

Fig. 4. Flutter modes;  $\gamma = 3$

have been described. Very similar qualitative results are obtained for a continuous system of a uniform elastic cantilever subjected to a follower load at its tip and having a pinned support with a rotational spring [8].

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Eigenvalues of the Elastoplastic Constitutive Operator

MSC (1991): 73E05

1. Introduction

The idea of non-associative flow-rules in elastoplasticity dates back to the works of MRÓZ [1] and MANDEL [2]. In the modern technical literature, non-associativity is considered as the key feature for modelling the behavior of porous metals [3], of metals showing the Strength-Differential effect [4], of concrete and geomaterials [5, 6]. Moreover, deviations from normality can model non-Schmid effects in crystals [7] and result to promote various kinds of material instability such as localization of deformation [8], surface instability [9] and plastic cavitation [10].

Due to its importance in the modelling of many materials of engineering importance, the elastoplastic operator with non-associative flow-rule has been extensively investigated. In particular, MAIER and HUECKEL [11] found the necessary and sufficient

condition for positive definiteness of the elastoplastic operator, RICE [8] for ellipticity and BIGONI and ZACCARIA [12] for strong ellipticity. Moreover, in the particular case when the tensors giving the plastic flow-mode and the yield surface gradient are co-axial, the conditions of loss of strong ellipticity [12] and loss of ellipticity [13, 14] can be given in an explicit form. Even though the elastoplastic operator has been studied in many details, few information seem to be known about its eigenvalues. The only contribution has been given by PREVOST in an unpublished note [15], where the eigenvalues are shortly derived, together with a condition for excluding complex eigenvalues.

A formal solution to the eigenvalue problem of the elastoplastic operator, different from that reported in [15], is presented in this paper. It is shown that more than two complex conjugate eigenvalues are not possible. The necessary and sufficient condition for complex eigenvalues derived in [15] is obtained and then re-written excluding the case of complex eigenvalues occurring for negative values of the plastic modulus. In this case it is shown that complex eigenvalues are excluded for deviatoric normality. Moreover, in the case of isotropic hardening, assuming the existence of a smooth plastic potential surface, it is shown that complex eigenvalues are not admitted, even in the case of deviatoric non-associativity. Finally, an application is presented for the model proposed in [4]. This model is suitable for the description of the behavior of metals showing the Strength-Differential effect.

2. Constitutive assumptions

Reference is made to an initially isotropic, time independent, elastoplastic solid having two tensorial zones. The behavior of this solid is governed by a constitutive operator which relates a generic

objective rate of any symmetric stress measure  $\overset{\Delta}{T}$  to the velocity of deformation  $\overset{\Delta}{D}$ , in the following form (details on the notation used in this paper can be found in [16]):

$$\overset{\Delta}{T} = \mathbb{E}[\overset{\Delta}{D}], \quad \text{if } \mathbf{N} \cdot \overset{\Delta}{D} \leq 0, \quad (1)$$

$$\overset{\Delta}{T} = \mathbb{H}[\overset{\Delta}{D}], \quad \text{if } \mathbf{N} \cdot \overset{\Delta}{D} \geq 0, \quad (2)$$

where  $\mathbb{E}: \text{Sym} \rightarrow \text{Sym}$  is the elastic fourth-order tensor,  $\mathbf{N}$  gives the direction of the yield surface gradient in the deformation space and the fourth-order tensor  $\mathbb{H}: \text{Sym} \rightarrow \text{Sym}$  defines the comparison solid corresponding to the loading branch of the constitutive operator:

$$\mathbb{H} = \mathbb{E} - \frac{1}{\varphi} \mathbf{M} \otimes \mathbf{N}, \quad (3)$$

in which the symbol  $\otimes$  denotes the dyadic product,  $\mathbf{M}$  gives the direction of the plastic flow in the deformation space and the plastic modulus  $\varphi$  is related to the hardening modulus  $h$  through:

$$\varphi = h + \mathbf{M} \cdot \mathbb{E}^{-1}[\mathbf{N}]. \quad (4)$$

It should be noted that no restrictions are placed on the choice of the type of hardening, as well as on the choice of  $\mathbf{N}$  and  $\mathbf{M}$ , which can be not coaxial.

The elastic tensor  $\mathbb{E}$  will be assumed to be an isotropic mapping  $\text{Sym} \rightarrow \text{Sym}$ . Consequently:

$$\mathbb{E} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I}, \quad (5)$$

where  $\mathbf{I}$  and  $\mathbb{I}$  denote the second and fourth-order identity tensors respectively and  $\lambda$  and  $\mu$  are the Lamé moduli. It is well-known [17] that (5) merely represent an assumption in finite-elasticity theory. Nevertheless, hypothesis (5) has been widely employed in bifurcation analyses of elastic and elastoplastic solids [4, 6, 8, 18, 19].

The eigenvalue problem for the elastic tensor  $\mathbb{E}$  has the following solution (see, e.g. [20], § 22):

Eigenvalue	Multiplicity	Eigenspaces
$2\mu + 3\lambda$	1	$\{\alpha \mathbf{I}, \forall \alpha \in \mathbb{R}\}$
$2\mu$	5	$\{\mathbf{V} \in \text{Sym} : \text{tr } \mathbf{V} = 0\}$

All the eigenvalues of the elastic tensor are also eigenvalues of the elastoplastic operator. In fact, by changing (if necessary) the sign

of the corresponding eigentensor, it is always possible to satisfy the unloading condition  $N \cdot D \leq 0$ . For the same reason, all eigenvalues of the tensor  $\mathbf{H}$  are also eigenvalues of the elastoplastic operator. Then, all eigenvalues of the elastoplastic operator are the union of the eigenvalues of  $\mathbf{E}$  and  $\mathbf{H}$ . The eigenvalues and corresponding eigentensors of  $\mathbf{H}$  will be obtained in the following.

3. Eigenvalue problem for  $\mathbf{H}$

The eigenvalue problem for the operator  $\mathbf{H}$  is posed as follows:

$$(\mathbf{H} - \eta \mathbf{I}) V = 0, \quad \eta \in \mathbb{R}, \quad V \in \text{Sym} - \{0\}. \tag{6}$$

Eqn. (6), by using (3) and (5), becomes

$$\lambda(\text{tr } V) I + 2\mu V - \frac{1}{g} (V \cdot N) M - \eta V = 0. \tag{7}$$

The following (non-orthogonal) basis of  $\text{Sym}$  is introduced:

$$E_i \quad (i = 1, 2, \dots, 6), \tag{8}$$

such that:

$$E_1 = I, \tag{9}$$

$$E_2 = N, \tag{10}$$

$$\text{tr } E_i = 0, \quad E_i \cdot E_i = 1, \quad (i = 3, \dots, 6), \tag{11}$$

$$E_i \cdot E_j = 0, \quad (i \neq j, \quad i, j = 2, \dots, 6). \tag{12}$$

The problem (7) can be solved through a projection onto the basis (8), thus obtaining:

$$\left. \begin{aligned} (3\lambda + 2\mu - \eta) \text{tr } V + \frac{1}{g} (V \cdot N) \text{tr } M &= 0 \\ \lambda(\text{tr } N) \text{tr } V + \left(2\mu - \frac{1}{g} M \cdot N - \eta\right) V \cdot N &= 0. \\ (2\mu - \eta) V \cdot E_i - \frac{1}{g} (M \cdot E_i) (V \cdot N) &= 0 \quad (i = 3, \dots, 6) \end{aligned} \right\} \tag{13}$$

The system (13) is the representation of (7) in the dual basis of (8) which is defined as:

$$E^i \cdot E_j = \delta_j^i, \tag{14}$$

where  $\delta_j^i$  is Kronecker symbol.

The characteristic equation is obtained from (13) in the form:

$$(2\mu - \eta)^4 \left[ (3\lambda + 2\mu - \eta) \left(2\mu - \frac{1}{g} M \cdot N - \eta\right) + \frac{\lambda}{g} (\text{tr } M) (\text{tr } N) \right] = 0. \tag{15}$$

Once the solutions to (15) are known, system (13) gives the components of the eigentensors in the dual basis defined by (14).

Thus:

$$V = \frac{(\text{tr } V) (N \cdot N) - (V \cdot N) \text{tr } N}{3N \cdot N - (\text{tr } N)^2} I + \frac{3V \cdot N - (\text{tr } V) (\text{tr } N)}{3N \cdot N - (\text{tr } N)^2} N + \sum_{i=3}^6 (V \cdot E_i) E_i. \tag{16}$$

From (15) it is seen that  $2\mu$  is an eigenvalue with algebraic multiplicity four. System (13) and equation (16) show that  $2\mu$  is associated to the eigenspace spanned by  $E_i$  ( $i = 3, \dots, 6$ ), which is the space of the deviatoric tensors orthogonal to  $N$ . Moreover, from (15) the other two eigenvalues result to be the solutions of the equation:

$$\eta^2 - \left(3\lambda + 4\mu - \frac{1}{g} M \cdot N\right) \eta + (3\lambda + 2\mu) \left(2\mu - \frac{1}{g} M \cdot N\right) + \frac{\lambda}{g} (\text{tr } M) (\text{tr } N) = 0. \tag{17}$$

If  $N = \alpha I$  ( $\alpha \in \mathbb{R}$ ) tensors (8) do not represent a basis of  $\text{Sym}$ . In this case, however, the eigenvalues of  $\mathbf{H}$  can be obtained by direct inspection as:

Eigenvalues	Multiplicity	Eigenspaces
$2\mu$	5	$\{V \in \text{Sym} : \text{tr } V = 0\}$
$3\lambda + 2\mu - \frac{\alpha}{g} \text{tr } M$	1	$\left\{ \beta V : V = \lambda I - \frac{\alpha}{g} M \right\}$
		$\forall \beta \in \mathbb{R}$

Alternatively, the above eigenvalues can be directly obtained from (15), which continues to hold, for continuity, in the case  $N = \alpha I$ .

4. Complex eigenvalues of  $\mathbf{H}$

Being non-symmetric, tensor  $\mathbf{H}$  may possess two complex conjugated eigenvalues. Such eigenvalues correspond to negative values of the discriminant of (17), which is

$$\Delta = \left(3\lambda + \frac{1}{g} M \cdot N\right)^2 - 4 \frac{\lambda}{g} (\text{tr } M) (\text{tr } N). \tag{18}$$

From (18) it is seen that  $\Delta < 0$  if and only if:

$$9\lambda^2 g^2 - 2\lambda [2(\text{tr } N) (\text{tr } M) - 3M \cdot N] g + (M \cdot N)^2 < 0. \tag{19}$$

Therefore  $\Delta$  is negative for values of  $g$  internal to the interval defined by  $g_1$  and  $g_2$ :

$$g_1 \left\{ \begin{aligned} &= \frac{1}{9\lambda} \left[ \sqrt{(\text{tr } N) (\text{tr } M)} \pm \sqrt{(\text{tr } N) (\text{tr } M) - 3M \cdot N} \right]^2. \end{aligned} \right. \tag{20}$$

It is concluded that a necessary and sufficient condition for the existence of complex eigenvalues is

$$(\text{tr } N) (\text{tr } M) [(\text{tr } N) (\text{tr } M) - 3M \cdot N] > 0. \tag{21}$$

It is important to note, however, that if  $(\text{tr } N) (\text{tr } M) < 0$  and  $(\text{tr } N) (\text{tr } M) - 3M \cdot N < 0$ , complex eigenvalues are possible for negative values of  $g_1$  and  $g_2$ . Thus, if (as usually accepted)  $g$  is restricted to strictly positive values, the necessary and sufficient condition (21) becomes:

$$(\text{tr } N) (\text{tr } M) > 0 \text{ \& } (\text{tr } N) (\text{tr } M) - 3M \cdot N > 0. \tag{22}$$

Now, by introducing the yield surface gradient  $\mathbf{Q}$  and the plastic flow-mode  $\mathbf{P}$  in the stress space:

$$N = \mathbb{E}[\mathbf{Q}] = \lambda(\text{tr } \mathbf{Q}) I + 2\mu \mathbf{Q}, \tag{23}$$

$$M = \mathbb{E}[\mathbf{P}] = \lambda(\text{tr } \mathbf{P}) I + 2\mu \mathbf{P}, \tag{24}$$

solutions (20) become:

$$g_1 \left\{ \begin{aligned} &= \frac{1}{9\lambda} \left[ (3\lambda + 2\mu) \sqrt{(\text{tr } \mathbf{Q}) (\text{tr } \mathbf{P})} \right. \\ &\quad \left. \pm 2\mu \sqrt{-3 \text{dev } (\mathbf{Q}) \cdot \text{dev } (\mathbf{P})} \right]^2, \end{aligned} \right. \tag{25}$$

where  $\text{dev } (*)$  denotes the deviatoric part of a tensor. Moreover, conditions (22) become:

$$(\text{tr } \mathbf{Q}) (\text{tr } \mathbf{P}) > 0 \text{ \& } \text{dev } (\mathbf{Q}) \cdot \text{dev } (\mathbf{P}) < 0. \tag{26}$$

Thus, in the case of deviatoric associativity, where  $\text{dev } (\mathbf{Q}) = \text{dev } (\mathbf{P})$ , complex eigenvalues are not possible.

However, even in the case of deviatoric non-associativity, it will be shown in the following that complex eigenvalues are excluded under the hypothesis of isotropic hardening and assuming convexity (and smoothness) of yield and plastic potential functions. In fact, due to the material isotropy (initial isotropy plus isotropy of hardening),  $\mathbf{Q}$  and  $\mathbf{P}$  must be coaxial and, furtherly, the yield and the plastic potential functions can be represented as convex surfaces in the Haigh-Westergaard stress space. In this space,  $\text{dev } \mathbf{Q}$  and  $\text{dev } \mathbf{P}$  can be represented as vectors lying on a deviatoric plane. In this plane,  $\text{dev } \mathbf{Q}$  is orthogonal to the section of the yield surface and  $\text{dev } \mathbf{P}$  to the section of the plastic potential surface. It is well-known that isotropy implies symmetry of the two sections about the orthogonal projections of the principal stress axes on the deviatoric section. Thus, considerations can be restricted to any

segment of the six 60° segments marked off by the three diameters in the deviatoric section. In this segment, the convexity and continuity of the yield and plastic potential functions imply that vectors  $dev \mathbf{Q}$  and  $dev \mathbf{P}$  lie between the two extreme cases, corresponding to the two unit vectors directed along two adjacent diameters (Fig. 1). These two vectors form a 60° angle and, therefore, it is concluded that:

$$\frac{1}{2} |dev \mathbf{Q}| |dev \mathbf{P}| \leq dev \mathbf{Q} \cdot dev \mathbf{P} \leq |dev \mathbf{Q}| |dev \mathbf{P}| \quad (27)$$

and thus (26<sub>2</sub>) is never satisfied.

### 5. Application

The isotropic elastoplastic model proposed by Needmann and Rice [4], for modelling metals showing the Strength-Differential effect, is based on the Drucker-Prager yield surface and on the Huber-Hencky-von Mises plastic potential. Tensors  $\mathbf{P}$  and  $\mathbf{Q}$  are given by:

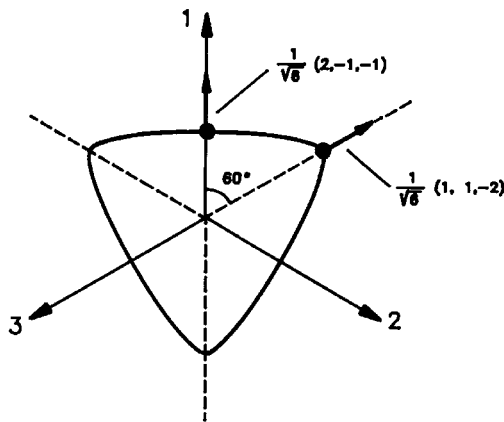


Fig. 1. Possible range of inclination of  $dev \mathbf{Q}$  and  $dev \mathbf{P}$  in the first 60° segment of a deviatoric section

$$\mathbf{P} = \frac{dev(\mathbf{T})}{2\sqrt{J_2}}, \quad (28)$$

$$\mathbf{Q} = \frac{dev(\mathbf{T})}{2\sqrt{J_2}} + \frac{SD}{2\sqrt{3}} \mathbf{I}, \quad (29)$$

where  $J_2 = dev(\mathbf{T}) \cdot dev(\mathbf{T})/2$  is the second invariant of the deviatoric stress and SD is defined as:

$$SD = 2 \frac{\sigma_c - \sigma_t}{\sigma_c + \sigma_t}. \quad (30)$$

In equation (30)  $\sigma_c$  and  $\sigma_t$  are the yield strengths in compression and tension, respectively. In this case eqn. (18) becomes

$$\Delta = \left( 3\lambda + \frac{2}{g} \mu^2 \sqrt{J_2} \right)^2 \quad (31)$$

and the solutions of (17) are

$$\begin{cases} \eta_1 = 3\lambda + 2\mu \\ \eta_2 = 2\mu - \frac{2}{g} \mu^2 \sqrt{J_2} \end{cases} \quad (32)$$

The eigenvalue  $\eta_1$  is the elastic eigenvalue associated to the eigentensor  $\mathbf{I}$ . It can be noted that the eigenvalues do not depend on SD. Moreover,  $\mathbf{Q}$  and  $\mathbf{P}$  are the gradients of a conical and a cylindrical surface, respectively, and the intersection with any deviatoric plane of the yield surface or the plastic potential surface in Haigh-Westergaard is a circle centred in the origin of axes. Therefore, during yielding, for a fixed value of the mean normal stress, the eigenvalues are also independent of the position of the stress point on the yield surface.

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### A Note on Representation for Isotropic Functions of 4th-Order Tensors in 2-Dimensional Space

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### 1. Introduction

The theory of representations for tensor functions constitutes a rational basis for a consistent mathematical modelling of complex mechanical behaviour of materials. However, contrary to that we have had good knowledge of tensor function representations which include scalars, vectors and 2nd-order tensors in 2- and 3-dimensional Euclidean spaces, representations which include 4th-order tensors are much less well understood [1–6].

This work substantiates the investigation by BETTEN et al. [1–3] on the entitled subject. Let  $\mathcal{A}$  denote a 4th-order tensor in

2-dimensional Euclidean space  $\mathcal{E}_2$  which fulfils the usual symmetry conditions

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}. \tag{1.1}$$

Here,  $A_{ijkl}$  denotes the components of  $A$  in a rectangular cartesian coordinate system  $X_i, i = 1, 2$ . An isotropic function or invariant of  $A$  is a function  $f(A_{ijkl})$  of the components  $A_{ijkl}$  for which  $f(\bar{A}_{ijkl}) \equiv f(A_{ijkl})$  with respect to the components  $\bar{A}_{ijkl}$  of  $A$  in any rectangular cartesian coordinate system  $\bar{X}_i$ . A set of isotropic invariants  $I_1, \dots, I_n$  of  $A$  is said to be an isotropic integrity basis or function basis of  $A$ , if every isotropic polynomial or function of  $A$  can be expressed as a polynomial or a single-valued function in numbers of this set; it is said to be irreducible, if none among  $I_1, \dots, I_n$  can be expressed as a single-valued function of the remainders. In literature an irreducible isotropic function basis is also named a complete and irreducible representation for isotropic functions. Up to now we have no precise knowledge of complete and irreducible representation for isotropic functions of  $A$ , although in [1–3] this problem has been discussed.

The set of all symmetric 2nd-order tensors in  $\mathcal{E}_2$  together with the usual definition of the inner product  $s:t = s_{ij}t_{ij}$  constitutes a 3-dimensional Euclidean space, say  $\mathcal{S}^{sym}$ . Thus,  $A$  is a symmetric linear transformation on  $\mathcal{S}^{sym}$ ; and in this sense we may employ the following notation:

$$\left. \begin{aligned} I &= A^0 (= \delta_{ij}), & A &= (A_{ijkl}), & A^2 &= (A_{ijmn}A_{mnkl}), \\ A^3 &= (A_{ijmn}A_{mnrsl}A_{rskl}); \end{aligned} \right\} \tag{1.2}$$

$$\left. \begin{aligned} I_1 &= \text{tr } A = A_{ijij}, & I_2 &= \text{tr } A^2 = A_{ijmn}A_{mnij}, \\ I_3 &= \text{tr } A^3 = A_{ijmn}A_{mnrsl}A_{rsij}; \end{aligned} \right\} \tag{1.3}$$

$$I_4 = \mathbf{1}:A\mathbf{1} = A_{ijij}, \quad I_5 = \mathbf{1}:A^2\mathbf{1} = A_{ijmn}A_{mnij}, \tag{1.4}$$

where  $\mathbf{1} (= \delta_{ij})$  with  $\delta_{ij}$  the Kronecker delta symbol denotes the 2nd-order identity tensor in  $\mathcal{E}_2$  and  $I (= A_{ijkl})$  with  $A_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$  denotes the identity linear transformation on  $\mathcal{S}^{sym}$ .

It is well known that a 4th-order tensor  $\Pi (= H_{ijkl})$  with the index symmetry (1.1) is isotropic, if and only if there are constants  $\lambda$  and  $\mu$  such that  $H_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ . In the theory of linear elasticity,  $A$  is known as the general elasticity tensor and  $\Pi$ , with Lamé elasticity constants  $\lambda$  and  $\mu$ , as the isotropic elasticity tensor. By the use of  $\Pi$ , BETTEN [1] has introduced the so-called extended eigenvalue problem of  $A$ ; that is  $As = \Pi s$ , or in its matrix form,

$$\begin{bmatrix} A_{1111} & A_{1122} & \sqrt{2}A_{1112} \\ A_{1122} & A_{2222} & \sqrt{2}A_{2212} \\ \sqrt{2}A_{1112} & \sqrt{2}A_{2212} & 2A_{1212} \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ \sqrt{2}s_{12} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ \sqrt{2}s_{12} \end{Bmatrix}. \tag{1.5}$$

From (1.5), the following extended characteristic polynomial:

$$P(\lambda, \mu) = \det(A - \Pi) = 8\mu^3 - 4J_1\mu^2 - 2J_2\mu - J_3 - 2K_1\lambda - 2K_2\mu\lambda + 8\mu^2\lambda, \tag{1.6}$$

is arisen, in which  $J_1, J_2, J_3, K_1$  and  $K_2$  (see [2, 3]) are invariants of  $A$ :

$$\begin{aligned} J_1 &= I_1 = \text{tr } A, & J_2 &= (I_2 - I_1^2)/2 = [\text{tr } A^2 - (\text{tr } A)^2]/2, \\ J_3 &= (2I_3 - 3I_1I_2 + I_1^3)/6 = \det A, \\ K_1 &= (A_{1112} - A_{2212})^2 - A_{1212}(A_{1111} + A_{2222} - 2A_{1122}), \\ K_2 &= A_{1111} + A_{2222} + 4A_{1212} - 2A_{1122}. \end{aligned} \tag{1.7}$$

BETTEN [1, 2] and BETTEN and HELISCH [3] have already discussed the irreducibility and the completeness of the system (1.7). For instance, with the aid of computer algebra, BETTEN and HELISCH [3] find that all the invariants of  $A$  with degree 4 (eg.  $A_{ijmn}A_{mnrsl}A_{rsuvw}A_{vwij}$ ) can be expressed as polynomials of the invariants given in (1.7). Thus, they suggested in [3] without proving that the invariants of  $A$  with the highest degree 3 would constitute an integrity basis of  $A$ .

In this short note we prove that the invariants  $J_1, J_2, J_3, K_1$ , and  $K_2$  of  $A$  constitute indeed an irreducible isotropic function basis (not only integrity basis) of  $A$ .

### 2. The isotropic function basis

An isotropic function basis of  $A$  may be constructed in such a way, as employed in [7–9], that we choose a rectangular cartesian coordinate system  $X_i$  in which we are able to determine all of the components  $A_{ijkl}$  of  $A$  by the use of the numbers of the basis.

As a symmetric linear transformation on the 3-dimensional Euclidean space  $\mathcal{S}^{sym}$ ,  $A$  can be expressed in its spectral form:

$$A = A_1e_1 \otimes e_1 + A_2e_2 \otimes e_2 + A_3e_3 \otimes e_3. \tag{2.1}$$

Here,  $A_1, A_2, A_3$  with  $A_1 \geq A_2 \geq A_3$  are the eigenvalues of  $A$  which are uniquely determined by the  $I_1, I_2, I_3$  in terms of the relations

$$\begin{aligned} I_1 &= A_1 + A_2 + A_3, & I_2 &= A_1^2 + A_2^2 + A_3^2, \\ I_3 &= A_1^3 + A_2^3 + A_3^3; \end{aligned} \tag{2.2}$$

and  $e_1, e_2, e_3$  are the characteristic vectors (in  $\mathcal{S}^{sym}$ ) of  $A$  which are unit and orthogonal:

$$e_\alpha : e_\beta = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3. \tag{2.3}$$

Consider the following possibilities for  $A_1, A_2$ , and  $A_3$ .

Case i:  $A_1 > A_2 > A_3$ . We can write

$$\left. \begin{aligned} e_1 \otimes e_1 &= (A_3 - A_2)[A_2A_3I - (A_2 + A_3)A + A^2]/A, \\ e_2 \otimes e_2 &= (A_1 - A_3)[A_3A_1I - (A_3 + A_1)A + A^2]/A, \\ e_3 \otimes e_3 &= (A_2 - A_1)[A_1A_2I - (A_1 + A_2)A + A^2]/A, \end{aligned} \right\} \tag{2.4}$$

in which  $A = (A_2 - A_3)(A_3 - A_1)(A_1 - A_2)$ . Let  $\bar{X}_i$  be a given rectangular cartesian coordinate system. From (2.4), we may further uniquely determine  $e_1, e_2, e_3$  by requiring that

$$\begin{aligned} \bar{e}_{211} &> 0 \text{ (if } \bar{e}_{211} \neq 0 \text{)}, & \bar{e}_{211} &> 0 \text{ (if } \bar{e}_{211} = 0 \text{ and } \bar{e}_{311} \neq 0 \text{)}, \\ \text{or } \bar{e}_{112} &> 0 \text{ (if } \bar{e}_{112} = \bar{e}_{222} = 0 \text{)}, \end{aligned} \tag{2.5}$$

in which  $\bar{e}_{\alpha ij}, \alpha = 1, 2, 3$ , are the components of  $e_\alpha$  in  $\bar{X}_i$ . Thus, we may choose a rectangular cartesian coordinate system  $X_i$  and determine the components  $e_{\alpha ij}$  of  $e_\alpha, \alpha = 1, 2, 3$ , instead of the components  $A_{ijkl}$  of  $A$  in order to determine the isotropic function basis of  $A$ . However, it is well known [10, 9] that the invariants

$$e_{\alpha ii}, e_{\alpha ij}e_{\beta ij}, \quad \alpha, \beta = 1, 2, 3 \text{ with } \alpha \leq \beta, \tag{2.6}$$

constitute a function basis of  $e_1, e_2$ , and  $e_3$ ; in addition, the  $e_{\alpha ij}e_{\beta ij} (= e_\alpha : e_\beta = \delta_{\alpha\beta}$ , see (2.3)) are constants. Thus, only the  $e_{\alpha ij}$  should be dealt with and from (2.4) and (2.5) the  $e_{\alpha ij}$  can uniquely be determined by

$$A_1, A_2, A_3, A_{ijij}, A_{iimn}A_{mnjj}. \tag{2.7}$$

Case ii:  $A_1 = A_2 > A_3$ . Instead of (2.1) and (2.4) we have  $A = A_1I + (A_3 - A_1)e_3 \otimes e_3$  and  $e_3 \otimes e_3 = (A - A_1I)/(A_3 - A_1)$ ; and we may determine  $e_3$  by means of (2.5). Similar to (2.6) and (2.7) we know that  $A_1, A_3, A_{ijij}$  determine uniquely the  $e_{3ij}$ . Analogously we can discuss the case:  $A_1 > A_2 = A_3$ .

Case iii:  $A_1 = A_2 = A_3$ . In this case  $A = A_1I$ .

The analysis given above in cases i, ii, and iii thus yields that the set of the five invariants  $I_1, \dots, I_5$  of  $A$ , as defined in (1.3) and (1.4), constitutes an isotropic function basis of  $A$ . Furthermore, from (1.7) and the relations (cf. (3.2))

$$2K_1 = I_2 - I_5 - I_1(I_1 - I_4), \quad K_2 = 2I_1 - I_4, \tag{2.8}$$

we see that  $J_1, J_2, J_3, K_1$ , and  $K_2$  are equivalent to  $I_1, \dots, I_5$  so that they constitute also a complete representation for isotropic functions of  $A$ .

### 3. The irreducibility of the isotropic function basis

We prove the irreducibility of  $I_1, \dots, I_5$  according to the method employed by PENNISI and TROVATO [11]. For convenience we give firstly the expressions of  $I_1, \dots, I_5$  in terms of the components of  $A$ :