

On the eigenvalues of the acoustic tensor in elastoplasticity

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ABSTRACT. – A technique is presented to perform the spectral analysis of the acoustic tensor for the loading branch of a generic elastic-plastic solid at finite strain, with general associative and non-associative flow-rule. The spectral analysis applies to general three-dimensional deformations, under the hypotheses that the elastic acoustic tensor be symmetric, and that an acoustical axis is always coincident with a neutral plastic wave amplitude. Moreover, the spectral analysis applies, without the latter restriction, to two-dimensional theories of elastoplasticity. Under these conditions, the occurrence of complex eigenvalues (flutter instability) is specified in terms of a necessary and sufficient condition. The onset of flutter (coalescence of two eigenvalues) is proved to be possible, under broad conditions. It is also shown that, for a fixed direction of wave propagation, if values of the plastic modulus exist for which flutter instability may occur, they are greater than the critical value of plastic modulus for the onset of stationary waves (strain localization into a planar band). Finally, it is shown that flutter instability can be triggered by a small perturbation of the direction of the plastic flow, even for associative plasticity. Under peculiar circumstances, this perturbation and the yield function gradient can be coaxial.

1. Introduction

The material stability in Mandel sense [Mandel, 1966] requires that the acceleration waves can propagate in a solid in every direction with finite speed. This local condition of stability is satisfied when the acoustic tensor possesses real and strictly positive eigenvalues ([Mandel, 1962]; [Raniecki, 1975]), *i.e.* as far as the dynamic equations of motion are hyperbolic. In the case of an associative (hyper)elastic-plastic solid, the use of work-conjugate measures of stress and strain yields a symmetric elastoplastic constitutive operator [Hill, 1978]. Therefore, in this context, the acoustic tensor is symmetric and the loss of hyperbolicity occurs at localization of deformation [Hill, 1962], *i.e.* at the vanishing of the speed of an acceleration wave. The use of a non-associative flow-rule, or (in certain cases) the use of hypoelasticity makes the acoustic tensor non-symmetrical and substantially changes the scenario of the local stability criteria. In this context, hyperbolicity can, as a matter of fact, be lost before strain localization, when two eigenvalues become complex conjugate. This possibility was pointed out by Rice [1976] who termed it “flutter instability”. The problem of flutter instability is considered by Loret *et al.* [1990] in the case of the infinitesimal theory. In that context, it has been

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proved that flutter instability is ruled out by using non-associative flow-rules satisfying deviatoric normality. On the other hand, examples of flutter instability have been found by Loret and Harireche [1991] and Vardoulakis [1992] for an infinitesimal mixture theory of elastoplastic porous media. In the context of plane strain-finite incremental theory, An & Schaeffer [1992] found a flutter instability as a consequence of the hypoelastic character of the elastic part of the deformation, together with the non-coaxiality of the yield function gradient and the Cauchy stress tensor. In a recent paper, Loret [1992] proves that flutter instability can be caused by an infinitesimal perturbation in the direction of plastic flow, for the infinitesimal theory in the presence of deviatoric associativity. However, in order to obtain flutter with a small perturbation, the non-coaxiality of this perturbation and the yield function gradient is required.

In the present work, a procedure is presented which allows the eigenvalues of the acoustic tensor to be calculated for the loading branch of a generic elastoplastic solid at finite strains, under the restrictive hypotheses that the elastic acoustic tensor be symmetric (as in the case of hyperelasticity) and that either a two-dimensional theory of plasticity is addressed (as in [Hill, 1979, 1980]) or an acoustical axis corresponding to the neutral plastic wave amplitude exists. The last hypothesis is a-priori satisfied for the particular case of the infinitesimal theory, and for certain simplified theories of plasticity at finite strain [Bigoni & Zaccaria, 1992]. Under these hypotheses it is shown that, for a fixed direction of wave propagation, if flutter occurs, it necessarily precedes strain localization into a planar band orthogonal to the direction of propagation. Moreover, a necessary and sufficient condition for flutter instability is given. It is shown that the condition of the onset of flutter, *i.e.* coalescence of two eigenvalues of the acoustic tensor, can always be satisfied, at a certain value of the plastic modulus. Finally, the problem of flutter as induced by a constitutive perturbation is analyzed. It is shown that flutter always becomes possible for a perturbation non coaxial with the yield function gradient. Moreover, under special circumstances (which can be encountered even in existing elastoplastic models), flutter can be triggered by an infinitesimal perturbation *coaxial* with the yield function gradient, even for associative elastoplasticity (the perturbation, of course, must break the symmetry of the constitutive operator). This last result appears not to contrast but is complementary to the results of Loret [1992], because of our slightly different definition of coaxiality, and of An and Schaeffer [1992], because of our more general elastoplastic model.

2. Notation and constitutive equations

Throughout the paper, a notation commonly used in continuum mechanics is adopted [Gurtin, 1981]. A second order tensor \mathbf{A} is a linear transformation over a (three-dimensional) inner product space \mathcal{V} . Lin denotes the set of all second order tensors and Sym the set of all symmetric second order tensors. The second order tensor \mathbf{I} indicates the identity over \mathcal{V} . The second order tensor \mathbf{S} denotes the first Piola-Kirchhoff stress tensor defined by:

$$(2.1) \quad \mathbf{t} = \mathbf{S} \mathbf{n}_o,$$

where \mathbf{t} is the nominal traction and \mathbf{n}_o the normal to the material surface element.

A fourth order tensor \mathbb{A} is a linear transformation over Lin or Sym. The fourth-order tensor \mathbb{I} denotes the identity over Sym. The symbol $\langle \cdot \rangle$ denotes McAulay brackets. The symbol \times denotes the exterior product over \mathcal{V} . The symbol \otimes denotes the tensor product over an inner product space (including \mathcal{V} and Lin). The symbol \boxtimes denotes the tensor product of transformations defined, for the specific case of $\mathbf{A}, \mathbf{B} \in \text{Lin}$, as [Del Piero, 1979]:

$$(2.2) \quad \mathbf{A} \boxtimes \mathbf{B} [\mathbf{C}] = \mathbf{A} \mathbf{C} \mathbf{B}^T, \quad \forall \mathbf{C} \in \text{Lin}.$$

Two symmetric, second order tensors \mathbf{A} and \mathbf{B} are coaxial if they commute [Del Piero, 1989; Ogden, 1984], *i.e.*: $\mathbf{A} \mathbf{B} = \mathbf{B} \mathbf{A}$. This definition of coaxiality is slightly more general than the requirement that \mathbf{A} and \mathbf{B} have identical eigenspaces. However, in elastoplasticity the former definition seems to be more appropriate. In fact, when the yield function gradient and the Cauchy stress are coaxial in the former sense, it is possible to represent the yield function gradient in Haigh-Westergaard stress space.

The relative Lagrangean description is assumed throughout the paper, and therefore the deformation gradient is equal to the identity ($\mathbf{F} = \mathbf{I}$) and, consequently, its determinant is unity ($J = 1$). The following constitutive equation is assumed, relating any objective flux of Kirchhoff stress \mathbf{K} to the velocity gradient \mathbf{L} :

$$(2.3) \quad \overset{\Delta}{\mathbf{K}} = \mathbb{E} [\mathbf{L}] - \frac{1}{g} \langle \mathbf{L} \cdot \mathbb{E} [\mathbf{Q}] \rangle \mathbb{E} [\mathbf{P}],$$

where \mathbf{P} and \mathbf{Q} are the flow-mode tensor and the yield function gradient in stress space, respectively. $\mathbb{E} : \text{Lin} \rightarrow \text{Lin}$ is the tensor of elastic moduli, having minor symmetries, and g is the plastic modulus. Constitutive Eq. (2.3) may be expressed in terms of the material derivative of the first Piola-Kirchhoff stress tensor \mathbf{S} :

$$(2.4) \quad \dot{\mathbf{S}} = \tilde{\mathbb{E}} [\mathbf{L}] - \frac{1}{g} \langle \mathbf{L} \cdot \mathbb{E} [\mathbf{Q}] \rangle \mathbb{E} [\mathbf{P}],$$

where the definition of $\tilde{\mathbb{E}}$ changes coherently with the particular choice of $\overset{\Delta}{\mathbf{K}}$. For instance, if $\overset{\Delta}{\mathbf{K}}$ is the Lie derivative of Kirchhoff stress, defined as in Marsden & Hughes [1983], then tensor $\tilde{\mathbb{E}}$ is given by $\tilde{\mathbb{E}} = \mathbb{E} + \mathbf{I} \boxtimes \mathbf{K}$. Moreover, $\tilde{\mathbb{E}}$ is assumed to possess the major symmetries, as in the case of hyperelasticity.

The acoustic tensor $\mathbf{A}(\mathbf{n})$, related to constitutive Eq. (2.4), is defined as:

$$(2.5) \quad \mathbf{A}(\mathbf{n}) \mathbf{v} = \mathbf{A}_E(\mathbf{n}) \mathbf{v} - \frac{1}{g} \langle \mathbf{v} \cdot \mathbb{E} [\mathbf{Q}] \mathbf{n} \rangle \mathbb{E} [\mathbf{P}] \mathbf{n}, \quad \forall \mathbf{v} \in \mathcal{V},$$

where \mathbf{n} is the unit vector giving the direction of wave propagation, and the elastic acoustic tensor $\mathbf{A}_E(\mathbf{n})$ has been introduced in the form:

$$(2.6) \quad \mathbf{A}_E(\mathbf{n}) \mathbf{v} = \tilde{\mathbb{E}} [\mathbf{v} \otimes \mathbf{n}] \mathbf{n}, \quad \forall \mathbf{v} \in \mathcal{V}.$$

It is worth noting that $\mathbf{A}_E(\mathbf{n})$ is symmetric as a consequence of the major symmetries of $\bar{\mathbb{E}}$.

3. Spectral analysis

The spectral analysis will be performed on the acoustic tensor corresponding to the incrementally linear comparison solid defined by the loading branch of the constitutive Eq. (2.4) (*Hill comparison solid*). Therefore, from (2.5) the following eigenvalue problem is considered:

$$(3.1) \quad \mathbf{A}_E(\mathbf{n}) \mathbf{v} - \frac{1}{g} (\mathbf{v} \cdot \mathbf{q}) \mathbf{p} - \eta \mathbf{v} = \mathbf{0}, \quad \forall \mathbf{v} \in \mathcal{V},$$

where:

$$(3.2) \quad \mathbf{p} = \mathbb{E}[\mathbf{P}] \mathbf{n}, \quad \mathbf{q} = \mathbb{E}[\mathbf{Q}] \mathbf{n}.$$

Let \mathbf{a}_1 , \mathbf{a}_2 and \mathbf{a}_3 denote three orthonormal eigenvectors of the (symmetric) elastic acoustic tensor \mathbf{A}_E defined in (2.6) and α_1 , α_2 and α_3 the corresponding eigenvalues. We now choose two eigenvectors \mathbf{a}_1 and \mathbf{a}_3 such that the triplet:

$$(3.3) \quad \mathbf{e}_1 = \mathbf{a}_1, \quad \mathbf{e}_2 = \mathbf{q}, \quad \mathbf{e}_3 = \mathbf{a}_3$$

forms a linearly independent system of vectors, which are, in general, non-orthogonal (the case of \mathbf{q} parallel to \mathbf{a}_1 is considered in Appendix 1). The dual basis of (3.3), defined by the condition $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$, can be written in the form:

$$(3.4) \quad \mathbf{e}^1 = -\frac{\mathbf{q} \cdot \mathbf{a}_1}{\mathbf{q} \cdot \mathbf{a}_2} \mathbf{a}_2 + \mathbf{a}_1, \quad \mathbf{e}^2 = \frac{1}{\mathbf{q} \cdot \mathbf{a}_2} \mathbf{a}_2, \quad \mathbf{e}^3 = -\frac{\mathbf{q} \cdot \mathbf{a}_3}{\mathbf{q} \cdot \mathbf{a}_2} \mathbf{a}_2 + \mathbf{a}_3.$$

With reference to the dual basis (3.4), a vector \mathbf{v} can be written in the form:

$$(3.5) \quad \mathbf{v} = v_1 \mathbf{e}^1 + v_2 \mathbf{e}^2 + v_3 \mathbf{e}^3.$$

A substitution of (3.5) in the eigenvalue problem (3.1) and a projection onto the basis (3.3) yields the following system:

$$(3.6) \quad \begin{cases} (\alpha_1 - \eta) v_1 - \frac{1}{g} (\mathbf{p} \cdot \mathbf{a}_1) v_2 = 0 \\ (\alpha_1 - \alpha_2) (\mathbf{q} \cdot \mathbf{a}_1) v_1 + \left(\alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} - \eta \right) v_2 + (\alpha_3 - \alpha_2) (\mathbf{q} \cdot \mathbf{a}_3) v_3 = 0. \\ -\frac{1}{g} (\mathbf{p} \cdot \mathbf{a}_3) v_2 + (\alpha_3 - \eta) v_3 = 0 \end{cases}$$

For a better understanding of the way of obtaining system (3.6), the following equalities should be noted:

$$(3.7) \quad \begin{cases} \mathbf{e}_i \cdot \mathbf{A}_E(\mathbf{n}) \mathbf{e}^j = \alpha_i (\mathbf{e}_i \cdot \mathbf{e}^j) & (i = 1 \text{ and } 3, j = 1, 2, 3) \\ \mathbf{e}_2 \cdot \mathbf{A}_E(\mathbf{n}) \mathbf{e}^j = (\alpha_j - \alpha_2) (\mathbf{q} \cdot \mathbf{a}_j) & (j = 1 \text{ and } 3). \\ \mathbf{e}_2 \cdot \mathbf{A}_E(\mathbf{n}) \mathbf{e}^2 = \alpha_2 \end{cases}$$

From (3.6), the characteristic equation of (3.1) becomes:

$$(3.8) \quad \det \begin{bmatrix} \alpha_1 - \eta & -\frac{1}{g} \mathbf{p} \cdot \mathbf{a}_1 & 0 \\ (\alpha_1 - \alpha_2) \mathbf{q} \cdot \mathbf{a}_1 & \alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} - \eta & (\alpha_3 - \alpha_2) \mathbf{q} \cdot \mathbf{a}_3 \\ 0 & -\frac{1}{g} \mathbf{p} \cdot \mathbf{a}_3 & \alpha_3 - \eta \end{bmatrix} = 0,$$

which yields:

$$(3.9) \quad (\alpha_1 - \eta) \left[\left(\alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} - \eta \right) (\alpha_3 - \eta) + \frac{1}{g} (\alpha_3 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_3) (\mathbf{q} \cdot \mathbf{a}_3) \right] + (\alpha_3 - \eta) \frac{1}{g} (\alpha_1 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) = 0.$$

If we assume that a particular wave amplitude corresponds to a neutral plastic wave, with velocity corresponding to the eigenvalue α_3 , the following relation must hold:

$$(3.10) \quad (\alpha_3 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_3) (\mathbf{q} \cdot \mathbf{a}_3) = 0.$$

It should be trivially noted that, if $\alpha_1 = \alpha_3$ and $\alpha_2 \neq \alpha_3$, the numeration of indices can obviously be changed to obtain $\alpha_1 \neq \alpha_3$ and $\alpha_2 = \alpha_3$. The trivial case of $\alpha_1 = \alpha_2 = \alpha_3$, is not considered.

Condition (3.10) is valid when the elastic acoustic tensor has two coincident eigenvalues, or for \mathbf{q} or \mathbf{p} in the plane orthogonal to \mathbf{a}_3 . Under condition (3.10), Eq. (3.9) becomes:

$$(3.11) \quad (\alpha_3 - \eta) \left[\eta^2 - \left(\alpha_1 + \alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} \right) \eta + \alpha_1 \left(\alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} \right) + \frac{1}{g} (\alpha_1 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) \right] = 0.$$

The second degree polynomial in brackets appearing in (3.11) can be directly obtained for two-dimensional theories of plasticity such as plane stress and plane strain situations [Hill, 1979, 1980]; therefore, all the following results hold for two-dimensional cases as well. The discriminant of the second degree polynomial appearing in (3.11) can be written as:

$$(3.12) \quad \Delta = \left(\alpha_1 - \alpha_2 + \frac{1}{g} \mathbf{p} \cdot \mathbf{q} \right)^2 - 4 \frac{1}{g} (\alpha_1 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1),$$

or as:

$$(3.13) \quad \Delta = \left(\alpha_1 + \alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q} \right)^2 - 4 \alpha_1 \alpha_2 \left(1 - \frac{g_{cr}}{g} \right),$$

where:

$$(3.14) \quad g_{cr} = \frac{\alpha_1 \mathbf{p} \cdot \mathbf{q} - (\alpha_1 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1)}{\alpha_1 \alpha_2},$$

is the *critical value of the plastic modulus for localization of deformation into a planar band of normal (unit) vector \mathbf{n}* [Rice, 1976]. It is worth noting that (3.14), which gives the localization condition *under hypothesis* (3.10), is more explicit than the general condition given by Rice [1976] (see also [Bigoni & Zaccaria, 1993]).

From (3.13) it is easily concluded that, if the elastic acoustic tensor is positive definite, *flutter is always excluded for values of the plastic modulus inferior than or equal to the critical plastic modulus for strain localization*. Therefore, if flutter occurs for a given direction \mathbf{n} , it must occur before strain localization in that direction. From (3.13) the following necessary and sufficient condition for flutter is obtained:

$$(3.15) \quad (\alpha_1 - \alpha_2)^2 (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) [(\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) - \mathbf{p} \cdot \mathbf{q}] > 0.$$

When (3.15) is satisfied, flutter instability occurs for values of the plastic modulus internal to the interval (g_1, g_2) :

$$(3.16) \quad \left. \begin{matrix} g_1 \\ g_2 \end{matrix} \right\} = \frac{1}{(\alpha_1 - \alpha_2)} \left[\sqrt{(\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1)} \pm \sqrt{(\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) - \mathbf{p} \cdot \mathbf{q}} \right]^2.$$

Therefore, the condition to have flutter for positive values of the plastic modulus g is obtained in the form:

$$(3.17) \quad (\alpha_1 - \alpha_2) (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) > 0 \ \& \ (\alpha_1 - \alpha_2) [(\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) - \mathbf{p} \cdot \mathbf{q}] > 0.$$

Inequality (3.17)₂ can be written in the useful form:

$$(3.18) \quad (\alpha_1 - \alpha_2) (\mathbf{p} \times \mathbf{a}_1) \cdot (\mathbf{a}_1 \times \mathbf{q}) > 0.$$

It is worth noting from (3.18) that conditions (3.17) are always violated for associative flow-rules ($\mathbf{p} = \mathbf{q}$).

Finally, in the particular case $\mathbf{q} \times \mathbf{a}_1 = \mathbf{0}$, the triplet (3.3) is no longer linearly independent. In this case, however, the direct solution of the eigenvalue problem (3.1) gives the same answer as obtained from (3.9) through substitution of $\mathbf{q} \times \mathbf{a}_1 = \mathbf{0}$ (see Appendix 1).

4. On the onset of flutter instability

Following the terminology introduced by An & Schaeffer [1992], we identify the *onset of flutter instability* with the coalescence of two eigenvalues of the acoustic tensor. It should be noted that conditions could be sought for which, under hypothesis (3.10), two elastoplastic eigenvalues are equal to α_3 , *i.e.* (3.11) has two solutions equal to α_3 . These are of course conditions for onset of flutter, but are not of interest in the present context. In fact, condition (3.10) a-priori excludes the development of flutter when this coalescence of eigenvalues is reached.

The onset of flutter occurs at a positive value of the plastic modulus, as soon as one of the two inequalities (3.17) is violated, *i.e.* when $(\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1) = 0$ or $(\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1) = \mathbf{p} \cdot \mathbf{q}$. Therefore, the onset of flutter may occur at the following critical value of the plastic modulus:

$$(4.1) \quad g_{\text{cr}}^o = \begin{cases} \frac{-\mathbf{p} \cdot \mathbf{q}}{\alpha_1 - \alpha_2}, & \text{if } (\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1) = 0 \\ \frac{(\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1)}{\alpha_1 - \alpha_2}, & \text{if } (\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1) - \mathbf{p} \cdot \mathbf{q} = 0, \end{cases}$$

An & Schaeffer have pointed out that the condition for onset of flutter can always be met in the infinitesimal theory of elastoplasticity. The same circumstance is expected to be verified in a more general context. However, due to the fact that all quantities \mathbf{q} , \mathbf{p} , \mathbf{a}_1 , α_1 and α_2 depend on \mathbf{n} , the proof that one of the terms $(\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1)$ or $(\mathbf{p} \cdot \mathbf{a}_1)(\mathbf{q} \cdot \mathbf{a}_1) - \mathbf{p} \cdot \mathbf{q}$ can always be made zero for a proper choice of \mathbf{n} may appear to be not straightforward. In the rest of this Section, we prove that the left hand side of inequality (3.17₂) can be made zero (for a proper choice of \mathbf{n} and g) under broad hypotheses. To this purpose, let us assume that tensors $\mathbb{E}[\mathbf{P}]$, $\mathbb{E}[\mathbf{Q}]$ are isotropic tensor functions of \mathbf{K} . Therefore, \mathbf{p} and \mathbf{q} can be represented as follows:

$$(4.2) \quad \begin{cases} \mathbf{q} = \gamma_1 \mathbf{n} + \gamma_2 \mathbf{K} \mathbf{n} + \gamma_3 \mathbf{K}^2 \mathbf{n}, \\ \mathbf{p} = \gamma_4 \mathbf{n} + \gamma_5 \mathbf{K} \mathbf{n} + \gamma_6 \mathbf{K}^2 \mathbf{n}, \end{cases}$$

where γ_i are isotropic scalar functions of the Kirchhoff stress \mathbf{K} . Moreover, we assume for \mathbb{E} the following representation (where ϕ_i are isotropic scalar functions of \mathbf{K}):

$$(4.3) \quad \begin{aligned} \mathbb{E} = & \phi_1 \mathbf{I} \otimes \mathbf{I} + \phi_2 (\mathbf{I} \otimes \mathbf{K} + \mathbf{K} \otimes \mathbf{I}) + \phi_3 (\mathbf{I} \otimes \mathbf{K}^2 + \mathbf{K}^2 \otimes \mathbf{I}) \\ & + \phi_4 \mathbf{K} \otimes \mathbf{K} + \phi_5 (\mathbf{K} \otimes \mathbf{K}^2 + \mathbf{K}^2 \otimes \mathbf{K}) \\ & + \phi_6 \mathbf{K}^2 \otimes \mathbf{K}^2 + \phi_7 (\mathbf{I} \boxtimes \mathbf{K} + \mathbf{K} \boxtimes \mathbf{I}) + \phi_8 (\mathbf{I} \boxtimes \mathbf{K}^2 + \mathbf{K}^2 \boxtimes \mathbf{I}) + \phi_9 \mathbb{I}, \end{aligned}$$

which corresponds to a symmetric hypoelastic constitutive law (Appendix 2). From (4.3), the elastic acoustic tensor can be written in the form:

$$(4.4) \quad \begin{aligned} \mathbf{A}_E(\mathbf{n}) = & c_1 \mathbf{I} + c_2 \mathbf{n} \otimes \mathbf{n} + c_3 \mathbf{K} \mathbf{n} \otimes \mathbf{K} \mathbf{n} \\ & + c_4 (\mathbf{n} \otimes \mathbf{K} \mathbf{n} + \mathbf{K} \mathbf{n} \otimes \mathbf{n}) + c_5 \mathbf{K} \\ & + c_6 \mathbf{K}^2 + c_7 (\mathbf{n} \otimes \mathbf{K}^2 \mathbf{n} + \mathbf{K}^2 \mathbf{n} \otimes \mathbf{n}) \\ & + c_8 (\mathbf{K} \mathbf{n} \otimes \mathbf{K}^2 \mathbf{n} + \mathbf{K}^2 \mathbf{n} \otimes \mathbf{K} \mathbf{n}) + c_9 \mathbf{K}^2 \mathbf{n} \otimes \mathbf{K}^2 \mathbf{n}, \end{aligned}$$

where c_i are isotropic functions of \mathbf{K} , which can be expressed in terms of functions ϕ_i , if the objective derivative of stress is specified. Now, let us denote the eigenvectors of \mathbf{K} as $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and the eigenvalues as k_1, k_2, k_3 and assume that the propagation direction \mathbf{n} coincides with an eigenvalue of \mathbf{K} , e.g. $\mathbf{n} = \mathbf{k}_1$. Then $\mathbf{A}_E(\mathbf{k}_1)$ has eigenvectors $\mathbf{a}_i = \mathbf{k}_i$. Moreover, condition (3.10) is satisfied for $\mathbf{a}_3 = \mathbf{k}_3$, and $(\mathbf{p} \cdot \mathbf{k}_1)(\mathbf{q} \cdot \mathbf{k}_1) - \mathbf{p} \cdot \mathbf{q} = 0$. However, (3.17₁) may be not satisfied (apart from in the associative case). Therefore, *in the case of finite strain elastoplasticity, under the hypotheses (4.2) and (4.3) the condition of coalescence of two eigenvalues of the acoustic tensors (onset of flutter) can always be met*. This circumstance occurs when the plastic modulus equals the following critical value, obtained from (4.1₂) and (4.2) by substitution of $\mathbf{n} = \mathbf{k}_1$:

$$(4.5) \quad g_{\text{cr}}^o = \frac{(\gamma_1 + \gamma_2 k_1 + \gamma_3 k_1)(\gamma_4 + \gamma_5 k_1 + \gamma_6 k_1)}{\alpha_1 - \alpha_2}.$$

The fact that the boundary between stability and flutter instability can always be reached at $g = g_{\text{cr}}^o$ is evident in the case of an associative flow-rule ($\mathbf{p} = \mathbf{q}$), where g_{cr}^o is positive. Note also that even if the condition of the onset are met, flutter instability is *a-priori* excluded by the symmetry of the constitutive operator in the case of associative elastoplasticity (this may not be true if $\tilde{\mathbf{E}}$ does not have the major symmetries).

5. On flutter instability

Let us consider a constitutive perturbation of the flow-rule, in the case of associative plasticity. To this purpose, we write:

$$(5.1) \quad \mathbf{p} = \mathbf{q} + \varepsilon \mathbf{B} \mathbf{n},$$

where ε is a (small) scalar perturbation parameter and the second order tensor \mathbf{B} specifies the "mode" of perturbation. Note that the perturbation breaks the symmetry of the constitutive operator and that the associative plasticity is recovered for $\varepsilon = 0$. Let us assume $\alpha_1 - \alpha_2 > 0$, so that condition (3.17₁) is always satisfied for sufficiently small ε . Condition (3.17₂) becomes, using (5.1):

$$(5.2) \quad [(\mathbf{q} \cdot \mathbf{a}_1)^2 - \mathbf{q} \cdot \mathbf{q}] + \varepsilon [(\mathbf{a}_1 \cdot \mathbf{B} \mathbf{n})(\mathbf{q} \cdot \mathbf{a}_1) - \mathbf{q} \cdot \mathbf{B} \mathbf{n}] > 0.$$

The first term in (5.2) is always negative and null if and only if $\mathbf{a}_1 \times \mathbf{q} = \mathbf{0}$. Thus, condition (5.2) can be verified for small ε , when \mathbf{a}_1 tends to be parallel to \mathbf{q} . Condition (5.2) can be written in the equivalent, useful form:

$$(5.3) \quad [\mathbf{q} + \varepsilon \mathbf{B} \mathbf{n}] \cdot [(\mathbf{q} \cdot \mathbf{a}_1) \mathbf{a}_1 - \mathbf{q}] > 0.$$

It is evident that the difference $(\mathbf{q} \cdot \mathbf{a}_1) \mathbf{a}_1 - \mathbf{q}$ tends to be orthogonal to \mathbf{q} , if \mathbf{a}_1 tends to be parallel to \mathbf{q} . Thus condition (5.3) can always be satisfied (for a proper choice of the sign of ε), if and only if, when \mathbf{a}_1 tends to be parallel to \mathbf{q} , $\mathbf{B} \mathbf{n}$ does not. In this circumstance, a *non coaxial perturbation can trigger flutter*.

As an example, let us assume the representations (4.2₁) and (4.4) and let \mathbf{n} be given by:

$$(5.4) \quad \mathbf{n} = \mathbf{k}_1 + \xi \mathbf{k}_2,$$

where ξ is a scalar parameter and \mathbf{k}_i denotes the eigenvectors of \mathbf{K} . Notice that the modulus of \mathbf{n} is 1 to the accuracy of terms of the order $o(\xi)$. A substitution of (5.4) into (4.4) yields:

$$(5.5) \quad \mathbf{A}_E(\xi) = b_1 \mathbf{k}_1 \otimes \mathbf{k}_1 + b_2 \mathbf{k}_2 \otimes \mathbf{k}_2 + b_3 \mathbf{k}_3 \otimes \mathbf{k}_3 \\ + \xi b_4 (\mathbf{k}_1 \otimes \mathbf{k}_2 + \mathbf{k}_2 \otimes \mathbf{k}_1) + \xi^2 (b_5 \mathbf{I} + b_6 \mathbf{k}_2 \otimes \mathbf{k}_2),$$

where coefficient b_i are scalar functions of coefficients c_i and of the eigenvalues k_i of \mathbf{K} . The eigenvectors and eigenvalues of (5.5) can be obtained in the form

$$(5.6) \quad \begin{cases} \mathbf{a}_1 = \mathbf{k}_1 + \xi s \mathbf{k}_2 + o(\xi), & \alpha_1 = b_1 + o(\xi) \\ \mathbf{a}_2 = \mathbf{k}_2 - \xi s \mathbf{k}_1 + o(\xi), & \alpha_2 = b_2 + o(\xi) \\ \mathbf{a}_3 = \mathbf{k}_3, & \alpha_3 = b_3 + \xi^2 b_5, \end{cases}$$

where $s = b_4/(b_1 - b_2)$.

If $\xi \rightarrow 0$, then $\mathbf{n} \rightarrow \mathbf{k}_1$, $\mathbf{q} \times \mathbf{n} \rightarrow 0$ and $\mathbf{a}_1 \rightarrow \mathbf{n}$, as can be seen from (5.4), (5.6) and (4.2₁), respectively. Therefore, $\mathbf{q} \times \mathbf{a}_1 \rightarrow 0$, if $\xi \rightarrow 0$, and thus a noncoaxial perturbation induces flutter. For a better understanding of this result, it should be noted that a substitution of (4.2₁) and (5.6) into (5.3) yields

$$(5.7) \quad \xi \varepsilon \mathbf{B} \mathbf{k}_1 \cdot [\gamma_1 (s - 1) + \gamma_2 (s k_1 - k_2) + \gamma_3 (s k_1^2 - k_2^2)] \mathbf{k}_2 + o(\xi) > 0,$$

which may always be satisfied if $\xi \rightarrow 0$ and \mathbf{k}_1 is not an eigenvector of \mathbf{B} .

On the other hand, there are possibilities of satisfying Eq. (5.2) in particular cases, even if \mathbf{B} and \mathbf{Q} are coaxial (as noted by [Bigoni & Zaccaria, 1992]). In fact, let us assume $\mathbf{B} = \mathbf{K}$ and \mathbf{q} is given by (4.2₁) with $\gamma_3 = 0$. Condition (5.2) can be written in the following form:

$$(5.8) \quad \gamma_1^2 [(\mathbf{n} \cdot \mathbf{a}_1)^2 - 1] + 2 \gamma_1 \gamma_2 [(\mathbf{n} \cdot \mathbf{a}_1) (\mathbf{a}_1 \cdot \mathbf{K} \mathbf{n}) - \mathbf{n} \cdot \mathbf{K} \mathbf{n}] \\ + \gamma_2^2 [(\mathbf{a}_1 \cdot \mathbf{K} \mathbf{n})^2 - \mathbf{K} \mathbf{n} \cdot \mathbf{K} \mathbf{n}] \\ + \varepsilon \{ \gamma_1 [(\mathbf{n} \cdot \mathbf{a}_1) (\mathbf{a}_1 \cdot \mathbf{K} \mathbf{n}) - \mathbf{n} \cdot \mathbf{K} \mathbf{n}] + \gamma_2 [(\mathbf{a}_1 \cdot \mathbf{K} \mathbf{n})^2 - \mathbf{K} \mathbf{n} \cdot \mathbf{K} \mathbf{n}] \} > 0.$$

If a direction \mathbf{n} satisfies $\mathbf{a}_1 = \mathbf{n}$ (note that in the case of the infinitesimal theory $\mathbf{a}_1 = \mathbf{n}$ for every \mathbf{n}), condition (5.8) becomes:

$$(5.9) \quad [\gamma_2^2 + \varepsilon\gamma_2] [(\mathbf{n} \cdot \mathbf{K} \mathbf{n})^2 - \mathbf{K} \mathbf{n} \cdot \mathbf{K} \mathbf{n}] > 0,$$

which can always be satisfied in the limit $\gamma_2 \rightarrow 0$, if \mathbf{n} does not coincide with an eigenvector of \mathbf{K} . In this case it is simple to verify that condition (3.17₁) is also satisfied. Condition $\gamma_2 \rightarrow 0$ implies that the yield surface gradient tends to become spherical. Therefore, *when the yield function gradient tends to become spherical and the stress does not, a perturbation of the yield function gradient coaxial with the stress is sufficient to trigger flutter* in the directions for which $\mathbf{a}_1 = \mathbf{n}$. In the particular case of the infinitesimal theory, a coaxial perturbation in the yield function gradient may induce flutter if the yield function gradient tends to become spherical and the perturbation does not. In this case, flutter may occur for every direction of propagation \mathbf{n} . In this case, flutter will necessarily precede strain localization, if a continuous loading program is followed. Examples in which \mathbf{Q} is spherical and the stress tensor is not can be encountered in plasticity. It is required, in fact, that the yield surface has planar portions quasi-orthogonal to the hydrostatic axis (see Fig. 1). For instance, the “caps” and tension cut-offs often used in the modelling of concrete and geomaterials are examples of this circumstance. Finally, the Hill yield surface in plane stress (Fig. 2) possesses orthogonal zones to the hydrostatic axis, and thus a small disturbance in the direction of plastic flow may be sufficient to trigger flutter.

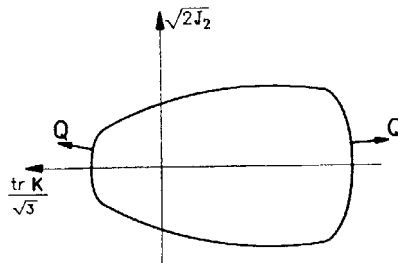


Fig. 1. – Meridian section of a yield surface for which the yield function gradient tends to become spherical and the stress does not.

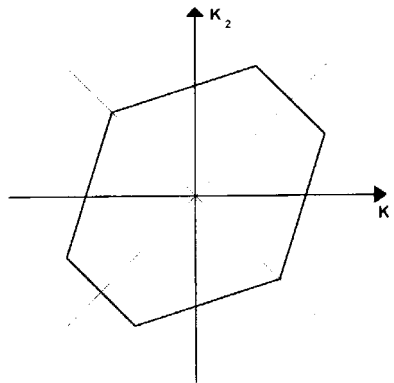


Fig. 2. – Hill yield surface in plane stress.

Analogous to the preceding case, flutter may occur when two eigenvalues of $\mathbb{E}[\mathbf{Q}]$ tend to coincide, and this tendency is not followed by \mathbf{K} . In fact, if $\mathbb{E}[\mathbf{Q}]$ is an isotropic function of \mathbf{K} , it is possible to construct the following representations

$$(5.10) \quad \mathbf{K} = k_1 \mathbf{k}_1 \otimes \mathbf{k}_1 + k_2 \mathbf{k}_2 \otimes \mathbf{k}_2 + k_3 \mathbf{k}_3 \otimes \mathbf{k}_3,$$

$$(5.11) \quad \mathbb{E}[\mathbf{Q}] = q_1 \mathbf{k}_1 \otimes \mathbf{k}_1 + q_2 \mathbf{k}_2 \otimes \mathbf{k}_2 + q_3 \mathbf{k}_3 \otimes \mathbf{k}_3,$$

where the coefficients q_i are the eigenvalues of $\mathbb{E}[\mathbf{Q}]$. If $\mathbf{a}_1 = \mathbf{n}$, for any \mathbf{n} belonging to the plane of \mathbf{k}_2 and \mathbf{k}_3 (this condition may always be satisfied in the case of the infinitesimal theory), condition (5.2) becomes, for $\mathbf{B} = \mathbf{K}$:

$$(5.12) \quad -[(q_2 - q_3)^2 + \varepsilon(k_2 - k_3)(q_2 - q_3)](\mathbf{k}_2 \cdot \mathbf{n})^2 (\mathbf{k}_3 \cdot \mathbf{n})^2 > 0,$$

which may always be satisfied if $(q_2 - q_3)$ tends to zero and $(k_2 - k_3)$ does not. This condition may be met for yield functions with deviatoric sections of the type shown in Figure 3. These yield functions are obtained by smoothing the deviatoric section of the Rankine criterion and have been proposed for the modelling of concrete ([William and Warnke, 1975]; [Ottosen, 1977]). A one-parameter family of yield surfaces can be introduced [Haythornthwaite, 1985], having cross sectional shapes in the π -plane ranging between the two extreme cases of Rankine and von Mises sections (Fig. 4). If the Rankine section is continuously approached, two eigenvalues of \mathbf{Q} tend to coincide and \mathbf{K} continues to possess distinct eigenvalues.

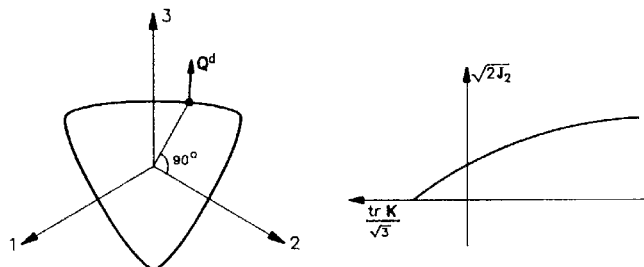


Fig. 3. – Yield function of the type obtained as a smoothing of the deviatoric section of the Rankine criterion.

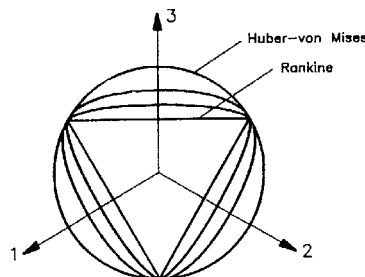


Fig. 4. – One-parameter family of yield surfaces, approaching the deviatoric section of the Rankine criterion.

6. An application to a simplified elastoplastic model

An application of the preceding results is performed in this Section. Reference is made to the elastoplastic model proposed by Hill [1962], slightly modified so as to include non-associativity. The incremental equation of the model can be obtained on the basis of the following four constitutive assumptions:

- 1) Existence of a *yield surface* in the second Piola-Kirchhoff stress space:

$$(6.1) \quad \mathcal{F}(\boldsymbol{\Sigma}, \mathcal{K}_i) = 0,$$

where \mathcal{K}_i denotes a set of *internal* (scalar or tensorial) *variables* describing the hardening of the material and $\boldsymbol{\Sigma}$ is the second Piola-Kirchhoff stress tensor, defined as:

$$(6.2) \quad \boldsymbol{\Sigma} = \mathbf{F}^{-1} \mathbf{K} \mathbf{F}^{-T},$$

where \mathbf{F} is the deformation gradient and \mathbf{K} the Kirchhoff stress tensor.

- 2) The incremental *stress-strain* relationship can be written in the following form:

$$(6.3) \quad \dot{\boldsymbol{\Sigma}} = \mathbb{E} [\dot{\mathbf{E}} - \dot{\mathbf{E}}^P],$$

where $\dot{\mathbf{E}}$ is the material derivative of the Green-Lagrange deformation tensor (conjugate of $\boldsymbol{\Sigma}$) and $\dot{\mathbf{E}}^P$ its plastic part. $\mathbb{E} : \text{Lin} \rightarrow \text{Lin}$ is the tensor of the elastic instantaneous moduli, possessing the minor symmetries. In the relative Lagrangean description, the following identities hold

$$(6.4) \quad \mathcal{L}_v \mathbf{K} = \dot{\boldsymbol{\Sigma}}, \quad \dot{\mathbf{E}} = \mathbf{D},$$

where \mathbf{D} is the velocity of deformation and

$$(6.5) \quad \mathcal{L}_v \mathbf{K} = \dot{\mathbf{K}} - \mathbf{L} \mathbf{K} - \mathbf{K} \mathbf{L}^T,$$

denotes the Lie derivative with respect to the velocity field \mathbf{v} . It is worthwhile to remember that the Lie derivative of the Kirchhoff stress with respect to the velocity field coincides with the Oldroyd derivative of the Kirchhoff stress, which, in turn, coincides with the Truesdell rate of the Cauchy stress. Discussions about the use of the Lie derivative in elastoplasticity can be found, e.g., in Simo [1988] and Perzyna [1993].

- 3) Existence of a “*flow-rule*” function $\mathbf{P} : (\boldsymbol{\Sigma}, \mathcal{K}_i) \rightarrow \text{Sym}$, such that:

$$(6.6) \quad \dot{\mathbf{E}}^P = \Lambda \mathbf{P},$$

where Λ is the (non-negative) plastic multiplier. Note that if the internal variables are of tensorial nature, as in the case of kinematic hardening, the material objectivity requirement does not generally imply coaxiality of \mathbf{P} and $\boldsymbol{\Sigma}$.

4) Existence of the “hardening modulus” function $h : (\boldsymbol{\Sigma}, \mathcal{K}_i) \rightarrow \mathbb{R}$, such that:

$$(6.7) \quad h \Lambda = -\Sigma_i \frac{\partial \mathcal{F}}{\partial \mathcal{K}_i} \cdot \dot{\mathcal{K}}_i.$$

Prager consistency condition requires that the material derivative of the yield function be null, *i.e.* $\dot{\mathcal{F}} = 0$, in the case of plastic loading. Thus, using (6.7):

$$(6.8) \quad \mathbf{Q} \cdot \dot{\boldsymbol{\Sigma}} - \Lambda h = 0,$$

where $\mathbf{Q} = \partial \mathcal{F} / \partial \boldsymbol{\Sigma}$ is the yield function gradient. Substitution of (6.5) into (6.8) yields the following expression for the plastic multiplier

$$(6.9) \quad \Lambda = \frac{\langle \dot{\mathbf{E}}^P \cdot \mathbf{E}[\mathbf{Q}] \rangle}{g},$$

where g is the plastic modulus, *i.e.*:

$$(6.10) \quad g = h + \mathbf{Q} \cdot \mathbf{E}[\mathbf{P}],$$

which is assumed to be strictly positive. Substitution of (6.6) into (6.3), together with (6.9), yields the incremental stress-strain relationship, which, in the relative Lagrangean description becomes

$$(6.11) \quad \mathcal{L}_v \mathbf{K} = \mathbf{E} \mathbf{S}[\mathbf{D}] - \frac{1}{g} \langle \mathbf{D} \cdot \mathbf{E}[\mathbf{Q}] \rangle \mathbf{E}[\mathbf{P}].$$

Constitutive Eq. (6.11) may be expressed in terms of the material derivative of the first Piola-Kirchhoff stress tensor \mathbf{S} (in the relative Lagrangean description):

$$(6.12) \quad \dot{\mathbf{S}} = \mathbf{E}[\mathbf{D}] - \frac{1}{g} \langle \mathbf{D} \cdot \mathbf{E}[\mathbf{Q}] \rangle \mathbf{E}[\mathbf{P}] + \mathbf{L} \mathbf{K}.$$

It is well-known [Truesdell & Noll, 1965, § 68-69] that, at large elastic strains, the tensor \mathbf{E} does not, in general, admit an isotropic representation. Nevertheless, the assumption of isotropic elastic deformation may be reasonable when the elastic strains remain small as, for instance, in the case of many local and global instability problems reported in the literature ([Hutchinson, 1973]; [Rice, 1976]; [An & Schaeffer, 1992]). In the following, the elastic tensor is assumed isotropic, *i.e.*:

$$(6.13) \quad \mathbf{E} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I},$$

where λ and μ are the Lamé moduli. When \mathbf{E} is given by (6.13) and $\mathbf{Q} = \text{dev}(\mathbf{K}) / (2\sqrt{J_2})$, constitutive Eq. (6.12) reduces to the finite strain version of the J_2

flow theory used by Hutchinson [1973]. The elastoplastic acoustic tensor can be written in the form (2.5), in which the elastic acoustic tensor is given by:

$$(6.14) \quad \mathbf{A}_E(\mathbf{n}) = (\lambda + \mu) \mathbf{n} \otimes \mathbf{n} + (\mu + \mathbf{n} \cdot \mathbf{K} \mathbf{n}) \mathbf{I}.$$

If \mathbf{s} is any vector orthogonal to \mathbf{n} , tensor \mathbf{A}_E has the following spectrum:

$$(6.15) \quad \mathbf{A}_E(\mathbf{n}) = (\lambda + 2\mu + \mathbf{n} \cdot \mathbf{K} \mathbf{n}) \mathbf{n} \otimes \mathbf{n} + (\mu + \mathbf{n} \cdot \mathbf{K} \mathbf{n}) \mathbf{s} \otimes \mathbf{s} + (\mu + \mathbf{n} \cdot \mathbf{K} \mathbf{n}) \mathbf{n} \times \mathbf{s} \otimes \mathbf{n} \times \mathbf{s},$$

thus \mathbf{n} is an eigenvalue of \mathbf{A}_E . Eq. (3.16) becomes, for $\mathbf{a}_1 = \mathbf{n}$:

$$(6.16) \quad \left. \begin{matrix} g_1 \\ g_2 \end{matrix} \right\} = \frac{1}{\lambda + \mu} [\sqrt{(\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n})} \pm \sqrt{(\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) - \mathbf{p} \cdot \mathbf{q}}]^2.$$

Conditions (3.17) are now given by:

$$(6.17) \quad (\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) > 0 \ \& \ (\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) - \mathbf{p} \cdot \mathbf{q} > 0.$$

Equation (6.17)₂ can be rewritten in the useful form (3.18)

$$(6.18) \quad (\mathbf{n} \times \mathbf{q}) \cdot (\mathbf{p} \times \mathbf{n}) > 0,$$

from which it is possible to verify that *for deviatoric associativity complex eigenvalues are excluded*. However, $\mathbf{n} \times \mathbf{q} = \mathbf{0}$ when \mathbf{n} and \mathbf{q} are parallel, and therefore the problem is not strictly hyperbolic, as noted by An & Schaeffer [1992]. Moreover, when \mathbf{n} is parallel to \mathbf{q} , the three eigenvalues of the acoustic tensor coincide, when:

$$(6.19) \quad g_{cr}^o = \frac{1}{\lambda + \mu} \sqrt{(\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n})}.$$

7. Conclusions

A spectral analysis of the acoustic tensor has been performed for general associative and non associative elastoplasticity at finite strain. The spectral analysis has general validity for two-dimensional theories of elastoplasticity. In the case of three dimensional theories, the spectral analysis is true under the hypothesis that an acoustical axis is coincident with a neutral plastic wave amplitude. Necessary and sufficient conditions for flutter instability are given. Moreover, the following results have been proved:

- 1) the boundary between stability and flutter instability (*i.e.* coalescence of two eigenvalues of the acoustic tensor) can be reached (under broad assumptions, at an opportune value of the plastic modulus);
- 2) if flutter occurs for a given propagation direction, it must precede strain localization into a planar band orthogonal to that direction;

3) a small perturbation of the direction of plastic flow (so as to break the symmetry of the constitutive operator) can induce flutter if this perturbation and the yield function gradient are not coaxial;

4) for special shapes of the yield surface, flutter can be triggered by a coaxial perturbation of the yield function gradient.

All results have been proved for a non-associative flow-rule coaxial with the Kirchhoff stress in the case of two-dimensional theories or under the hypothesis that a wave amplitude always corresponds to neutral loading. This last hypothesis was not used to obtain result (2). Moreover, results (3) and (4) have been proved under the hypothesis that a wave propagation direction coincides with an eigenvector of the elastic acoustic tensor, as in the case of the infinitesimal theory. Therefore, in the infinitesimal theory, a coaxial perturbation of the yield surface gradient may induce flutter in certain circumstances. We expect, however, that any instability which manifests itself in the case of coaxial plasticity will not be canceled for more complicated constitutive models. Results (1) and (3) are a generalization of the results due to An & Schaeffer [1992] (restricted to a particular constitutive law in a two dimensional theory) and to Loret *et al.* [1990] and Loret [1992] (restricted to the infinitesimal theory). Results (2) and (4) seem to pass unobserved.

As a general remark, we note that the occurrence of flutter instability is promoted by the non-associativity of the flow-rule (especially deviatoric non-associativity) and by the non-coaxiality of the yield function gradient and the stress tensor. Generally, the same effects promote other kinds of material instability, such as localization of deformation ([Needleman, 1979]; [Rice, 1976]), surface instability ([Hori & Nemat-Nasser, 1982]; [Vardoulakis, 1984]; [Benallal *et al.*, 1989]) and elastoplastic cavitation ([Bigoni & Laudiero, 1989]). In comparison to these types of local instabilities, the flutter instability is much less understood. In fact, all the analyses reported in the literature ([Rice, 1976]; [Loret *et al.*, 1990]; [An & Schaeffer, 1992]; [Loret & Harireche, 1991]; [Loret, 1992]; including the present one) refer to the comparison solid "in loading". Nothing is known about real elastoplastic behavior. Moreover, no specific examples are available in the literature of the behavior of a continuous medium exhibiting flutter.

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APPENDIX 1

Condition $\mathbf{q} \times \mathbf{a}_1 = \mathbf{0}$ is equivalent to $\mathbf{q} = (\mathbf{q} \cdot \mathbf{a}_1) \mathbf{a}_1$. Thus $\mathbf{p} \cdot \mathbf{q} = (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1)$ and the characteristic Eq. (3.9), obtained under the hypothesis $\mathbf{q} \times \mathbf{a}_1 \neq \mathbf{0}$ becomes,

for $\mathbf{q} \times \mathbf{a}_1 = \mathbf{0}$:

$$(A1.1) \quad (\alpha_2 - \eta)(\alpha_3 - \eta) \left[\alpha_1 - \frac{1}{g} (\mathbf{p} \cdot \mathbf{a}_1) (\mathbf{q} \cdot \mathbf{a}_1) - \eta \right] = 0.$$

The solutions to (A1.1) are:

$$(A1.2) \quad \eta_1 = \alpha_2, \quad \eta_2 = \alpha_3, \quad \eta_3 = \alpha_1 - \frac{1}{g} (\mathbf{p} \cdot \mathbf{q}).$$

On the other hand, the same solutions (A1.2) are solutions of (3.1) when $\mathbf{a}_1 \times \mathbf{q} = \mathbf{0}$. In fact, if $\mathbf{a}_1 \times \mathbf{q} = \mathbf{0}$ Eq. (3.1) is satisfied for η_1 , η_2 and η_3 given by (A1.2). The value η_3 in (A1.2) can be directly verified to be the eigenvalue of the elastoplastic acoustic tensor associated with the left eigenvector \mathbf{a}_1 and to the right eigenvector:

$$\mathbf{a}_1 - \frac{1}{g} (\mathbf{q} \cdot \mathbf{a}_1) \left[\frac{\mathbf{p} \cdot \mathbf{a}_2}{\alpha_1 - \alpha_2 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q}} \mathbf{a}_2 + \frac{\mathbf{p} \cdot \mathbf{a}_3}{\alpha_1 - \alpha_3 - \frac{1}{g} \mathbf{p} \cdot \mathbf{q}} \mathbf{a}_3 \right].$$

In the case when $\mathbf{q} \times \mathbf{a}_3 = \mathbf{0}$, an analogous result holds.

APPENDIX 2

Derivation of representation (4.3)

An hypoelastic material can be characterized as follows:

$$(A2.1) \quad \overset{\Delta}{\mathbf{K}} = \mathbf{f}(\mathbf{K}, \mathbf{D}),$$

where \mathbf{f} is an isotropic function of \mathbf{K} and \mathbf{D} , linear in \mathbf{D} , and $\overset{\Delta}{\mathbf{K}}$ is any objective derivative of the Kirchhoff stress [Truesdell & Noll, 1965]. The representation theorem for isotropic tensorial functions of two symmetric tensor, implies [Truesdell & Noll, 1965]:

$$(A2.2) \quad \overset{\Delta}{\mathbf{K}} = \psi_0 \mathbf{I} + \psi_1 \mathbf{K} + \psi_2 \mathbf{D} + \psi_3 \mathbf{K}^2 + \psi_4 \mathbf{D}^2 + \psi_5 (\mathbf{K}\mathbf{D} + \mathbf{D}\mathbf{K}) \\ + \psi_6 (\mathbf{D}^2 \mathbf{K} + \mathbf{K}\mathbf{D}^2) + \psi_7 (\mathbf{D}\mathbf{K}^2 + \mathbf{K}^2 \mathbf{D}) + \psi_8 (\mathbf{D}^2 \mathbf{K}^2 + \mathbf{K}^2 \mathbf{D}^2),$$

where the coefficients ψ_i are isotropic scalar functions of \mathbf{K} and \mathbf{D} :

$$(A2.3) \quad \psi_i = \tilde{\psi}_i (\text{tr } \mathbf{K}, \text{tr } \mathbf{K}^2, \text{tr } \mathbf{K}^3, \\ \text{tr } \mathbf{D}, \text{tr } \mathbf{D}^2, \text{tr } \mathbf{D}^3, \text{tr } \mathbf{D}\mathbf{K}, \text{tr } \mathbf{D}\mathbf{K}^2, \text{tr } \mathbf{K}\mathbf{D}^2, \text{tr } \mathbf{K}^2 \mathbf{D}^2).$$

If the non-linear terms in \mathbf{D} are eliminated from (A2.2), the following incremental constitutive law is obtained:

$$(A2.4) \quad \overset{\Delta}{\mathbf{K}} = \mathbb{E}_{\mathbf{K}} [\mathbf{D}],$$

where:

$$\begin{aligned}
 \text{(A2.5)} \quad \mathbf{E}_{\mathbf{K}} = & \phi_1 \mathbf{I} \otimes \mathbf{I} + \phi_2 \mathbf{I} \otimes \mathbf{K} + \bar{\phi}_2 \mathbf{K} \otimes \mathbf{I} + \phi_3 \mathbf{I} \otimes \mathbf{K}^2 + \bar{\phi}_3 \mathbf{K}^2 \otimes \mathbf{I} \\
 & + \phi_4 \mathbf{K} \otimes \mathbf{K} + \phi_5 \mathbf{K} \otimes \mathbf{K}^2 + \bar{\phi}_5 \mathbf{K}^2 \otimes \mathbf{K} \\
 & + \phi_6 \mathbf{K}^2 \otimes \mathbf{K}^2 + \phi_7 (\mathbf{K} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{K}) + \phi_8 (\mathbf{K}^2 \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{K}^2) + \phi_9 \mathbf{I},
 \end{aligned}$$

where ϕ_i and $\bar{\phi}_i$ are isotropic tensorial functions of \mathbf{K} . In order to obtain the representation (4.3) it is now sufficient to impose the symmetry of $\mathbf{E}_{\mathbf{K}}$, *i.e.* $\phi_2 = \bar{\phi}_2$, $\phi_3 = \bar{\phi}_3$, $\phi_5 = \bar{\phi}_5$.

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