } i = 1, 2. \tag{29}$$

For example, the reference stress T_1 represents the part of the Piola-Kirchhoff stress $T_0 = S$ without disarrangements at level 1, while T_2 represents the part of T_1 without disarrangements at level 2. To establish (29) we note that $G_i = G_{i-1}K_i$, so that $G_i^* = G_{i-1}^*K_i^*$ and therefore we have

$$N_i = (N \circ g)G_i^* = (N \circ g)G_{i-1}^*K_i^* = N_{i-1}K_i^* = N_{i-1.\backslash}.$$

Inasmuch as the additive decomposition (28) is the counterpart for fluxes of the geometrical decomposition $G_{i-1} = G_i + M_i$, it is natural to seek a counterpart for fluxes of the full geometrical decomposition (14): $G_0 = G_2 + M_2 + M_1$. In fact, it is not difficult to arrive at the corresponding full decomposition of fluxes in the following form:

$$(\det K_1)(\det K_2)\, N_0 = N_{1.\backslash} + N_{1.d} + (\det K_2)N_{0.d}\ . \tag{30}$$

Indeed, the decomposition (28) of fluxes at level i with $i = 2$, the definition $(27)_2$ with $i = 1$, and the identification (29) of N_i with $i = 1$ permit us to write

$$\begin{aligned} & N_{1.\backslash} + N_{1.d} + (\det K_2)N_{0.d} \\ = \ & (\det K_2)N_1 + (\det K_2)((\det K_1)N_0 - N_{0.\backslash}) \\ = \ & (\det K_2)N_1 + (\det K_2)((\det K_1)N_0 - N_1) \end{aligned}$$

and this verifies (30). We refer to the decompositions (14) of G_0 and (30) of N_0 as *parallel decompositions*, because corresponding terms on the right-hand sides refer to the same disarrangement status and to the same level at which disarrangements occur.

2.4 Universal consistency relations for tensor fluxes at submacroscopic levels

For $i = 1, 2$ the definition $(27)_1$ of $N_{i-1.\backslash}$ is a multiplicative relation, while the decomposition (28) is an additive relation. Together these two relations yield:

$$N_{i-1.\backslash} + N_{i-1.d} = (\det K_i)N_{i-1} = N_{i-1.\backslash}K_i^T$$

Multiplying on the right the left-most and the right-most members by $K_i^{-T} = (G_i^{-1}G_{i-1})^{-T} = G_i^T G_{i-1}^{-T}$ we find that

$$N_{i-1.\backslash}(G_i^T G_{i-1}^{-T} - I) + N_{i-1.d}G_i^T G_{i-1}^{-T} = 0.$$

Multiplying both sides on the right by G_{i-1}^T and using the definition $M_i = G_i - G_{i-1}$ we obtain the *consistency relation at submacroscopic level* i:

$$N_{i-1.\backslash}M_i^T + N_{i-1.d}G_{i-1}^T = 0 \qquad \text{for } i = 1, 2. \tag{31}$$

The consistency relations (31) show that, for a given three-level structured deformation and a given tensor field N on $g(\mathcal{B})$, the parts of N_{i-1} with and without disarrangments at each submacroscopic level i cannot be assigned independently. In the context of two-level structured deformations and when N is taken to be the Cauchy stress tensor T, the consistency relation for $i = 1$ can be rewritten in terms of $S = T_0$ and reads

$$S_{\backslash}M_1^T + S_d(\nabla g)^T = 0.$$

It was derived in [2] and provides an essential relation among the field relations that govern elastic bodies undergoing disarrangements at one submacroscopic level.

It is important to keep in mind that the consistency relations (31) are universal, in the sense that that apply to every body, whatever its material composition, undergoing a structured deformation (g, G_1, G_2) in the presence of a given tensor flux N. The universality property as well as a uniqueness property of the associated decompositons (27) and (28) were studied in the context of two-level structured deformations in [42].

It is helpful to rewrite the two consistency relations (31) in terms of the three tensor fluxes $N_{1.\backslash}$, $N_{1.d}$, and $N_{0.d}$ that appear in the full decomposition (30) of the reference flux N_0 at the macroscopic level. Putting $i = 1$ in (31) and multiplying both sides by $\det K_2$ we have

$$\begin{aligned}
0 &= (\det K_2)N_{0.\backslash}M_1^T + (\det K_2)N_{0.d}G_0^T \\
&= (\det K_2)N_1 M_1^T + (\det K_2)N_{0.d}G_0^T \\
&= (N_{1.\backslash} + N_{1.d})M_1^T + (\det K_2)N_{0.d}G_0^T
\end{aligned}$$

where we have used (29) and (28) in the second and third relations. This relation together with (31) for $i = 2$ and the relation $G_1 = G_2 + M_2$ yield the desired form of the consistency relations:

$$N_{1.\backslash}M_1^T + N_{1.d}M_1^T + (\det K_2)N_{0.d}G_0^T = 0 \tag{32}$$

$$N_{1.\backslash}M_2^T + N_{1.d}M_2^T + N_{1.d}G_2^T = 0. \tag{33}$$

3 Motions and a decomposition of the stress-power

We follow some earlier treatments of motions in the context of two-level structured deformations ([2], [32], [33]) by requiring that, at each time t in a given interval, there is specified a three-level structured deformation $(g(\cdot,t), G_1(\cdot,t), G_2(\cdot,t))$ of the body. We require for every X in $\mathcal{B}$ that the field $t \longmapsto g(X,t)$ is twice continuously differentiable, while $t \longmapsto G_i(X,t)$ for $i = 0,1,2$ is continuously differentiable. Thus, structured motions here may be regarded as smoothly time-parameterized families of three-level structured deformations. (See [33] for an alternative approach in the context of two-level structured deformations in which time-like disarrangements are admitted.) It is convenient in what follows to denote derivatives with respect to t via Newtonian notation (superposed dots) and, without risk of ambiguity, to omit frequently the explicit writing of the variable t.

We may now use the parallel decompositions (14) of $G_0 = \nabla g$ and the full decomposition of fluxes (30) with N equal to T, the Cauchy stress, to compute $(\det K_1)(\det K_2)T_0 \cdot \dot{G}_0$, the *stress-power* $\mathcal{P}$ in a given motion (measured per unit volume at submacroscopic level two):

$$\begin{aligned}
\mathcal{P} \quad : \quad &= (\det K_1)(\det K_2)T_0 \cdot \dot{G}_0 \\
&= \left\{T_{1.\backslash} + T_{1.d} + (\det K_2)T_{0.d}\right\} \cdot \left\{\dot{G}_2 + \dot{M}_2 + \dot{M}_1\right\} \\
&= T_{1.\backslash} \cdot \dot{G}_2 + T_{1.d} \cdot \dot{M}_2 + (\det K_2)T_{0.d} \cdot \dot{M}_1 + \mathcal{P}_{mix}.
\end{aligned} \tag{34}$$

The stress-power $\mathcal{P}$ is an inner product of two factors, each with three terms, and so can be written as a sum of nine terms. Here, each of the first three terms displayed explicitly in the last expression in (34) contains a stress measure and a measure of rate of deformation that match both in terms of submacrosopic levels and in terms of whether or not the effects of disarrangements are captured. For example, in the inner product $T_{1.\backslash} \cdot \dot{G}_2$, the factor $T_{1.\backslash}$ is the part of T_1 without level 2 disarrangements and $\dot{G}_2$ is the rate of change of deformation without level 2 disarrangements. The sum of the remaining six terms is designated in (34) by $\mathcal{P}_{mix}$. These six terms include, for example, the terms $T_{1.\backslash} \cdot \dot{M}_2$ and $T_{1.d} \cdot \dot{M}_1$, and each such term has the property that there is at least one "mismatch" between its factors with regard to submacroscopic level or with regard to the inclusion or not of effects of disarrangement. Thus, in the term $T_{1.\backslash} \cdot \dot{M}_2$ in the sum $\mathcal{P}_{mix}$, $T_{1.\backslash}$ is the part of T_1 *without* level two disarrangements, while $\dot{M}_2$ is the rate of deformation *due to* level two disarrangements. Consequently, we follow the terminology introduced in [2] in the context of two-level structured motions and call $\mathcal{P}_{mix}$ the *mixed power* in the given motion (again measured per unit volume at submacroscopic level two). The refined geometry of hierarchical structured deformations here permits us to distinguish the mixed power $\mathcal{P}_{mix}$ from the total stress power $\mathcal{P}$, and this distinction will allow us to identify explicitly, in both physical and mathematical terms, the sources of dissipation that arise through the constitutive prescriptions that we make in Subsection 5.1.

4 Dynamical processes, constitutive classes, and the dissipation inequality

We adapt the discussion in [2] to specify here a dynamical process by giving the fields g, G_1, and G_2 associated with a structured motion as well as the Cauchy stress field $T \circ g$, the volume density ψ of the Helmholtz free energy in the macroscopic reference configuration, and the mass density ρ_0 in that configuration. The fields g, G_1, G_2 , $T \circ g$, and ψ are defined on pairs (X, t) with $X \in \mathcal{B}$ and t in a time interval that can depend upon g, while ρ_0 is defined on points $X \in \mathcal{B}$. Henceforth, we consider the density ρ_0 as fixed and omit it when listing the quantities associated with dynamical processes. The formulas from above (22) for $K_i = G_{i-1}^{-1}G_i$ with $i = 1, 2$, accompanied by the relations (23) and (27) with $N = T$, show that the stresses $T_{i-1.\backslash}$ and $T_{i-1.d}$ with and without level i disarrangements are also specified once a dynamical process is specified. The same is true for the stress power $\mathcal{P}$ and the mixed power $\mathcal{P}_{mix}$ (see the formula (34)), as well as the time derivative $\dot{\psi}$ of the free energy density.

A constitutive class for the body [34] is simply a collection $\mathcal{C}$ of dynamical processes of the body, and a particular choice of constitutive class then limits the dynamical processes that are to be considered. In practice, a constitutive class is specified by giving a list of response functions and requiring that the fields g, G_1, G_2 , $T \circ g$, and ψ in each dynamical process be related through equations and/or inequalities involving the response functions in the given list.

Another limitation on the collection of dynamical processes in the present isothermal setting is thermodynamical in nature and here takes the form of a dissipation inequality:

$$\dot{\psi}(X, t) \le T_0(X, t) \cdot \dot{G}_0(X, t) \tag{35}$$

asserting that the rate of change of free energy density cannot exceed the stress power in the macroscopic reference configuration. We denote by $\mathcal{T}$ the collection of dynamical processes that satisfy the dissipation inequality, and we impose the requirement $\mathcal{C} \subset \mathcal{T}$ on possible choices of constitutive classes $\mathcal{C}$. This requirement suffices to assure that our constitutive relations are compatible with the Second Law. The standard, Coleman-Noll procedure for assuring compatibility of constitutive relations with the Second Law here proves to be too restrictive in that it rules out non-zero internal dissipation in smooth processes. (For a discussion of this shortcoming of the Coleman-Noll procedure in the context of elastic bodies undergoing two-level structured deformations, see Sections 6 and 7 of the article [2].)

5 Elasticity with hierarchical disarrangements

5.1 Constitutive assumptions

The constitutive assumptions that we make here in the context of three-level structured deformations are adaptations of those made for the field theory elasticity with disarrangements [2] in the context of two-level structured deformations. It is important to keep in mind that, other than the free energy density ψ itself, all of the fields that enter into our constitutive relations have been identified through the multilevel geometry of structured deformations and through the decompositions of deformations and of fluxes that this geometry provides. Consequently, we can avoid the pitfall of confusing, say, a constitutive equation containing a particular stress field with a relation that defines that field.

We first assume that the density ψ of free energy per unit volume in the macroscopic reference configuration is determined at each time t and point X in $\mathcal{B}$ by the values of the fields G_2, M_2, and M_1:

$$\psi(X,t) = \Psi(G_2(X,t), M_2(X,t), M_1(X,t)) \tag{36}$$

where Ψ is a smooth function of its three tensor arguments. Next, we consider the decomposition (30) again with $N = T$, and we make constitutive assumptions on the stresses $T_{1.\backslash}$, $T_{1.d}$, and $T_{d.1}$ appearing in that decomposition:

$$T_{1.\backslash}(X,t) = (\det K_1)(\det K_2)\mid_{(X,t)} D_{G_2}\Psi(G_2(X,t), M_2(X,t), M_1(X,t)), \tag{37}$$

$$T_{1.d}(X,t) = (\det K_1)(\det K_2)\mid_{(X,t)} D_{M_2}\Psi(G_2(X,t), M_2(X,t), M_1(X,t)), \tag{38}$$

$$T_{0.d}(X,t) = (\det K_1)\mid_{(X,t)} D_{M_1}\Psi(G_2(X,t), M_2(X,t), M_1(X,t)). \tag{39}$$

These three constitutive assumptions reflect the refined geometry of three-level structured deformations by linking the submacroscopic level and the disarrangement status of a given stress field on the left-hand side with the variable with respect to which the free energy is differentiated on the right-hand side. This feature of the constitutive relations (37) - (39) tell us in language found in the physics literature that the previously defined stress fields $T_{1.\backslash}$, $T_{1.d}$, and $T_{0.d}$ now play the role of "driving forces" corresponding to the geometrical fields G_2, M_2, and M_1. (As we remarked at the beginning of this section, the stresses $T_{1.\backslash}$, $T_{1.d}$, and $T_{0.d}$ were not *defined* to be driving forces.)

The factors $(\det K_1)(\det K_2)$ and $(\det K_1)$ involving determinants of K_1 and K_2 in (37) - (39) adjust for the fact that Ψ measures free energy per unit volume in the macroscopic reference configuration, while the stresses $T_{1.\backslash}$, $T_{1.d}$ refer to level two disarrangements and $T_{0.d}$ refers to level one disarrangements. Moreover, the factors $\det K_1$ and $\det K_2$ in the three constitutive relations (37) - (39) facilitate the following computation:

$$\begin{aligned} & (\det K_1)(\det K_2)\dot{\psi} \\ = \; & (\det K_1)(\det K_2)\, D_{G_2}\Psi(G_2, M_2, M_1) \cdot \dot{G}_2 + \end{aligned}$$

$$
\begin{aligned}
& (\det K_1)(\det K_2)\, D_{M_2}\Psi(G_2, M_2, M_1)\cdot \dot{M}_2 + \\
& (\det K_1)(\det K_2)\, D_{M_1}\Psi(G_2, M_2, M_1)\cdot \dot{M}_1 \\
= \;& T_{1.\backslash}\cdot \dot{G}_2 + T_{1.d}\cdot \dot{M}_2 + \det K_2 T_{0.d}\cdot \dot{M}_1 \\
= \;& \mathcal{P} - \mathcal{P}_{mix},
\end{aligned} \tag{40}
$$

where $\mathcal{P}$ and $\mathcal{P}_{mix}$ are the stress-power and the mixed stress-power defined in the previous section. Because $\mathcal{P} = (\det K_1)(\det K_2)T_0\cdot\dot{G}_0$, we conclude that the relation (40) is equivalent to

$$
T_0\cdot\dot{G}_0 - \dot{\psi} = \frac{\mathcal{P}_{mix}}{(\det K_1)(\det K_2)}. \tag{41}
$$

Thus, our constitutive assumptions (37) - (39) imply that the *rate of internal dissipation* per unit volume in the macroscopic reference configuration

$$
\Gamma := T_0\cdot\dot{G}_0 - \dot{\psi} = \frac{\mathcal{P}_{mix}}{(\det K_1)(\det K_2)} \tag{42}
$$

is given, to within the positive factor $(\det K_1)(\det K_2)$, by the six "mismatch" terms that comprise the mixed power $\mathcal{P}_{mix}$:

$$
\begin{aligned}
(\det K_1)(\det K_2)\Gamma \;=\;& T_{1.\backslash}\cdot\dot{M}_2 + T_{1.\backslash}\cdot\dot{M}_1 + \\
& T_{1.d}\cdot\dot{G}_2 + T_{1.d}\cdot\dot{M}_1 + \\
& (\det K_2)\, T_{0.d}\cdot\dot{G}_2 + (\det K_2)\, T_{0.d}\cdot\dot{M}_2.
\end{aligned} \tag{43}
$$

The first two mismatch terms $T_{1.\backslash}\cdot\dot{M}_2 + T_{1.\backslash}\cdot\dot{M}_1$ represent the power expended by stresses $T_{1.\backslash}$ without level two disarrangement against rates of change of deformation $\dot{M}_2$ and $\dot{M}_1$ due to level 2 and to level 1 disarrangements, respectively. The third and fourth terms $T_{1.d}\cdot\dot{G}_2 + T_{1.d}\cdot\dot{M}_1$ represent the power expended by stresses $T_{1.d}$ due to level two disarrangements against a rate of change of deformation without (level one or level two) disarrangements $\dot{G}_2$ or against a rate of deformation $\dot{M}_1$ due to level one disarrangements. Finally, the fifth and sixth mismatch terms $(\det K_2)\, T_{0.d}\cdot\dot{G}_2 + (\det K_2)\, T_{0.d}\cdot\dot{M}_2$ represent stresses $T_{0.d}$ due to level one disarrangments expending power against rates of deformation $\dot{G}_2$ and $\dot{M}_2$ at level 2. Thus, *our constitutive assumptions tell us that internal dissipation arises only when the contact forces and the geometrical changes against which they do work are mismatched in terms of submacroscopic level or in terms of whether or not disarrangements are tracked.*

We note from the formula (43) and from the three constitutive relations (37) - (39) that we may write the rate of internal dissipation Γ in the following form:

$$
\begin{aligned}
\Gamma \;=\;& D_{G_2}\Psi\cdot\dot{M}_2 + D_{G_2}\Psi\cdot\dot{M}_1 + \\
& D_{M_2}\Psi\cdot\dot{G}_2 + D_{M_2}\Psi\cdot\dot{M}_1 + \\
& D_{M_1}\Psi\cdot\dot{G}_2 + D_{M_1}\Psi\cdot\dot{M}_2,
\end{aligned} \tag{44}
$$

and this formula renders transparent the mismatches described above. In view of (44) our final *constitutive assumption* takes the form of a restriction on the fields G_2, M_2, M_1, and on their time derivatives $\dot{G}_2$, $\dot{M}_2$, $\dot{M}_1$:

$$0 \le \Gamma(X,t) \tag{45}$$

for every $X \in \mathcal{B}$ and for every time t. Of course, the dissipation inequality (45) just imposed is a sufficient condition on the dynamical processes available to the body for satisfaction of the second law of thermodynamics in the present, isothermal context.

Our constitutive assumptions (36), (37) - (39), and (45) thus specify a collection $\mathcal{E}_{hd}$ of dynamical processes (namely those that satisfy (36), (37) - (39), and (45)) and, thereby, specify an elastic body undergoing disarrangements at two submacroscopic levels. The notation $\mathcal{E}_{hd}$ and the term *elasticity with hierarchical disarrangements* are used to distinguish the present choice of the constitutive class from our choice $\mathcal{E}_d$ made when only one submacroscopic level was considered [2]. As we pointed out in the previous paragraph, our constitutive assumptions guarantee that all of the (isothermal) dynamical processes available to such a body also are compatible with the second law of thermodynamics, so that our constitutive class is included in the collection of dynamical processes satisfying the second law, i.e., $\mathcal{E}_{hd} \subset \mathcal{T}$, as required in the previous section.

The decomposition (30) with $N = T$ along with the constitutive equations (37) - (39) imply

$$\begin{aligned}(\det K_1)(\det K_2)T_0 &= T_{1.\backslash} + T_{1.d} + (\det K_2)T_{0.d} \\ &= (\det K_1)(\det K_2)D_{G_2}\Psi + (\det K_1)(\det K_2)D_{M_2}\Psi \\ &\quad +(\det K_2)(\det K_1)D_{M_1}\Psi\end{aligned}$$

and thereby yield the *stress relation* for the Piola-Kirchhoff stress $S = T_0$:

$$T_0 = D_{G_2}\Psi + D_{M_2}\Psi + D_{M_1}\Psi. \tag{46}$$

An analogous stress relation

$$T_1 = (\det K_1)(D_{G_2}\Psi + D_{M_2}\Psi) \tag{47}$$

for the reference stress T_1 at level 1 follows from the relations :

$$\begin{aligned}(\det K_2)T_1 &= T_{1.\backslash} + T_{1.d} \\ &= (\det K_1)(\det K_2)D_{G_2}\Psi + (\det K_1)(\det K_2)D_{M_2}\Psi.\end{aligned}$$

Finally, the stress relation

$$T_2 = (\det K_1)(\det K_2)D_{G_2}\Psi \tag{48}$$

follows from the constitutive assumption (37) and from (29) with $i = 2$ and with $N = T$.

We note also from the relations (23), (29) with $i = 2$, and from (37) that the Cauchy stress field is given by

$$\begin{aligned} T \circ g &= (\det G_2)^{-1} T_2 G_2^T = (\det G_2)^{-1} T_{1.\backslash} G_2^T \\ &= (\det K_1 \det K_2)(\det G_2)^{-1} D_{G_2}\Psi\, G_2^T \\ &= (\det G_0)^{-1} D_{G_2}\Psi\, G_2^T. \end{aligned} \tag{49}$$

Similarly, we have

$$\begin{aligned} T \circ g &= (\det G_1)^{-1} T_1 G_1^T = (\det G_1)^{-1} (\det K_1)(D_{G_2}\Psi + D_{M_2}\Psi)(G_2 + M_2)^T \\ &= (\det G_0)^{-1} (D_{G_2}\Psi + D_{M_2}\Psi)(G_2 + M_2)^T \end{aligned} \tag{50}$$

and

$$\begin{aligned} T \circ g &= (\det G_0)^{-1} T_0 G_0^T \\ &= (\det G_0)^{-1} (D_{G_2}\Psi + D_{M_2}\Psi + D_{M_1}\Psi)(G_2 + M_2 + M_1)^T. \end{aligned} \tag{51}$$

5.2 Universal consistency relations for stresses revisited

As a consequence of the constitutive equations (37) - (39), the consistency relations (32) and (33) with $N = T$ now take the form

$$(D_{G_2}\Psi + D_{M_2}\Psi)\,|_{(G_2,M_2,M_1)}\ M_1^T + D_{M_1}\Psi\,|_{(G_2,M_2,M_1)}\ G_0^T = 0, \tag{52}$$

$$D_{G_2}\Psi\,|_{(G_2,M_2,M_1)}\ M_2^T + D_{M_2}\Psi\,|_{(G_2,M_2,M_1)}\ G_1^T = 0. \tag{53}$$

These relations are two tensorial equations that, in view of the decompositions $G_1 = G_2 + M_2$ and $G_0 = G_1 + M_1 = G_2 + M_2 + M_1$, restrict the tensors of deformation G_2, M_2, and M_1 at each point X in the body and at each time t. Alternatively, the tensors of deformation G_2, M_2, and M_1 can be expressed in terms of the original tensors G_2, G_1, and $G_0 = \nabla g$ associated with the three-level structured deformation (g, G_1, G_2), so that the consistency relations (52) and (53) provide two tensorial relations that relate the fields G_1 and G_2, associated with the two submacroscopic levels, to the gradient ∇g of the macroscopic deformation field g.

We note that an alternative derivation of the consistency relations for stresses (52), (53) is provided by using the three different equations (49) - (51) for $T \circ g$ in terms of the partial derivatives of Ψ. In fact, subtracting (49) from (50) yields the consistency relation (53), while subtracting (50) from (51) recovers (52). This alternative derivation should provide the simplest derivation of the consistency relations applicable when more than two submacroscopic levels are to be considered.

5.3 Material frame indifference

At this point, we have shown that the response of an elastic body undergoing hierarchical disarrangements is completely determined by the smooth response function $(G_2, M_2, M_1) \longmapsto \Psi(G_2, M_2, M_1)$ whose values provide the free energy per unit volume in the macroscopic reference configuration at each pair (X, t) :

$$\psi(X,t) = \Psi(G_2(X,t), M_2(X,t), M_1(X,t)).$$

In order to guarantee that the response of such a body is frame-indifferent, we first must consider how three-level structured motions may transform under a change of observer $g \longrightarrow g^{obs}$, where the motions g and g^{obs} are related by

$$g^{obs}(X,t) = x(t) + Q(t)(g(X,t) - X_0) \tag{54}$$

with $t \longmapsto x(t)$ a smooth point-valued mapping and $t \longmapsto Q(t)$ a smooth, orthogonal tensor-valued mapping. The definition of change of observer (54) and the chain rule tell us that the macroscopic deformation gradient $G_0 = \nabla g$ satisfies the transformation rule $G_0 \to QG_0$ under change of observer. The identification relations (6) and (7) for G_1 and G_2 tell us that these quantities have the same transformation property as G_0, and the definitions of M_1 and M_2 as differences formed from G_0, G_1, and G_2 yield the transformation rules:

$$G_i \to QG_i \text{ for } i = 0, 1, 2 \tag{55}$$

$$M_i \to QM_i \text{ for } \ i = 1, 2. \tag{56}$$

Thus, *the deformations with and without disarrangements at all levels transform in the same manner under change of observer.* Consequently, we say that the three-level structured motion $(g^{obs}, G_1^{obs}, G_2^{co})$ is obtained from (g, G_1, G_2) by means of a change of observer if g^{obs} is given by (54) and

$$G_i^{obs}(X,t) = Q(t)G_i(X,t) \quad \text{for all pairs } (X,t) \text{ and for } i = 1, 2. \tag{57}$$

We use the notation $(g, G_1, G_2) \to (g^{obs}, G_1^{obs}, G_2^{obs})$ to indicate that the three-level structured motion $(g^{obs}, G_1^{obs}, G_2^{obs})$ is obtained from (g, G_1, G_2) by means of a change of observer.

It is standard in continuum physics to require that the free energy density $\psi(X, t)$ be unchanged and that the Cauchy stress $T(g(X, t), t)$ be replaced by $Q(t)T(g^{obs}(X, t), t)Q(t)^T$ under a change of observer. We adopt these transformation rules here and so arrive at the transformation rule for dynamical processes under change of observer:

$$(g, G_1, G_2, \psi, T \circ g) \to (g^{obs}, G_1^{obs}, G_2^{obs}, \psi^{obs}, T^{obs} \circ g^{obs}) \tag{58}$$

where for all X and t: $\psi^{obs}(X,t) = \psi(X,t)$, $(T^{obs} \circ g^{obs})(X,t) = Q(t)T(g(X,t),t)Q(t)^T$, and where $(g, G_1, G_2) \to (g^{obs}, G_1^{obs}, G_2^{obs})$. We recall the definition $\Gamma :=$

$T_0 \cdot \dot{G}_0 - \dot{\psi}$ of the rate of internal dissipation in a dynamical process, and we may now determine the transformation rule for Γ under a change of observer:

$$\begin{aligned}
\Gamma^{co} &= T_0^{obs} \cdot (G_0^{obs})^{\cdot} - (\psi^{obs})^{\cdot} \\
&= (T^{obs} \circ g^{obs})(G_0^{obs})^{*} \cdot (G_0^{obs})^{\cdot} - \dot{\psi} \\
&= Q(T \circ g)Q^T(QG_0)^{*} \cdot (QG_0)^{\cdot} - \dot{\psi} \\
&= Q(T \circ g)Q^T(QG_0^{*}) \cdot (\dot{Q}\, G_0 + Q\, \dot{G}_0) - \dot{\psi} \\
&= T_0 \cdot \dot{G}_0 - \dot{\psi} + T_0 G_0^T \cdot Q^T \dot{Q} = \Gamma + T_0 G_0^T \cdot Q^T \dot{Q}.
\end{aligned}$$

Thus we have: $\Gamma \to \Gamma + T_0 G_0^T \cdot Q^T \dot{Q}$ under a change of observer, and, because $Q^T \dot{Q}$ is skew-valued, we may also write

$$\Gamma \to \Gamma + sk(T_0 G_0^T) \cdot Q^T \dot{Q} \tag{59}$$

where $sk(A) := (1/2)(A - A^T)$ is the skew part of a tensor A. When N is taken to be the Cauchy stress T , the following additional transformation rules of the type (55) and (56) follow from the definitions and relations established in Section 2.3 on tensor fluxes :

$$\begin{aligned}
(T_i)^{obs} &= QT_i \quad \text{for } i = 0, 1, 2 \\
(T_{i-1.\backslash})^{obs} &= QT_{i-1.\backslash} \quad \text{for } i = 1, 2 \\
(T_{i-1.d})^{obs} &= QT_{i-1.d} \quad \text{for } i = 1, 2.
\end{aligned} \tag{60}$$

We say that the response of an elastic body undergoing hierarchical disarrangements is *frame-indifferent* if the constitutive class $\mathcal{E}_{hd}$ is "closed under a change of observer", i.e., if for every change of observer there holds

$$(g, G_1, G_2, \psi, T \circ g) \in \mathcal{E}_{hd} \Longrightarrow (g^{obs}, G_1^{obs}, G_2^{obs}, \psi^{obs}, T^{obs} \circ g^{obs}) \in \mathcal{E}_{hd} \tag{61}$$

In other words, modification of a dynamical process in $\mathcal{E}_{hd}$ by a change of observer cannot produce a dynamical process lying outside of $\mathcal{E}_{hd}$. To obtain the consequences of frame-indifference (61) we first note that the constitutive relation (45) tells us that for every dynamical process in $\mathcal{E}_{hd}$, both Γ and Γ^{obs} must be non-negative. Because the orthogonal-valued mapping $t \longmapsto Q(t)$ is arbitrary and $Q^T(t)\dot{Q}(t)$ is skew, the formula (59) is equivalent to the symmetry of the tensor field $T_0 G_0^T = (\det G_0)(T \circ g)$, i.e., the Cauchy stress $T(X, t)$ is symmetric for every X and t in every dynamical process in $\mathcal{E}_{hd}$. According to the relation (49), the preservation of the dissipation inequality (45) under change of observer is equivalent to the symmetry condition

$$(D_{G_2}\Psi(G_2, M_2, M_1)G_2^T)^T = D_{G_2}\Psi(G_2, M_2, M_1)G_2^T \tag{62}$$

for all three-level structured motions. Similarly, the constitutive relation (36) and the transformation rules (55), (56) tell us that the invariance of the free energy under change of observer is equivalent to the condition:

$$\Psi(QG_2, QM_2, QM_1) = \Psi(G_2, M_2, M_1) \tag{63}$$

for all three-level structured motions and for all orthogonal-valued mappings $t \longmapsto Q(t)$. If we restrict this condition to the case of fields that are constant in space and time, then Q becomes an arbitrary orthogonal tensor, and G_2, M_2, and M_1 are tensors satisfying

$$0 < \det G_2 \leq \det(G_2 + M_2) \leq \det(G_2 + M_2 + M_1). \tag{64}$$

If we restrict attention to tensors G_2, M_2, and M_1 for which all the inequalities in (64) are strict and assume that Ψ is smooth, then we may differentiate both sides of (63) in turn with respect to G_2, M_2, and M_1 to find that

$$D_A\Psi(QG_2, QM_2, QM_1) = QD_A\Psi(G_2, M_2, M_1) \tag{65}$$

for $A = G_2, M_2, M_1$. Continuity of the derivatives of Ψ then permits us to extend (65) to tensors satisfying (64), including the equalities. Therefore, we have obtained transformation rules for the partial derivatives of Ψ under change of observer, and they agree with the rules (60) for the stresses $T_{1.\backslash}$, $T_{1.d}$, and $T_{0.d}$. Because the tensors $K_i = G_{i-1}^{-1}G_i$ are unchanged under a change of observer, it follows that each of the three remaining consitutive relations (37) - (39) is satisfied on a dynamical process if and only if it is satisfied on every dynamical process obtained from it by means of a change of observer. These considerations establish the following characterization of frame-indifference in the present context: *the constitutive class $\mathcal{E}_{hd}$ is closed under changes of observer if and only if the response function Ψ satisfies (62) and (63) for all tensors G_2, M_2, and M_1 satisfying (64) and for all orthogonal tensors Q.* In view of the formula (49) , the condition (62) is equivalent to the symmetry of the Cauchy stress and, hence, implies the balance of angular momentum in local form.

5.4 Stable disarrangement phases

Among the geometrical and constitutive requirements that we have imposed on a given elastic body with free energy response function Ψ, we focus in this subsection on the accommodation inequalities (3), on the consistency relations (52) and (53), and on the frame-indifference condition (62). The relation (3) is a system of inequalities restricting the determinants of the geometrical fields G_0, G_1, and G_2, while (52), (53), and (62) are tensor equations in terms of the response function Ψ that restrict G_0, G_1, and G_2. For a point X in the body and a time t, we denote by F the macroscopic deformation gradient $G_0(X,t) = \nabla g(X,t)$ which we consider for the moment as fixed. By virtue of (3), (14), (52), (53), and (62) we then seek tensors G_2, M_2 and M_1 satisfying the relations

$$G_2 + M_2 + M_1 = F. \tag{66}$$

$$0 < \det G_2 \leq \det(G_2 + M_2) \leq \det F \tag{67}$$

$$D_{G_2}\Psi(G_2, M_2, M_1)M_1^T + D_{M_2}\Psi(G_2, M_2, M_1)M_1^T + D_{M_1}\Psi(G_2, M_2, M_1)\,F^T = 0 \tag{68}$$

$$D_{G_2}\Psi(G_2, M_2, M_1)\, M_2^T + D_{M_2}\Psi(G_2, M_2, M_1) M_2^T + D_{M_2}\Psi(G_2, M_2, M_1) G_2^T = 0. \tag{69}$$

$$(D_{G_2}\Psi(G_2, M_2, M_1) G_2^T)^T = D_{G_2}\Psi(G_2, M_2, M_1) G_2^T. \tag{70}$$

For a given macroscopic deformation gradient tensor F, these four tensor equations and three inequalities are the requirements that our notion of elastic body imposes on the deformation without disarrangements G_2 at level 2 and on the deformations due to disarrangements M_2 and M_1 at levels 2 and 1. (Of course, the equation (66) can be used to eliminate say G_2 from the remaining relations leaving only the disarrangement tensors M_1 and M_2 to be determined in terms of F, but we prefer to retain the full system (66) - (70) for present considerations.)

Given a macroscopic deformation gradient F, a triple (G_2, M_2, M_1) that satisfies the system (66) - (70) is called a *(three-level) disarrangement phase corresponding to* F. In the context of three-level structured deformations, only the three-level geometry and the free energy response function Ψ are employed in determining the system (66) - (70). It is clear from the consistency relations (68) and (69) and from the symmetry condition (70) that the disarrangement phases for F include all triples (G_2, M_2, M_1) that satisfy the geometrical conditions (66), (67) and also that are stationary points of Ψ, i.e., all three partial derivatives of Ψ vanish at (G_2, M_2, M_1).

We call a disarrangement phase (G_2, M_2, M_1) corresponding to F *stable* if

$$\Psi(G_2, M_2, M_1) \leq \Psi(G_2^{'}, M_2^{'}, M_1^{'})$$

for all disarrangement phases $(G_2^{'}, M_2^{'}, M_1^{'})$ corresponding to F. The problem of determining stable disarrangement phases then becomes the problem of minimizing the free energy subject to the constraints (66) - (70) imposed by the additive decomposition of F, the accommodation inequalities, and the consistency relations. It is important to note that, because of the constraints included in (66) - (70), a stable disarrangement phase need not be a stationary point of the free energy and, therefore, the stress need not vanish at a stable disarrangement phase and (as examples in the two-level case show) need not be hydrostatic.

We note that "disarrangement phase" and "stable disarrangement phase" are constitutively based concepts that do not refer to balance laws. Such notions of material stability were introduced and studied for two-level structured deformations in the context of elasticity with disarrangements ([5], [6], [43]) with a view toward applications to granular media and, more generally, to "elastic aggregates", i.e., continuous bodies that submacroscopically are composed of a large number of elastic bodies. In the present context of three-level structured deformations, the notions of disarrangement phase and stable disarrangement phase permit the description of the loss of material stability during the course of a dynamical process as well as the identification of the submacroscopic level at which destabilizing disarrangements arise.

5.5 Field relations

We collect together and record now the full system of field relations that govern motions $(X,t) \longmapsto (g(X,t), G_1(X,t), G_2(X,t))$ of an elastic body undergoing disarrangements at two submacroscopic levels with free energy response $(G_2, M_2, M_1) \longmapsto \Psi(G_2, M_2, M_1)$ that is invariant under change of observer:

$$\rho_0 \ddot{g} = \operatorname{div}(D_{G_2}\Psi + D_{M_2}\Psi + D_{M_1}\Psi) + b \tag{71}$$

$$D_{G_2}\Psi\, M_1^T + D_{M_2}\Psi\, M_1^T + D_{M_1}\Psi\,(\nabla g)^T = 0 \tag{72}$$

$$D_{G_2}\Psi\, M_2^T + D_{M_2}\Psi M_2^T + D_{M_2}\Psi\, G_2^T = 0 \tag{73}$$

$$(D_{G_2}\Psi\, G_2^T)^T = D_{G_2}\Psi\, G_2^T. \tag{74}$$

$$0 < \det G_2 \leq \det(G_2 + M_2) \leq \det(G_2 + M_2 + M_1) \tag{75}$$

$$\nabla g = G_2 + M_2 + M_1 \tag{76}$$

$$\begin{aligned} 0 \quad \leq \quad & D_{G_2}\Psi \cdot \dot{M}_2 + D_{G_2}\Psi \cdot \dot{M}_1 + \\ & D_{M_2}\Psi \cdot \dot{G}_2 + D_{M_2}\Psi \cdot \dot{M}_1 + \\ & D_{M_1}\Psi \cdot \dot{G}_2 + D_{M_1}\Psi \cdot \dot{M}_2 \end{aligned} \tag{77}$$

Here, ρ_0 denotes the density of the body in the macroscopic reference configuration (level 0), (71) is the balance of linear momentum in that configuration, (72) and (73) are the consistency relations for stresses at submacroscopic levels 1 and 2, (74) expresses the invariance of the internal dissipation under change of observer or, equivalently, the symmetry of the Cauchy stress T which, in turn, guarantees that the balance of angular momentum is satisfied, (75) contains the accommodation inequalities, (76) is the additive decomposition of ∇g, and (77) is the dissipation inequality. Of course, the field G_1 needed to specify the three level structured deformation (g, G_1, G_2) can be recovered from the fields G_2 and M_2 from the relation $G_1 = G_2 + M_2$.

The system of field relations (71) - (77) amounts to thirty-three scalar equations and two sets of inequalities that are to be satisfied by the thirty scalar components of g, G_2, M_2, and M_1. When Ψ does not depend upon M_1 and M_2 and when one sets $M_1 = M_2 = 0$, then $G_1 = G_2 = \nabla g$, so that only classical deformations are admitted, and the field relations (71) - (77) then reduce to those of finite elasticity:

$$\rho_0 \ddot{g} = \operatorname{div}(D\Psi(\nabla g)) + b \tag{78}$$

$$(D\Psi(\nabla g)(\nabla g)^T)^T = D\Psi(\nabla g)(\nabla g)^T \tag{79}$$

where the remaining relations are satisfied identically. In particular, the inequalities are satisfied with equality throughout, while the assumed invariance of Ψ under change of observer in the special setting of finite elasticity implies that (79) is satisfied. If we require $D_{M_2}\Psi$ is identically zero and consider motions in which $M_2 = 0$, but in which M_1 need not vanish, then the field relations (71) - (77) for the three-level case reduce to those of elasticity with disarrangements in the context of two-level structured deformations [2]. Consequently, it would be appropriate in studies of the field relations (71) - (77) to consider counterparts of the notions of submacroscopically stable equilibria [6] and of moving phase boundaries [23] already investigated for the two-level case.

6 The case of purely dissipative disarrangements

In general, disarrangements such as slips and separations at submacroscopic levels 1 and 2 can contribute to the free energy through dependence of Ψ upon M_1 and M_2. For example, the disarrangements at both submacroscopic levels in the three-level shear introduced in Section 2 can be captured via approximating sequences in terms of the slips between parallel faces of small pieces of the body ("cards"). In single crystals, such slips can arise across crystallographic planes and may produce a dependence of energy stored on amount of slip that is periodic [30]. A significantly different dependence of energy upon disarrangements arises in the case of elastic aggregates, i.e., continua that are comprised of many small elastic bodies, where we can consider interactions between different pieces of the aggregate as they move relative to one another, while maintaining the same level of deformation in the individual pieces. Such interactions correspond to short-range forces across boundaries of the pieces and may or may not result in additional changes in free energy of the aggregate. In the absence of a crystalline structure of the pieces or other sources of cohesive forces between pieces of the aggregate, one expects that disarrangements will not alter the free energy of the aggregate, and it is appropriate now to adapt a notion of "purely dissipative disarrangements" (introduced earlier for two-level structured deformations of elastic bodies ([6], [23]) to the present three-level setting.

A physical context in which purely dissipative, hierarchical disarrangements might arise would be an aggregate consisting of tiny bundles, each of which is an aggregate of thin, rectangular elastic sheets held loosely together by an inextensible string (see Figure 1). Assuming then that the only mechanism for storage of energy is the deformation of the elastic sheets, the collection of tiny bundles would conform to the notion of a body undergoing purely dissipative disarrangements at two submacroscopic levels, because the relative motion of the elastic sheets within a bundle and the relative motion of different bundles would, themselves, not store energy. Thus we are led to consider the case in which the constitutive relation (36) takes the special form

$$\psi(X,t) = \Psi(G_2(X,t)), \tag{80}$$

while we continue to allow the disarrangement fields M_1 and M_2 to be non-zero. In this case, we have $D_{M_2}\Psi = D_{M_1}\Psi = 0$ and $D_{G_2}\Psi = D\Psi$, so that the field relations (71) - (74) simplify as follows:

$$\rho_0 \ddot{g} = \mathrm{div}(D\Psi(G_2)) + b \tag{81}$$

$$D\Psi(G_2)\, M_1^T = 0 \tag{82}$$

$$D\Psi\,(G_2)M_2^T = 0 \tag{83}$$

$$0 < \det G_2 \leq \det(G_2 + M_2) \leq \det(G_2 + M_2 + M_1) \tag{84}$$

$$\nabla g = G_2 + M_2 + M_1 \tag{85}$$

$$0 \leq D\Psi\,(G_2) \cdot (\dot{M}_1 + \dot{M}_2) \tag{86}$$

We note that the last relation (74) in (71) - (74) automatically is satisfied in the present case in view of (80) and in view of the assumed invariance (63) of free energy under change of observer. In the present case of purely dissipative disarrangements, the field relations amount to thirty scalar equations and two sets of inequalities for thirty unknown components of the fields g, G_2, M_2, M_1. In this context the Piola-Kirchhoff stress $S = T_0$ in the stress relation (46) is given by

$$S = D\Psi(G_2) = D\Psi(\nabla g - M_1 - M_2). \tag{87}$$

An interesting aspect of the special field relations (81) - (86) is the following: the equations (81), (82), (83), (85), as well as the stress relation (87) and the dissipation inequality (86) are unchanged under interchange of M_1 and M_2, while only the system of inequalities (84) changes under that interchange. Consequently, if for given g and G_2, the unchanged relations are satisfied for the pair of disarrangement fields (M_1, M_2), then they are satisfied by the pair (M_2, M_1), and the level at which the disarrangements associated with M_1 and M_2 can occur is controlled soley by which, if any, of the following two systems of inequalities may be satisfied:

$$\det G_2 \leq \det(G_2 + M_2) \leq \det(\nabla g) = \det(G_2 + M_2 + M_1) \tag{88}$$

$$\det G_2 \leq \det(G_2 + M_1) \leq \det(\nabla g) = \det(G_2 + M_1 + M_2). \tag{89}$$

If only the first system of inequalities (88) is satisfied, then the disarrangements corresponding to the field M_2 can occur at submacroscopic level two but not at level one, and the field relations all are satisfied by the structured motion $(g, G_2 + M_2, G_2)$. Similarly, if only the second system (89) is satisfied, then the disarrangements associated with the field M_1 can occur at level two but not at level one, and the field relations are satisfied by the structured motion $(g, G_2 + M_1, G_2)$. If both systems of inequalities are satisfied then either set of disarrangements may occur at either level, and the field relations are satisfied by both of the motions $(g, G_2 + M_2, G_2)$ and $(g, G_2 + M_1, G_2)$.

For an example of the last type in which either disarrangement field may correspond to either hierarchical level, we put $G_2 = I$ and assume also that the the derivative $D\Psi(I)$ of the free energy response at the identity has non-trivial nullspace. We choose a unit vector v in the nullspace of $D\Psi(I)$ and a unit vector u perpendicular to v, we put $M_1 = -u \otimes v$, $M_2 = u \otimes v$, and we choose g to be a homogeneous deformation with $\nabla g = I + M_1 + M_2 = I$ to obtain, as is easily verified, a time-independent solution of the field relations (81) - (86). The two systems of inequalities (88) and (89) become

$$1 \leq \det(I + u \otimes v) \leq \det(I - u \otimes v + u \otimes v) = 1, \tag{90}$$

and

$$1 \leq \det(I - u \otimes v) \leq \det(I - u \otimes v + u \otimes v) = 1. \tag{91}$$

Thus, both systems are satisfied (with equality), and we may assign either the submacroscopic level *one* to the disarrangements associated with $u \otimes v$ or the submacroscopic level *two* to the disarrangements associated with $u \otimes v$. If say we assign level one to $u \otimes v$, then the resulting solution of the field equations corresponds to the three-level shear (g, G_1, G_2) defined in (8), (9) with $\mu_0 = \mu_2 = 0$ and $\mu_1 = 1$, so that $G_1 = I + u \otimes v$, $G_2 = I$. The alternative assignment of level two to the disarrangements associated with $u \otimes v$ results in the three-level shear (g, G_1, G_2) in (8), (9) with $\mu_0 = \mu_2 = 0$ and $\mu_1 = -1$, where now we simply replace u by $-u$ in all of the formulas for the previous choice.

In order to provide an example in which only one of the two systems of inequalities (88) and (89) is satisfied, we assume now that $D\Psi(I)$ has nullspace of dimension at least two, and we choose perpendicular unit vectors u and v in the nullspace. We let ε with $0 < \varepsilon < 1/2$ be given, we put $G_2 = I$, and define

$$\begin{aligned} M_1 \quad &: \quad = -\varepsilon u \otimes u - (1-\varepsilon) v \otimes u \\ M_2 \quad &: \quad = u \otimes v. \end{aligned} \tag{92}$$

As above, we choose g to be a time-independent homogeneous deformation with gradient $G_2 + M_2 + M_1$. Again, the equations (81), (82), (83), (85) and the dissipation inequality (86) -- all invariant under interchange of M_1 and M_2 -- are satisfied for the time-independent three-level structured motion (g, G_1, G_2). The two systems of inequalities (88) and (89) now read

$$\begin{aligned} \det I \quad \leq \quad & \det(I + u \otimes v) \leq \\ & \det(I + -\varepsilon u \otimes u - (1-\varepsilon) v \otimes u - \varepsilon u \otimes u - (1-\varepsilon) v \otimes u) \end{aligned}$$

and

$$\begin{aligned} \det I \quad \leq \quad & \det(I - \varepsilon u \otimes u - (1-\varepsilon) v \otimes u) \leq \\ & \det(I + u \otimes v - \varepsilon u \otimes u - (1-\varepsilon) v \otimes u). \end{aligned}$$

The first system reduces to $1 \leq 1 \leq 2(1-\varepsilon)$, which validates the choice $u \otimes v$ for the level 2 disarrangement tensor because $0 < \varepsilon < 1/2$. The second reads $1 \leq 1 - \varepsilon \leq 2(1-\varepsilon)$ which eliminates the choice $-\varepsilon u \otimes u - (1-\varepsilon) v \otimes u$ for the level 2 disarrangement tensor, since the first inequality is false. The elimination of the second choice shows that the choice $-\varepsilon u \otimes u - (1-\varepsilon) v \otimes u$ for the level 2 disarrangement tensor to construct the triple (g, G_1, G_2) eliminates the possibility of approximating that triple by *injective*, piecewise smooth mappings in the sense of the Approximation Theorem. In more geometrical terms, the possibility of zooming out from level two to level one by means of the combination of shrinkage and shearing associated with the disarrangement tensor $-\varepsilon u \otimes u - (1-\varepsilon) v \otimes u$ is eliminated by the accommodation inequalities, whereas the same combination may occur in zooming out from level 1 to level 0 if the tensor $u \otimes v$ accounts for the disarrangements in zooming out from level 2 to level 1.

7 Boundary conditions

We have obtained a detailed system of field relations (71) - (77) for the fields g, G_1, and G_2 that determine a structured motion in an elastic body undergoing hierarchical disarrangements at two submacroscopic levels. These relations are formulated in terms of the fields g, G_2, M_2, and M_1, because these fields provide the refined decomposition $\nabla g = G_2 + M_2 + M_1$ of the macroscopic deformation gradient as well as the parallel decomposition

$$\begin{aligned} S &= T_0 = D_{G_2}\Psi + D_{M_2}\Psi + D_{M_1}\Psi \\ &= (\det K_1 \det K_2)^{-1}(T_{1.\backslash} + T_{1.d} + (\det K_2)T_{0.d}) \end{aligned}$$

of the Piola-Kirchhoff stress (see Section 5.1).

The formulation of boundary conditions of place for (71) - (77) appears to be straightforward, because, even in the refined geometry of three-level structured deformations, the only field available for the computation of position is the macroscopic deformation field g. The refined decomposition $\nabla g = G_2 + M_2 + M_1$ does not provide a decomposition of g, itself, and therefore does not provide refined versions of boundary conditions of place. Thus, we single out a portion $\partial\mathcal{B}_p$ of the boundary of the body and require for all times t and $X \in \partial\mathcal{B}_p$:

$$g(X,t) = g_p(X,t) \tag{93}$$

where g_p is a given field on $\partial\mathcal{B}_p$.

Formulation of boundary conditions of traction is more interesting, and the starting point for our discussion is the well-known fact that the contact force $\int_{g(\mathcal{S},t)} T(x,t)n(x)dA_x$ on an oriented surface $g(\mathcal{S},t)$ in the deformed configuration is given by

$$\int_{\mathcal{S}} T_0(X,t)N(X)dA_X = \int_{\mathcal{S}} T(g(X,t),t)G_0^*(X,t)N(X)dA_X \tag{94}$$

where $N(X)dA_X$ is the vector element of area at the point X in the surface $\mathcal{S}$ in $\mathcal{B}$ (and $n(x)dA_x$ is the corresponding vector element of area for $g(\mathcal{S},t)$ at the point $g(X,t)$ in $g(\mathcal{S},t)$). In other words, the reference stress T_0 carries the same information about contact forces supported by the smooth surface $\mathcal{S}$ in $\mathcal{B}$ as does the Cauchy stress T for the smooth surface $g(\mathcal{S},t)$ in the deformed configuration $g(\mathcal{B},t)$. This observation simply restates the equivalence of the two stress measures T_0 and T of contact forces in continuum theories.

In the present context of three-level structured motions (g, G_1, G_2), there are two additional measures of stress obtained when $i = 1, 2$ from the definition (23) with $N = T$: $T_1 = (T \circ g)G_1^*$ and $T_2 = (T \circ g)G_2^*$. Like T_0, both are defined at each time t on $\mathcal{B}$ and both allow us to compute for each smooth surface $\mathcal{S}$ in $\mathcal{B}$ the surface integrals:

$$\int_{\mathcal{S}} T_i(X,t)N(X)dA_X = \int_{\mathcal{S}} T(g(X,t),t)G_i^*(X,t)N(X)dA_X \quad \text{for } i = 1, 2. \tag{95}$$

The integrals on the right-hand sides of (94) and (95) differ only by the oriented element of area $G_i^*(X,t)N(X)dA_X$ in the integrand. The continuity of the adjugate operation * and the formulas (6) and (7) here yield for every double sequence $(n_1, n_2) \longmapsto f_{n_1 n_2}$ as in the Approximation Theorem:

$$G_1^*(X,t)N(X) = ((\lim_{n_1\to\infty} \nabla \lim_{n_2\to\infty} f_{n_1 n_2})(X,t))^* N(X) \tag{96}$$

$$G_2^*(X,t)N(X) = ((\lim_{n_1\to\infty} \lim_{n_2\to\infty} \nabla f_{n_1 n_2})(X,t))^* N(X) \tag{97}$$

while (5) yields

$$G_0^*(X,t)N(X) = (\nabla \lim_{n_1\to\infty} \lim_{n_2\to\infty} f_{n_1 n_2})^* N(X). \tag{98}$$

Based on our discussion of (6) - (5) in Section 2, we may assert that the choice of index i then influences the measure of contact force $\int_{\mathcal{S}} T_i(X,t)N(X)dA_X = \int_{\mathcal{S}} T(g(X,t),t)G_i^*(X,t)N(X)dA_X$ by adjusting the element of area on $\mathcal{S}$ to account for no disarrangements whatsoever at sites on the given surface ($i = 2$), disarrangements only at submacroscopic level 2 ($i = 1$), or disarrangements at both submacroscopic levels 1 and 2 ($i = 0$). We conclude that the reference stresses T_1 and T_2 at the submacroscopic levels each carries different information about contact forces supported by the smooth surface $\mathcal{S}$ in $\mathcal{B}$ than does T_0, and the difference is accounted for by whether or not the effect of disarrangements at the submacroscopic levels is included in the element of area on $\mathcal{S}$.

We observe that prescription of the traction vector $\tau(X,t)$ at each point on a portion $\partial\mathcal{B}_\tau$ leads, for example, to the following three traction boundary conditions: choose $i \in \{0,1,2\}$ and require

$$T_i(X,t)N(X) = \tau(X,t) \quad \text{for every } X \in \partial\mathcal{B}_\tau \text{ and for every } t. \tag{99}$$

Here, $N(X)$ denotes the unit outward normal to $\partial\mathcal{B}_\tau$ at X, and we call (99) the *level i traction boundary condition*. The equivalence of T_0 and T as measures of contact forces assures us that the level 0 traction boundary condition is equivalent to specifying a traction boundary condition on a portion of $g(\mathcal{B},t)$ using the Cauchy stress field. Moreover, the stress relation (46) permits us to rewrite the level 0 traction boundary condition in terms of the response function Ψ: for every $X \in \partial\mathcal{B}_\tau$ and for every t

$$(D_{G_2}\Psi + D_{M_2}\Psi + D_{M_1}\Psi)\mid_{(X,t)} N(X) = \tau(X,t). \tag{100}$$

Similarly, (47) permits us to rewrite the level 1 traction condition as: for every $X \in \partial\mathcal{B}_\tau$ and for every t

$$(\det K_1)(D_{G_2}\Psi + D_{M_2}\Psi)\mid_{(X,t)} N(X) = \tau(X,t) \tag{101}$$

while (48) yields for the level 2 traction condition: for every $X \in \partial\mathcal{B}_\tau$ and for every t

$$(\det K_1)(\det K_2) D_{G_2}\Psi \mid_{(X,t)} N(X) = \tau(X,t). \tag{102}$$

We recall from Section 2, $K_i := G_{i-1}^{-1}G_i$ for $i = 1, 2$. Relation (102) shows that satisfaction of the level 2 traction boundary condition amounts to the statement that only the driving traction associated with G_2 is activated in balancing the applied traction τ, while (100) tells us that the the driving tractions corresponding to G_2, M_2 and M_1 add together in balancing τ. For the intermediate, level 1 boundary condition (101), the two driving tractions corresponding to G_2 and M_2 combine to balance τ.

Up to this point, we have not provided any guidance as to which of the traction conditions (100) - (102) should be employed to impose traction boundary conditions. The formulation of our field theory in terms of a free energy response that depends upon G_2, M_2, and M_1 or, equivalently, in terms of G_i for $i = 0, 1, 2$ presupposes knowledge for the given body about the number of submacroscopic levels available to the body and the manner in which the body stores energy in response to geometrical changes with and without disarrangements at these levels. It is consistent with this presupposition to suppose available some information about the device that provides the tractions τ to the body on the surface $\partial\mathcal{B}_\tau$.

For example, if the body is an aggregate composed of small elastic grains, then it is natural to identify the submacroscopic level 2 with a collection of such grains large enough to justify calculating the gradients $\nabla f_{n_1,n_2}$ that approximate G_2 by averaging deformations of individual grains in such a collection. We presume then that it is known whether or not the tractions τ provided by a given loading device can vary appreciably over a surface of area comparable to the cross section of such a collection. If so, it is natural to impose the level 2 traction condition (102). On the other hand, if τ varies appreciably only over surfaces of area comparable to the cross section of larger submacroscopic features such as shear bands or such as load-supporting, geometrically irregular chains of grains, then it is natural to impose the level 1 traction condition (101). Finally, if the imposed traction τ varies only over surfaces of area large compared to the cross sections of the larger submacroscopic features, then it is natural to impose the level 0 traction condition (100). We conclude that the choices of traction boundary condition available in this multilevel approach allow us to match the choice of level of the traction condition to characteristics of the loading device, and this matching capability provides flexibility beyond that afforded by a single level approach.

While there is no qualitiative theory yet available for the field equations of our field theory, we indicate here initial conditions that appear natural in view of the form of the field equations. Consequently, for a given initial time t_I we suggest that specification of the fields $g(\cdot, t_I)$, $\dot{g}(\cdot, t_I)$, $G_1(\cdot, t_I)$, and $G_2(\cdot, t_I)$ on the body $\mathcal{B}$ provides appropriate initial conditions to supplement the boundary conditions considered above in order to determine the evolution of the body in the form of a three-level structured motion.

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8 Appendix: Proof of the Approximation Theorem for three-level structured deformations

In this appendix we use the terms "simple deformation" and "piecewise-fit region" in the sense of [1]. Roughly speaking, a simple deformation is a piecewise smooth, injective mapping, and a piecewise-fit region is a finite union of regions without unopened cracks and with finite surface area.

Theorem 1 *For each three-level structured deformation* (g, G_1, G_2) *from a piecewise-fit region* $\mathcal{B}$ *there exists a double sequence* $(n_1, n_2) \longmapsto f_{n_1,n_2}$ *of simple deformations from* $\mathcal{B}$ *for which*

$$\begin{aligned} \lim_{n_1\to\infty} \lim_{n_2\to\infty} f_{n_1,n_2} &= g \\ \lim_{n_1\to\infty} \nabla \lim_{n_2\to\infty} f_{n_1,n_2} &= G_1 \\ \lim_{n_1\to\infty} \lim_{n_2\to\infty} \nabla f_{n_1,n_2} &= G_2 \end{aligned}$$

where each of the iterated limits $\lim_{n_1\to\infty}$ *and* $\lim_{n_2\to\infty}$ *is taken in the sense of* L^∞*-convergence.*

Proof. Let a three-level stuctured deformation (g, G_1, G_2) and a positive integer n_1 be given. The three-level accommodation inequality (3) implies that the pair (g, G_1) is a (two-level) structured deformation. By the Approximation Theorem for two-level structured deformations and properties of the determinant mapping, we may choose a constant $\bar{C} > 0$ (that depends only upon the field G_1) and a simple deformation f_{n_1} such that

$$\|g - f_{n_1}\|_\infty < \frac{1}{n_1}, \ \|G_1 - \nabla f_{n_1}\|_\infty < \frac{\bar{C}}{n_1}, \text{ and } \|\det G_1 - \det \nabla f_{n_1}\|_\infty < \frac{1}{n_1}, \tag{103}$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm. By the three-level accommodation inequality (3) and (103) we conclude that

$$\det \nabla f_{n_1} > \det G_1 - \frac{1}{n_1} \geq \det G_2 - \frac{1}{n_1} > c - \frac{1}{n_1}, \tag{104}$$

and we take from this point on $n_1 > c^{-1}$. (The inequality (104) can be written in terms of fields because the accommodation inequality holds at almost every point in $\mathcal{B}$ for one and the same positive number c.) We define

$$\varphi(n_1) := 2(1 - (1 - \frac{1}{n_1 c})^{\frac{1}{3}}) > 0 \tag{105}$$

and note that

$$\lim_{n_1\to\infty}\varphi(n_1)=0\ , \tag{106}$$

as well as $\varphi(n_1)>1-(1-\frac{1}{n_1c})^{\frac{1}{3}}$, so that

$$1-\frac{1}{n_1c}>(1-\varphi(n_1))^3. \tag{107}$$

We infer from (104), (105), and (107) that

$$\begin{aligned}\det\nabla f_{n_1} &> (1-\frac{1}{n_1\det G_2})\det G_2\\ &> (1-\frac{1}{n_1c})\det G_2>(1-\varphi(n_1))^3\det G_2\\ &= \det((1-\varphi(n_1))G_2).\end{aligned} \tag{108}$$

By (106) and the inequality $\det G_2>c$, we may choose $N_1>c^{-1}$ such that

$$(1-\varphi(n_1))^3>\frac{1}{2}\quad\text{for all } n_1>N_1.$$

The estimate (108) then implies

$$\det\nabla f_{n_1}>\det((1-\varphi(n_1))G_2)>\frac{c}{2}\quad\text{for all } n_1>N_1,$$

and we conclude that the pair $(f_{n_1},(1-\varphi(n_1))G_2)$ satisfies the accommodation inequality for (two-level) structured deformation for all $n_1>N_1$. Thus, for each $n_1>N_1$ and positive integer n_2, the Approximation Theorem for (two-level) structured deformations permits us to choose a simple deformation f_{n_1,n_2} such that

$$\|f_{n_1,n_2}-f_{n_1}\|_\infty<\frac{1}{n_2},\quad\text{and}\quad\|\nabla f_{n_1,n_2}-(1-\varphi(n_1))G_2\|_\infty<\frac{1}{n_2}.$$

Therefore, for each $n_1>N_1$ we may let n_2 tend to ∞ in the last two inequalities to obtain

$$\lim_{n_2\to\infty}f_{n_1,n_2}=f_{n_1}\quad\text{and}\quad\lim_{n_2\to\infty}\nabla f_{n_1,n_2}=(1-\varphi(n_1))G_2,$$

and, in view of (106), we may then let n_1 tend to ∞ in each of the last two relations to conclude from (103)

$$\begin{aligned}\lim_{n_1\to\infty}\lim_{n_2\to\infty}f_{n_1,n_2} &= \lim_{n_1\to\infty}f_{n_1}=g\\ \lim_{n_1\to\infty}\nabla\lim_{n_2\to\infty}f_{n_1,n_2} &= \lim_{n_1\to\infty}\nabla f_{n_1}=G_1\\ \lim_{n_1\to\infty}\lim_{n_2\to\infty}\nabla f_{n_1,n_2} &= \lim_{n_1\to\infty}((1-\varphi(n_1))G_2)=G_2.\end{aligned}$$

In all of these relations, the limits are taken in the sense of L^∞. ■

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