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ON FLUTTER INSTABILITY IN ELASTOPLASTIC CONSTITUTIVE MODELS

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Abstract—Two conditions of non-propagation of wave modes are analyzed: flutter instability as described by Rice (1976) and non propagation due to different algebraic and geometric multiplicity in the eigenvalues of the acoustic tensor. Explicit reference is made to elastoplastic constitutive operators at finite strains. Both loading and unloading branches of the constitutive operator are analyzed, but they are treated independently (we disregard the interaction between loading and unloading). The spectral analysis of Bigoni and Zaccaria (1994) is generalized to examine an unsymmetric acoustic tensor for the unloading branch of the constitutive operator. Two constitutive laws for finite-strain elastoplasticity are considered, one of which is widely in use (Rudnicki and Rice 1975). In both constitutive laws, unloading of the material follows a specific grade 1-hypo-elasticity, lacking in any stress-rate potential. For these materials, we show that instabilities are excluded in the unloading branch, whereas they remain possible in the loading branch of the elastoplastic constitutive operator. Therefore, the geometrical terms of the constitutive equations (when small compared to the elastic shear modulus) provide examples of perturbations which induce flutter and non-propagation instability in elastoplasticity, yet have no effect on infinitesimal, three-dimensional, isotropic elasticity (where two wave speeds always coincide).

1. INTRODUCTION

There are four conditions under which some acceleration wave modes cannot propagate; each condition is related to the eigenvalues of the acoustic tensor (which can be unsymmetric in the present context). These are:

one eigenvalue becomes zero,
one eigenvalue becomes negative,
two eigenvalues become complex conjugate,
an eigenvalue has algebraic multiplicity greater than geometric multiplicity.

The first possibility coincides with the condition for strain localization, which has been analyzed thoroughly. In the present paper, only the last two possibilities are considered.

The condition for which an eigenvalue exhibits geometric multiplicity less than its algebraic multiplicity passed unnoticed until the work by Brannon and Drugan (1993). We interpret this phenomenon in the spirit of Mandel (1962, 1966) as a type of instability (denoted here as *non-propagation instability*). Little is known about this instability. In particular, Brannon and Drugan (1993) showed that geometric and algebraic multiplicity of the acoustic tensor eigenvalues coincide in the infinitesimal theory of plasticity in the presence of deviatoric associativity.

The occurrence of complex conjugate eigenvalues in the acoustic tensor defines flutter instability in continuous media. This terminology, derived from structural mechanics (Ziegler, 1956), was proposed by Rice (1976) with reference to the theory of nonassociative plasticity. More recently, this instability has been analyzed for the infinitesimal theory of plasticity (Loret *et al.*, 1990; Loret, 1992), for specific finite strains theories of plasticity (An and Schaeffer, 1992; Bigoni and Zaccaria, 1992, 1994) and for an elastoplasticity theory of fluid-saturated, porous media (Loret and Harireche, 1991). In each of these works, the conditions for flutter are analyzed with reference to the loading branch of the elastoplastic constitutive operator.

For a class of widely used two-dimensional, infinitesimal theories of plasticity, An and Schaeffer (1992) demonstrated that the condition for *onset of flutter*, i.e. coalescence of two eigenvalues, can be encountered frequently. When two eigenvalues coincide, an *appropriate*, arbitrarily small, perturbation can give rise to complex conjugate eigenvalues, and so trigger flutter (from which the terminology "onset of flutter" is derived). However, for a three-dimensional, infinitesimal theory of isotropic elasticity, two eigenvalues of the acoustic tensor always coincide. Therefore, the question arises as to the relevant distinctions between elastoplasticity and infinitesimal, isotropic elasticity in the context of flutter analysis. The answer lies in the definition of the specific form of the perturbation inducing flutter. In particular, Loret (1992) introduced the concept of a perturbation in the direction of the plastic potential gradient. Such a perturbation, when non-coaxial with the yield function gradient, can induce flutter even in the case of deviatoric associativity. An and Schaeffer (1992) considered the *geometrical terms* of the constitutive laws at finite strains as perturbations. Finite strains theories of plasticity based on simple hypotheses have been successfully employed in global and local stability analyses (Hill, 1962; Hutchinson, 1973; Hutchinson and Miles, 1974; Rudnicki and Rice, 1975; Stören and Rice, 1975; Asaro and Rice, 1977; Needleman and Rice, 1978; Vardoulakis *et al.*, 1978; Vardoulakis, 1981; Raniecki, 1979; Raniecki and Bruhns, 1980; Vardoulakis, 1983; Chau and Rudnicki, 1990; Bardet, 1991; An and Schaeffer, 1992; Chau, 1992). These theories have a common structure which originates from the *replacement* of the usual stress rate in the constitutive law of infinitesimal elastoplasticity by an objective rate of a symmetric stress measure. This produces the so-called geometrical (or corotational) terms. Such terms are usually small by comparison to the elastic shear modulus of the material; here they are interpreted as perturbations to the infinitesimal theory. In flutter analysis, we are interested in the geometrical terms which provide unsymmetrical perturbations in the acoustic tensor. For this purpose, the model proposed by Rice and Rudnicki (1975) and later generalized in (Rice, 1976; Needleman and Rice, 1978; Rice and Rudnicki, 1980; Chau and Rudnicki, 1990, Chau, 1992) represents an interesting candidate for study.

As a general remark, we note that the physical meaning of flutter instability is generally unknown (apart from the non-propagation of some wave modes). Rice (1976) and Truesdell and Noll (1965), argue that flutter in a continuous medium might correspond to an increasing-with-time disturbance. Bigoni and Willis (1994) provide a confirmation and clarification of this issue; they also show that flutter is connected to non-integrability of governing differential equations.

In the present paper, the possibility of flutter and wave non-propagation instabilities is analyzed for the loading and unloading branches of elastoplastic solids. However, the branches are analyzed independently, i.e. loading/unloading interaction during instability is not considered. *Geometrical terms* which appear in approximate theories of finite strains elastoplasticity are identified with the perturbations which can promote instability. When considered as perturbations to the infinitesimal theory of isotropic elasticity, they are shown to be incapable of inducing either flutter or non-propagation instability (Section 3). In contrast, the same perturbations are shown to generate instability on the loading branch of elastoplastic constitutive operators (Sections 5 and 6). This result may clarify the difference between the coincidence of two wave speeds in elastoplasticity or in other contexts (isotropic, infinitesimal elasticity). Sections 4–6 examine the loading branch of constitutive laws of elastoplastic solids. In particular, a generalization of the results by Bigoni and Zaccaria (1994) to an unsymmetric acoustic tensor of the unloading branch of the constitutive operator (Section 4) yields examples of flutter (Section 5) and wave non-propagation instabilities (Section 6), presently induced by geometrical terms. The examples show that both instabilities can manifest themselves in the Rudnicki and Rice model. This occurs even for the associative flow rule, when the yield function gradient and the Cauchy stress are not coaxial, e.g. in the important case of kinematic hardening. The instabilities only take place when the hardening modulus reaches a critical value which may be greater than the critical value for loss of second order work positive definiteness (Maier and Hueckel, 1979) or for strain localization (Rice, 1976). Moreover, special cases will be shown in which the instabilities can occur when the yield function gradient and Cauchy stress are coaxial.

Examples of wave non-propagation instability occur even when flutter is *a priori* excluded (Sections 3, 6 and Appendix A).

2. NOTATION AND CONSTITUTIVE EQUATIONS

Reference is made to the Gurtin (1981) notation, which is briefly summarized hereafter.

A second order tensor is a linear transformation from the vector space $\mathscr{V}$ (associated to the three-dimensional euclidean point space) into itself. The set of second order tensors is denoted by Lin, and the subset of symmetric second order tensors by Sym. Elements of $\mathscr{V}$ are denoted by bold face, small letters ($\mathbf{a}, \mathbf{b}, \ldots$), whereas members of Lin and Sym are denoted by bold face, capital letters ($\mathbf{A}, \mathbf{B}, \ldots$). The linear transformations from Lin into Lin are fourth order tensors which are denoted by special capital letters ($\mathscr{A}, \mathscr{B}, \ldots$). The (right-handed) cross product of vectors is denoted by symbol $\times$ and the modulus of a vector (or of a scalar) by $|-|$. The tensor product of vectors **a** and **b** is defined by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}, \quad \forall \mathbf{v} \in \mathscr{V}, \tag{1}$$

where $\cdot$ denotes the natural inner product of $\mathscr{V}$. The tensor product [eqn (1)] is extended to second order tensors in the usual way:

$$(\mathbf{A} \otimes \mathbf{B})\mathbf{V} = (\mathbf{B} \cdot \mathbf{V})\mathbf{A}, \quad \forall \mathbf{V} \in \text{Lin}, \tag{2}$$

where $\cdot$ denotes the natural inner product of Lin, defined as $\mathbf{A} \cdot \mathbf{B} = \text{tr}(\mathbf{A}^T\mathbf{B})$. In addition to eqn (2) we will make use of the following tensor product, which has been introduced by Del Piero (1979):

$$(\mathbf{A} \boxtimes \mathbf{B})\mathbf{V} = \mathbf{A}\mathbf{V}\mathbf{B}^T, \quad \forall \mathbf{V} \in \text{Lin}. \tag{3}$$

If $\mathbf{e}_i$ is a non-orthogonal basis and $\mathbf{e}^j$ its dual basis (i.e. $\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$, δ^i_j being the Kronecker delta), the following representation of a second order tensor is possible

$$\mathbf{A} = \sum_{ij=1}^{3} (\mathbf{e}^i \cdot \mathbf{A}\mathbf{e}_j)\mathbf{e}_i \otimes \mathbf{e}^j, \tag{4}$$

where $\mathbf{e}^i \cdot \mathbf{A}\mathbf{e}_j = A^i_j$ are the mixed components of **A** onto the dual bases $\mathbf{e}_i$, $\mathbf{e}^j$.

The second order and the fourth-order identity tensors will be denoted by **I** and $\mathscr{I}$, respectively. The symmetryzing operator $\mathscr{S}$ is defined by

$$\mathscr{S}\mathbf{A} = (\mathbf{A} + \mathbf{A}^T)/2, \quad \forall \mathbf{A} \in \text{Lin}. \tag{5}$$

Two second order symmetric tensors **A** and **B** are defined coaxial if and only if they commute (Ogden, 1984; Del Piero, 1989), i.e.:

$$\mathbf{A}, \mathbf{B} \in \text{Sym coaxial} \Leftrightarrow \mathbf{AB} = \mathbf{BA}. \tag{6}$$

Finally, the spectral radius $\rho(\mathbf{A})$ of a second order tensor $\mathbf{A} \in \text{Lin}$ is defined as

$$\rho(\mathbf{A}) = \text{Max}\,\{|\alpha_1|, |\alpha_2|, |\alpha_3|\}, \tag{7}$$

where α_i are the eigenvalues of **A**. It may be interesting to note that $\rho(\mathbf{A})$ is always not greater than *any* matrix norm of [**A**] (Wilkinson, 1965; Salce, 1993).

A hypoelastic material of grade 1, in the *relative Lagrangean description*, is defined by the following constitutive tensor which relates the material derivative of the first Piola–Kirchhoff stress tensor **S** (transpose of the nominal stress tensor used by Hill, 1978) to the velocity gradient **L**:

$$\dot{\mathbf{S}} = \mathscr{E}[\mathbf{L}], \tag{8}$$

$$\mathscr{E} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathscr{S} + \psi_1 \mathbf{I} \otimes \mathbf{T} + \psi_2 \mathbf{T} \otimes \mathbf{I} + \psi_3 (\mathbf{I} \boxtimes \mathbf{T} + \mathbf{T} \boxtimes \mathbf{I}) \mathscr{S} + \mathbf{I} \boxtimes \mathbf{T}, \tag{9}$$

where $\mathbf{T}$ is the Cauchy stress, λ and μ are linear functions of $\mathrm{tr}\,\mathbf{T}$, ψ_i are material constants and $\mathscr{E}$ the fourth-order constitutive tensor.

The acoustic tensor of an hypoelastic material of grade 1 is defined, for every vector $\mathbf{g}$, by:

$$\mathbf{A}_E(\mathbf{n})\mathbf{g} = \mathscr{E}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}. \tag{10}$$

Therefore, from eqn (9):

$$\mathbf{A}_E(\mathbf{n}) = (\lambda + \mu)\mathbf{n} \otimes \mathbf{n} + \bar{\mu}\mathbf{I} + \gamma_1 \mathbf{n} \otimes \mathbf{Tn} + \gamma_2 \mathbf{Tn} \otimes \mathbf{n} + \gamma_3 \mathbf{T}, \tag{11}$$

where:

$$\bar{\mu} = \mu + \left(\frac{\psi_3}{2} + 1\right)\mathbf{n} \cdot \mathbf{Tn}, \quad \gamma_1 = \psi_1 + \frac{\psi_3}{2}, \quad \gamma_2 = \psi_2 + \frac{\psi_3}{2}, \quad \gamma_3 = \frac{\psi_3}{2}. \tag{12}$$

The following elastoplastic constitutive equation (valid in the *relative Lagrangean description*) incorporates, as special cases, most of the simplified theories of finite-strain elastoplasticity proposed in the literature:

$$\dot{\mathbf{S}} = \mathscr{E}[\mathbf{L}] - \frac{1}{g}\langle \mathbf{L} \cdot \mathbf{Q} \rangle \mathbf{P}, \tag{13}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the flow-mode tensor and the yield function gradient in the deformation space, respectively. Moreover, $\mathscr{E}$ is tensor (9), which does not possess the minor and may not possess the major symmetries as well. Finally, the scalar g is the plastic modulus and the symbol $\langle\ \rangle$ denotes the Macaulay brackets, i.e. the operator $\mathscr{R} \to \mathscr{R}^+$: $(\forall \alpha \in \mathscr{R})$ $\langle \alpha \rangle = \mathrm{Sup}\{\alpha, 0\}$.

It should be noted that, eqn (13) reduces to the hypoelastic law [eqn (8)] in case of unloading ($\mathbf{L} \cdot \mathbf{Q} < 0$) or neutral loading ($\mathbf{L} \cdot \mathbf{Q} = 0$). In case of plastic loading ($\mathbf{L} \cdot \mathbf{Q} > 0$), eqn (13) can once again be interpreted as a hypoelastic law, when $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{T}$ are coaxial. In the following, the loading and unloading branches of eqn (13) will be analyzed separately.

3. ANALYSIS OF THE ACOUSTIC TENSOR FOR HYPOELASTIC MATERIALS OF GRADE 1

In this section, the unloading branch of the constitutive equation (13), i.e. the hypoelastic grade 1 material [eqn (8)] is analyzed. In general, a hypoelastic material of grade 1 may have complex conjugate eigenvalues. Moreover, in cases of eigenvalues with algebraic multiplicity 2 (and 3), the geometric multiplicity may be 1 (and 2 or 1).

When coefficients γ_i are null and $\bar{\mu} = \mu$, the acoustic tensor [eqn (11)] formally coincides with that of the infinitesimal theory of isotropic elasticity. The tensor (11) represents the acoustic tensor of the unloading branch of the constitutive equation in many approximate theories of plasticity. In these theories $\lambda + \mu$ is *preponderant* with respect to the coefficients γ_i multiplied by the spectral radius of the Cauchy stress tensor. It may therefore be interesting to know whether flutter and non-propagation instabilities are possible under this condition. In other words, we will consider terms $\gamma_i T_{jk}$ as perturbations to the infinitesimal theory of isotropic elasticity and we will find, in the present section, that if these terms are sufficiently small with respect to $\lambda + \mu$, flutter and non-propagation instabilities are not possible. Moreover, we will quantify the *smallness* of $\gamma_i \rho(\mathbf{T})$ with respect to $\lambda + \mu$, as a condition sufficient to exclude flutter and non-propagation instability. In closure of the section, some results for hypoelastic materials of grade 1 will be presented, which hold regardless of the magnitude of $\gamma_i \rho(\mathbf{T})$.

In the following, the case that **n** lies in an eigenspace of **T** is never considered. In this case, in fact, the acoustic tensor (11) becomes symmetric and therefore flutter and non-propagation instabilities are excluded. Now, when **n** is not an eigenvector of **T**, we can introduce the following orthonormal basis of $\mathscr{V}$:

$$\begin{cases} \mathbf{e}_1 = \mathbf{n}, \\ \mathbf{e}_2 = \mathbf{l} = \dfrac{\mathbf{n}\times(\mathbf{n}\times\mathbf{Tn})}{|\mathbf{n}\times\mathbf{Tn}|}, \\ \mathbf{e}_3 = \mathbf{m} = \dfrac{\mathbf{n}\times\mathbf{Tn}}{|\mathbf{n}\times\mathbf{Tn}|}. \end{cases} \tag{14}$$

In this basis†, the hypoelastic acoustic tensor $\mathbf{A}_E(\mathbf{n})$ has components:

$$[\mathbf{A}_E] = \begin{bmatrix} \lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn} & (\gamma_1+\gamma_3)\mathbf{n}\cdot\mathbf{Tl} & 0 \\ (\gamma_2+\gamma_3)\mathbf{l}\cdot\mathbf{Tn} & \bar{\mu}+\gamma_3\mathbf{l}\cdot\mathbf{Tl} & \gamma_3\mathbf{l}\cdot\mathbf{Tm} \\ 0 & \gamma_3\mathbf{m}\cdot\mathbf{Tl} & \bar{\mu}+\gamma_3\mathbf{m}\cdot\mathbf{Tm} \end{bmatrix}. \tag{15}$$

The characteristic equation can be written as:

$$[\lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}-\eta]\left\{\eta^2-2\eta\left[\bar{\mu}+\gamma_3\frac{\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm}}{2}\right]\right.$$
$$\left.+\bar{\mu}^2+\gamma_3^2[(\mathbf{l}\cdot\mathbf{Tl})(\mathbf{m}\cdot\mathbf{Tm})-(\mathbf{m}\cdot\mathbf{Tl})^2]+\bar{\mu}\gamma_3(\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm})\right\}$$
$$-(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\mathbf{l}\cdot\mathbf{Tn})^2(\bar{\mu}+\gamma_3\mathbf{m}\cdot\mathbf{Tm}-\eta) = 0. \tag{16}$$

Now we assume that $\lambda+\mu$ be preponderant with respect to coefficients γ_i multiplied by any component of Cauchy stress, or in other words, that any term $\gamma_i\rho(\mathbf{T})$ be small with respect to $\lambda+\mu$. Under this condition, one of the solutions to eqn (16) must be close to $\lambda+\mu+\bar{\mu}$, the others to $\bar{\mu}$.

The next theorem is the central result of the section.

Proposition I.

Flutter and non-propagation instability are excluded if coefficients γ_i are sufficiently small with respect to $\lambda+\mu$ divided by the spectral radius of the Cauchy stress tensor: in particular, a sufficient condition to exclude flutter and wave non-propagation for any value of coefficients γ_i and components T_{jk} is

$$\lambda+\mu \geqslant 9.25\gamma\rho(\mathbf{T}), \tag{17}$$

where γ is the maximum of the absolute values of coefficients γ_i, i.e. $\gamma = \mathrm{Max}\{|\gamma_1|, |\gamma_2|, |\gamma_3|\}$.

† Unit vectors $\mathbf{e}_i$ depend on **T**, and the following relations hold true:

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{Tn}) = (\mathbf{n}\cdot\mathbf{Tn})\mathbf{n}-\mathbf{Tn},$$
$$|\mathbf{n}\times(\mathbf{n}\times\mathbf{Tn})| = |\mathbf{n}\times\mathbf{Tn}| = \sqrt{(\mathbf{n}\cdot\mathbf{T}^2\mathbf{n})-(\mathbf{n}\cdot\mathbf{Tn})^2},$$
$$\mathbf{n}\cdot\mathbf{Tl} = -\sqrt{(\mathbf{n}\cdot\mathbf{T}^2\mathbf{n})-(\mathbf{n}\cdot\mathbf{Tn})^2},\quad \mathbf{m}\cdot\mathbf{Tl} = \frac{-\mathbf{T}^2\mathbf{n}\cdot(\mathbf{n}\times\mathbf{Tn})}{(\mathbf{n}\cdot\mathbf{T}^2\mathbf{n})-(\mathbf{n}\cdot\mathbf{Tn})^2}$$
$$\mathbf{l}\cdot\mathbf{Tl} = \frac{(\mathbf{n}\cdot\mathbf{Tn})^3-2(\mathbf{n}\cdot\mathbf{Tn})(\mathbf{n}\cdot\mathbf{T}^2\mathbf{n})+\mathbf{n}\cdot\mathbf{T}^3\mathbf{n}}{(\mathbf{n}\cdot\mathbf{T}^2\mathbf{n})-(\mathbf{n}\cdot\mathbf{Tn})^2}.$$

Note that the third relationship implies that $\mathbf{n}\cdot\mathbf{Tl}\neq 0$ if and only if **n** is not in an eigenspace of **T**. Moreover, $\mathbf{m}\cdot\mathbf{Tl}$ can be zero even when $\mathbf{n}\cdot\mathbf{Tl}\neq 0$.

Proof:
The polynomial of η, say $\mathscr{P}(\eta)$, at the left hand side of eqn (16) can be written as follows:

$$\begin{aligned}\mathscr{P}(\eta) = {} & [\lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}-\eta] \\ & \times\left[\eta-\bar{\mu}-\gamma_3\mathbf{m}\cdot\mathbf{Tm}-\gamma_3\left(\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}+\sqrt{\left(\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}\right)^2+(\mathbf{m}\cdot\mathbf{Tl})^2}\right)\right] \\ & \times\left[\eta-\bar{\mu}-\gamma_3\mathbf{m}\cdot\mathbf{Tm}-\gamma_3\left(\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}-\sqrt{\left(\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}\right)^2+(\mathbf{m}\cdot\mathbf{Tl})^2}\right)\right] \\ & +(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\mathbf{l}\cdot\mathbf{Tn})^2(\eta-\bar{\mu}-\gamma_3\mathbf{m}\cdot\mathbf{Tm}),\end{aligned} \tag{18}$$

i.e. as the sum of a third and a first degree polynomial (Fig. 1). The 3rd degree polynomial has always three real roots (it is in fact the characteristic equation of the symmetric matrix obtained from matrix (15) by eliminating the non-symmetric terms), one of which is $\lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}$, and the other two are

$$\left.\begin{matrix}\Theta_1\\ \Theta_2\end{matrix}\right\} = \bar{\mu}+\gamma_3\mathbf{m}\cdot\mathbf{Tm}+\gamma_3\left[\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}\pm\sqrt{\left(\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}\right)^2+(\mathbf{m}\cdot\mathbf{Tl})^2}\right]. \tag{19}$$

The first degree polynomial in eqn (18) vanishes for a value of η internal to the closed interval defined by the two roots Θ_1 and Θ_2. Thus, continuity of $\mathscr{P}(\eta)$ implies that *a root of the characteristic equation* (16) *is always internal to the closed interval defined by* Θ_1 *and* Θ_2 (see Fig. 1). Noting that

$$\Theta_1 \leqslant \bar{\mu}+\gamma_3\frac{\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm}}{2} \leqslant \Theta_2 \leqslant \bar{\mu}+\gamma_3\frac{\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm}}{2}+\sqrt{2}\gamma\rho(\mathbf{T}), \tag{20}$$

and introducing the change of variable: $\eta = \bar{\mu}+\gamma_3(\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm})/2+\gamma\rho(\mathbf{T})x$, $\mathscr{P}(\eta)$ becomes:

$$\begin{aligned}\mathscr{P}(x) = {} & \left[\lambda+\mu+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}-\gamma_3\frac{\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm}}{2}-x\gamma\rho(\mathbf{T})\right] \\ & \times\left[(\gamma\rho(\mathbf{T})x)^2-\left(\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}\right)^2-(\mathbf{m}\cdot\mathbf{Tl})^2\right] \\ & +(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\mathbf{l}\cdot\mathbf{Tn})^2\left[\gamma\rho(\mathbf{T})x+\gamma_3\frac{\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm}}{2}\right].\end{aligned} \tag{21}$$

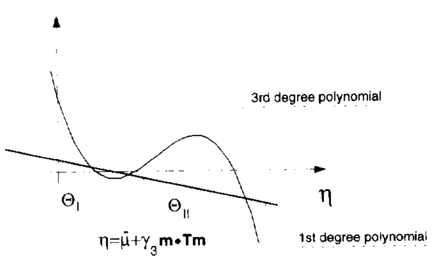

Fig. 1. Schematic representation of the two polynomials which define $\mathscr{P}(\eta)$.

For $\sqrt{2} \leqslant x \leqslant [\lambda+\mu-4\gamma\rho(\mathbf{T})]/(\gamma\rho(\mathbf{T}))$, the following inequality holds true

$$\mathscr{P}(x) \geqslant [\lambda+\mu-\gamma\rho(\mathbf{T})(x+4)][x^2-2][\gamma\rho(\mathbf{T})]^2-4[\gamma\rho(\mathbf{T})]^3(x+1). \tag{22}$$

Now, for $x = 3.25$, the right hand side of inequality (22) is not less than zero if $\lambda+\mu \geqslant 9.235\gamma\rho(\mathbf{T})$. For $x = 3.25$ and $\lambda+\mu \geqslant 9.235\gamma\rho(\mathbf{T})$, the inequality $\sqrt{2}\gamma\rho(\mathbf{T}) \leqslant x\gamma\rho(\mathbf{T}) \leqslant \lambda+\mu-4\gamma\rho(\mathbf{T})$ is verified. Therefore, if $\lambda+\mu \geqslant 9.235\gamma\rho(\mathbf{T})$, $\mathscr{P}(\eta) = +\infty$ for $\eta \to -\infty$, $\mathscr{P}(\eta) = 0$ for $\eta \in [\Theta_1, \Theta_2]$, $\mathscr{P}(\eta) \geqslant 0$ for $\eta > \Theta_2$ and, finally, $\mathscr{P}(\eta) = -\infty$ for $\eta \to +\infty$. This means that eqn (16) has three real roots if condition (17) holds true. Now we have to consider the possibility of different geometric and algebraic multiplicities of eigenvalues of matrix (15).

When inequality (17) holds true, the only possibility of coincident roots of eqn (16) is that $\mathscr{P}(\eta)$ has a minimum in correspondence of a root of eqn (16) not greater than $\bar{\mu}+\gamma_3(\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm})/2+\sqrt{2}\gamma\rho(\mathbf{T})$. This root is necessarily equal to $\bar{\mu}+\gamma_3\mathbf{m}\cdot\mathbf{Tm}$ (see Fig. 1). This may occur if and only if

$$\gamma_3 = 0 \quad \text{or} \quad \mathbf{l}\cdot\mathbf{Tm} = 0. \tag{23}$$

In this condition, it can be verified by direct inspection of matrix (15) that the algebraic multiplicity of the double eigenvalue must coincide with the geometric one. Therefore, non-propagation instability is excluded. ■

The order of magnitude of $\gamma\rho(\mathbf{T})/(\lambda+\mu)$ is typically 0.001–0.01 in many finite strains plasticity theories, therefore the bound (17) can be considered satisfactory.

It should be noted that $\rho(\mathbf{T})$ can be replaced in inequality (17) by *any matrix norm* of $[\mathbf{T}]$ (thus obtaining a less stringent condition). The following remarks of this section concern the possibility of flutter and non-propagation instability in an hypoelastic material of grade 1 without any assumption about the magnitude of coefficients $\gamma_i T_{jk}$. The next three remarks are examples of flutter and non-propagation instabilities in hypoelastic materials.

Remark 1. For a generic hypoelastic material of grade 1, flutter and non propagation instability may occur. In particular, we can construct examples of flutter and wave non-propagation, for certain values of coefficients γ_i, components T_{jk}, when

$$\lambda+\mu \leqslant 2\gamma\rho(\mathbf{T}). \tag{24}$$

Proof: This remark can be proved just showing an example. To this purpose, let us introduce the orthogonal basis defined presently by **n**, **s** and **r**, where **s** and **r** are arbitrary unit vectors mutually orthogonal and orthogonal to **n**. The acoustic tensor [eqn (11)] can be projected onto this basis, thus obtaining

$$[\mathbf{A}_E] = \begin{bmatrix} \lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn} & (\gamma_1+\gamma_3)\mathbf{n}\cdot\mathbf{Ts} & (\gamma_1+\gamma_3)\mathbf{n}\cdot\mathbf{Tr} \\ (\gamma_2+\gamma_3)\mathbf{s}\cdot\mathbf{Tn} & \bar{\mu}+\gamma_3\mathbf{s}\cdot\mathbf{Ts} & \gamma_3\mathbf{s}\cdot\mathbf{Tr} \\ (\gamma_2+\gamma_3)\mathbf{r}\cdot\mathbf{Tn} & \gamma_3\mathbf{r}\cdot\mathbf{Ts} & \bar{\mu}+\gamma_3\mathbf{r}\cdot\mathbf{Tr} \end{bmatrix}, \tag{25}$$

which, for $\mu = \bar{\mu}$, $\lambda = \gamma_3 = \mathbf{n}\cdot\mathbf{Tn} = \mathbf{s}\cdot\mathbf{Ts} = \mathbf{r}\cdot\mathbf{Tr} = \mathbf{n}\cdot\mathbf{Tr} = \mathbf{s}\cdot\mathbf{Tr} = 0$, $\gamma_1 = \mathbf{n}\cdot\mathbf{Ts} = 1$, $\gamma_2 = -1$ becomes

$$\begin{bmatrix} 2\mu & 1 & 0 \\ -1 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{26}$$

Matrix (26) is positive defined and has complex eigenvalues, when $\mu < 2$ and has an eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1, when $\mu = 2$. Note that (being $\mathbf{n}\cdot\mathbf{Tn} = \mathbf{s}\cdot\mathbf{Ts} = \mathbf{r}\cdot\mathbf{Tr} = \mathbf{n}\cdot\mathbf{Tr} = \mathbf{s}\cdot\mathbf{Tr} = 0$ and $\mathbf{n}\cdot\mathbf{Ts} = 1$) $\rho(\mathbf{T}) = 1$; moreover it is $\gamma = 1$, and, therefore, flutter is possible when condition (24) is satisfied. ■

Remark 2. Even when flutter is *a priori* excluded, the geometric multiplicity of some eigenvalues may be less than the algebraic multiplicity.

Proof: This remark can be proved just showing an example. To this purpose, let us assume:

$$\gamma_2 = \gamma_3 = 0, \tag{27}$$

so that flutter is *a priori* excluded. Under conditions (27), the characteristic equation (16) yields:

$$\eta_{\mathrm{I}} = \lambda+\mu+\bar{\mu}+\gamma_1\mathbf{n}\cdot\mathbf{Tn}, \quad \eta_{\mathrm{II}} = \eta_{\mathrm{III}} = \bar{\mu}. \tag{28}$$

The eigenvector problem associated to η_{I} can be formulated as follows:

$$[(\lambda+\mu)\mathbf{n}\otimes\mathbf{n}+\bar{\mu}\mathbf{I}+\gamma_1\mathbf{n}\otimes\mathbf{Tn}]\mathbf{v} = [\lambda+\mu+\bar{\mu}+\gamma_1\mathbf{n}\cdot\mathbf{Tn}]\mathbf{v}, \tag{29}$$

which has the solution $\mathbf{v} = \mathbf{n}$. The eigenvector problem associated to η_{II} can be formulated as follows:

$$[(\lambda+\mu)\mathbf{n}\otimes\mathbf{n}+\bar{\mu}\mathbf{I}+\gamma_1\mathbf{n}\otimes\mathbf{Tn}]\mathbf{v} = \bar{\mu}\mathbf{v}, \tag{30}$$

which can be projected onto $\mathbf{n}$ to obtain

$$(\lambda+\mu)\mathbf{v}\cdot\mathbf{n} = -\gamma_1\mathbf{n}\cdot\mathbf{Tv}, \tag{31}$$

i.e.:

$$[(\lambda+\mu)\mathbf{n}+\gamma_1\mathbf{Tn}]\cdot\mathbf{v} = 0. \tag{32}$$

From eqn (32) it can be concluded that the (right) eigenvectors are orthogonal to the following vector:

$$(\lambda+\mu)\mathbf{n}+\gamma_1\mathbf{Tn}. \tag{33}$$

Now, if $\eta_{\mathrm{I}} = \eta_{\mathrm{II}} = \eta_{\mathrm{III}}$, i.e. if:

$$\lambda+\mu+\gamma_1\mathbf{n}\cdot\mathbf{Tn} = 0, \tag{34}$$

the eigenplane defined by eqn (32) becomes:

$$\gamma_1[\mathbf{n}\times(\mathbf{n}\times\mathbf{Tn})]\cdot\mathbf{v} = 0, \tag{35}$$

and, therefore, it contains $\mathbf{n}$. Thus, the eigenvalue η has now algebraic multiplicity three and geometric multiplicity two. ■

Remark 3. For hypoelasticity of grade 1, the acoustic tensor may have one eigenvalue of algebraic multiplicity 3 and geometric multiplicity 1. In this condition only one wave amplitude is possible, corresponding to the only eigenvector.

Proof: This remark can be proved just showing an example. The representation (25) of the acoustic tensor becomes (for an appropriate choice of $\mathbf{T}$, $\mathbf{n}$, $\mathbf{s}$, $\mathbf{r}$ and λ, μ, $\bar{\mu}$, γ_i):

$$\begin{bmatrix} 0 & 7 & 0 \\ -4 & 40/7 & -(24^{3/2}+864^{1/2})/35 \\ 0 & -(24^{3/2}+864^{1/2})/35 & 2/7 \end{bmatrix}, \tag{36}$$

which has only one eigenvalue, equal to 2, of algebraic multiplicity 3 and geometric multiplicity 1. The only (right) eigenspace is the straight line corresponding to the direction $[-7\sqrt{6}, -\sqrt{24}, 12]$. ■

Remark 4. Any condition

$$\begin{aligned} \gamma_1 &= \gamma_2, \\ \gamma_1+\gamma_3 &= 0, \\ \gamma_2+\gamma_3 &= 0, \\ (\gamma_1+\gamma_3)(\gamma_2+\gamma_3) &> 0, \end{aligned} \tag{37}$$

is sufficient to exclude flutter.

Proof: $(37)_1$–$(37)_3$ directly follow from representation (15) or (25). $(37)_4$ can be obtained by observing that polynomial (18) is positive when $\eta \to -\infty$ and negative when $\eta \to +\infty$. Moreover, when $(\gamma_1+\gamma_3)(\gamma_2+\gamma_3) > 0$, polynomial (18) is negative at $\eta = \Theta_1$ and positive at $\eta = \Theta_2$, therefore it must have three real roots. ■

It should be noted that for the unloading branch of the elastoplastic model introduced by Rudnicki and Rice (1975), in which the Jaumann derivative of Cauchy stress is isotropically related to the velocity of deformation, coefficients γ_i result to be: $\gamma_1 = \gamma_3 = -\gamma_2 = -1/2$. Therefore, flutter is excluded (but not for the plastic branch, as will be discussed in Section 5).

Let us now turn the attention to the following submatrix obtained from matrix (15):

$$\begin{bmatrix} \bar{\mu}+\gamma_3 \mathbf{l}\cdot\mathbf{T}\mathbf{l} & \gamma_3 \mathbf{l}\cdot\mathbf{T}\mathbf{m} \\ \gamma_3 \mathbf{m}\cdot\mathbf{T}\mathbf{l} & \bar{\mu}+\gamma_3 \mathbf{m}\cdot\mathbf{T}\mathbf{m} \end{bmatrix}, \tag{38}$$

which is symmetric and has (real) eigenvalues, which are Θ_1 and Θ_2 given by eqn (19). Therefore, a rotation matrix does exist, for which:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{12} \\ 0 & r_{21} & r_{22} \end{bmatrix} [\mathbf{A}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_{11} & r_{21} \\ 0 & r_{12} & r_{22} \end{bmatrix} = \begin{bmatrix} \lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{T}\mathbf{n} & r_{11}(\gamma_1+\gamma_3)\mathbf{n}\cdot\mathbf{T}\mathbf{l} & r_{21}(\gamma_1+\gamma_3)\mathbf{n}\cdot\mathbf{T}\mathbf{l} \\ r_{11}(\gamma_2+\gamma_3)\mathbf{l}\cdot\mathbf{T}\mathbf{n} & \Theta_1 & 0 \\ r_{21}(\gamma_2+\gamma_3)\mathbf{l}\cdot\mathbf{T}\mathbf{n} & 0 & \Theta_2 \end{bmatrix}. \tag{39}$$

In this case, the characteristic equation (16) becomes:

$$(\Theta_1-\eta)(\Theta_2-\eta)[\lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{T}\mathbf{n}-\eta] - (\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\mathbf{l}\cdot\mathbf{T}\mathbf{n})^2[r_{11}^2(\Theta_2-\eta)+r_{21}^2(\Theta_1-\eta)] = 0, \tag{40}$$

from which it can be inferred the following remark.

Remark 5. Any condition

$$\begin{aligned}\gamma_3 &= 0,\\ \gamma_1+\gamma_3 &= 0,\\ \gamma_2+\gamma_3 &= 0,\\ \mathbf{l}\cdot\mathbf{Tm} &= 0,\\ \Theta_1 &= \Theta_2,\\ \gamma_1 &= \gamma_2.\end{aligned} \tag{41}$$

is sufficient to exclude geometric multiplicity 1 for an eigenvalue of algebraic multiplicity 3.

Proof: Cases $\gamma_3 = 0$, $\gamma_1 = \gamma_2$ and $\mathbf{l}\cdot\mathbf{Tm} = 0$ can be trivially deduced from matrix (15), so that cases $\gamma_1+\gamma_3 = 0$, $\gamma_2+\gamma_3 = 0$ and $\Theta_1 = \Theta_2$ only merit to be analyzed.

In the case $\gamma_1+\gamma_3 = 0$, the characteristic equation (40) has solutions $\eta_{\rm I} = \Theta_1$, $\eta_{\rm II} = \Theta_2$ and $\eta_{\rm III} = \lambda+\mu+\bar{\mu}+\gamma_2\mathbf{n}\cdot\mathbf{Tn}$. When $\eta_{\rm I} = \eta_{\rm II} = \eta_{\rm III}$, the eigenvector problem in the same basis of matrix (39) is

$$\begin{bmatrix} 0 & 0 & 0\\ r_{11}(\gamma_2+\gamma_3)\mathbf{l}\cdot\mathbf{Tn} & 0 & 0\\ r_{21}(\gamma_2+\gamma_3)\mathbf{l}\cdot\mathbf{Tn} & 0 & 0\end{bmatrix}\begin{bmatrix} v_x\\ v_y\\ v_z\end{bmatrix} = [0]. \tag{42}$$

Therefore, the two linearly independent solutions [0, 1, 0] and [0, 0, 1] are always possible. The case $\gamma_2+\gamma_3 = 0$ is analogous.

In the case $\Theta_1 = \Theta_2$, from eqn (40) it is observed that $\eta_{\rm I} = \Theta_1$ is an eigenvalue and thus matrix (38) is diagonal. Therefore $r_{11} = r_{22} = 1$ and $r_{21} = r_{12} = 0$. In addition to $\eta_{\rm I} = \Theta_1$, the characteristic equation (40) has now solutions:

$$\left.\begin{matrix}\eta_{\rm II}\\ \eta_{\rm III}\end{matrix}\right\} = \tfrac{1}{2}[\lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}+\Theta_1]\pm\sqrt{\Delta}, \tag{43}$$

where $\Delta = [\lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}-\Theta_1]^2+4(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\mathbf{l}\cdot\mathbf{Tn})^2$. If $\eta_{\rm I} = \eta_{\rm II} = \eta_{\rm III} = \Theta_1$, the conditions $\Theta_1 = \lambda+\mu+\bar{\mu}+(\gamma_1+\gamma_2+\gamma_3)\mathbf{n}\cdot\mathbf{Tn}$ and $\Delta = 0$ must hold true, which imply:

$$(\gamma_1+\gamma_3)(\gamma_2+\gamma_3)(\mathbf{l}\cdot\mathbf{Tn})^2 = 0, \tag{44}$$

and therefore one of conditions $(41_{2,3})$ must be verified. This means that (41_5) is equivalent to one of conditions $(41_{2,3})$. ■

Remark 6. From Remarks (4) and (5), any condition

$$\begin{aligned}\gamma_1 &= \gamma_2,\\ \gamma_1+\gamma_3 &= 0,\\ \gamma_2+\gamma_3 &= 0,\end{aligned} \tag{45}$$

is sufficient to exclude flutter and geometric multiplicity 1 for triple eigenvalues.

Note however that conditions $(45)_2$ and $(45)_3$ are not sufficient to exclude multiplicity 1 (2) for double (triple) eigenvalues, see Remark 2.

In closure of this section we want to stress that, if coefficients $\gamma_i T_{jk}$ of grade 1 hypoelastic materials are taken as pertubations to the infinitesimal theory of elasticity, flutter and non-propagation instabilities are excluded. This result is of some relevance for the theories of plasticity which will be presented in the next section. For these theories, flutter and non-propagation instability are excluded in the unloading branch of the constitutive operator, but may occur in the loading branch.

4. EIGENVALUE PROBLEM OF THE ACOUSTIC TENSOR FOR FINITE STRAINS ELASTOPLASTICITY

In this section the solution to the eigenvalue problem proposed by Bigoni and Zaccaria (1994) for generic finite strains elastoplasticity is generalized by taking into account the possibility of a non-symmetric acoustic tensor of the unloading branch of the constitutive operator, as in the case of the acoustic tensor (11). The analysis which follows yields the eigenvalues of the acoustic tensor and the condition for flutter for a quite general elasto-plastic solid *under the hypothesis that an acoustical axis for waves travelling in the direction* **n** *corresponds to a neutral plastic wave.* Moreover, the analysis is valid, without restrictions, for two-dimensional theories of plasticity. Reference is made to the loading branch of constitutive equation (13) (comparison solid of Hill, 1958). The eigenvalue problem for the acoustic tensor corresponding to the loading branch of the constitutive equation (13) can be written as:

$$\mathbf{A}_{\mathrm{E}}(\mathbf{n})\mathbf{v}-\frac{1}{g}(\mathbf{v}\cdot\mathbf{q})\mathbf{p}-\eta\mathbf{v}=0, \quad (\forall \mathbf{v}\in\mathscr{V}), \tag{46}$$

where:

$$\mathbf{q}=\mathbf{Q}\mathbf{n}, \quad \mathbf{p}=\mathbf{P}\mathbf{n}, \quad \mathbf{A}_E(\mathbf{n})\mathbf{v}=\mathscr{E}[\mathbf{v}\otimes\mathbf{n}]\mathbf{n} \quad (\forall \mathbf{v}\in\mathscr{V}). \tag{47}$$

It is assumed that flutter and non-propagation instability are excluded for the unloading branch of the constitutive operator. In other words, it is assumed that the hypoelastic acoustic tensor $\mathbf{A}_{\mathrm{E}}$ possesses the left linearly independent eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and thus the corresponding right eigenvectors $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$. The scalars α_1, α_2 and α_3 denote the corresponding eigenvalues, which are supposed to be real. Moreover, it is assumed that the triplet:

$$\mathbf{e}_1=\mathbf{a}_1, \quad \mathbf{e}_2=\mathbf{q}, \quad \mathbf{e}_3=\mathbf{a}_3 \tag{48}$$

forms a linearly independent system of vectors (see Appendix A for the case when this condition is not satisfied). The dual basis of basis (48) can be written in the form:

$$\begin{aligned}
\mathbf{e}^1 &= -\frac{\mathbf{a}^1\cdot\mathbf{q}}{(\mathbf{a}_1\cdot\mathbf{a}^1)(\mathbf{a}^2\cdot\mathbf{q})}\mathbf{a}^2+\frac{1}{(\mathbf{a}_1\cdot\mathbf{a}^1)}\mathbf{a}^1,\\
\mathbf{e}^2 &= \frac{1}{\mathbf{a}^2\cdot\mathbf{q}}\mathbf{a}^2,\\
\mathbf{e}^3 &= -\frac{\mathbf{a}^3\cdot\mathbf{q}}{(\mathbf{a}_3\cdot\mathbf{a}^3)(\mathbf{a}^2\cdot\mathbf{q})}\mathbf{a}^2+\frac{1}{(\mathbf{a}_3\cdot\mathbf{a}^3)}\mathbf{a}^3.
\end{aligned} \tag{49}$$

It is important to note that vectors $\mathbf{a}_i$ and $\mathbf{a}^i$ are not restricted in modulus, so that $\mathbf{a}_i\cdot\mathbf{a}^i$ can be different from unity. Expressing the generic vector **v** in the basis (49) and projecting the eigenvalue problem (46) onto the basis (48), the characteristic equation is obtained in the form:

$$\det \begin{bmatrix} \alpha_1 - \eta & -\frac{1}{g}\mathbf{p}\cdot\mathbf{a}_1 & 0 \\ (\alpha_1-\alpha_2)\frac{\mathbf{q}\cdot\mathbf{a}^1}{\mathbf{a}_1\cdot\mathbf{a}^1} & \alpha_2 - \frac{1}{g}\mathbf{p}\cdot\mathbf{q} - \eta & (\alpha_3-\alpha_2)\frac{\mathbf{q}\cdot\mathbf{a}^3}{\mathbf{a}_3\cdot\mathbf{a}^3} \\ 0 & -\frac{1}{g}\mathbf{p}\cdot\mathbf{a}_3 & \alpha_3 - \eta \end{bmatrix} = 0. \tag{50}$$

From condition (50), a third order algebraic equation can be easily obtained which, in general, has three solutions η_1, η_2, η_3 different from the eigenvalues of the elastic acoustic tensor α_1, α_2, α_3. If a neutral plastic wave is assumed to be possible, corresponding, say, to the eigenvalue α_3 (and thus to amplitude $\mathbf{a}^3$), this eigenvalue must be a solution of eqn (50). This circumstance occurs if:

$$(\alpha_3-\alpha_2)(\mathbf{p}\cdot\mathbf{a}_3)(\mathbf{q}\cdot\mathbf{a}^3) = 0. \tag{51}$$

Condition (51) is valid when the hypoelastic acoustic tensor has two coincident eigenvalues, or for $\mathbf{p}$ or $\mathbf{q}$ ranging in the planes orthogonal to $\mathbf{a}_3$ and $\mathbf{a}^3$, respectively. Using eqn (51), eqn (50) yields the following equation:

$$[\alpha_3-\eta]\left[\eta^2 - \left[\alpha_1+\alpha_2-\frac{1}{g}\mathbf{p}\cdot\mathbf{q}\right]\eta + \alpha_1\left[\alpha_2 - \frac{1}{g}\mathbf{p}\cdot\mathbf{q}\right] + \frac{1}{g}(\alpha_1-\alpha_2)\mathbf{p}\cdot\mathbf{a}_1\frac{\mathbf{q}\cdot\mathbf{a}^1}{\mathbf{a}_1\cdot\mathbf{a}^1}\right] = 0. \tag{52}$$

The second degree polynomial in eqn (52) could be directly obtained for two-dimensional theories of plasticity. The discriminant of the second degree polynomial in eqn (52) can be written as:

$$\Delta = \left[\alpha_1-\alpha_2+\frac{1}{g}\mathbf{p}\cdot\mathbf{q}\right]^2 - 4\frac{1}{g}(\alpha_1-\alpha_2)\mathbf{p}\cdot\mathbf{a}_1\frac{\mathbf{q}\cdot\mathbf{a}^1}{\mathbf{a}_1\cdot\mathbf{a}^1}, \tag{53}$$

or, equivalently:

$$\Delta = \left[\alpha_1+\alpha_2-\frac{1}{g}\mathbf{p}\cdot\mathbf{q}\right]^2 - 4\alpha_1\alpha_2\left(1-\frac{g_{cr}}{g}\right), \tag{54}$$

where:

$$g_{cr} = \frac{1}{\alpha_2}\mathbf{p}\cdot\mathbf{q} - \frac{(\alpha_1-\alpha_2)}{\alpha_1\alpha_2}\mathbf{p}\cdot\mathbf{a}_1\frac{\mathbf{q}\cdot\mathbf{a}^1}{\mathbf{a}_1\cdot\mathbf{a}^1} \tag{55}$$

is the *critical value of the plastic modulus for the localization of deformation into planar bands of normal (unit) vector* **n** (Rice, 1976).

Equation (54) shows that, if $\alpha_1\alpha_2 > 0$ (as in the case when the elastic acoustic tensor is positive definite), *flutter in a given direction* **n** *cannot occur for values of the plastic modulus less than or equal to the critical plastic modulus for strain localization in that direction*†. From eqn (53) the following necessary and sufficient condition for flutter is obtained:

† It may be important to stress that strain localization might occur before flutter in a different direction.

$$(\alpha_1-\alpha_2)^2\frac{(\mathbf{a}_1\cdot\mathbf{p})(\mathbf{a}^1\cdot\mathbf{q})}{\mathbf{a}_1\cdot\mathbf{a}^1}\left[\frac{(\mathbf{a}_1\cdot\mathbf{p})(\mathbf{a}^1\cdot\mathbf{q})}{\mathbf{a}_1\cdot\mathbf{a}^1}-\mathbf{p}\cdot\mathbf{q}\right]>0. \qquad (56)$$

When flutter occurs for positive values of the plastic modulus g, condition (56) becomes:

$$(\alpha_1-\alpha_2)\frac{(\mathbf{a}_1\cdot\mathbf{p})(\mathbf{a}^1\cdot\mathbf{q})}{\mathbf{a}_1\cdot\mathbf{a}^1}>0 \quad \text{and} \quad (\alpha_1-\alpha_2)\left[\frac{(\mathbf{a}_1\cdot\mathbf{p})(\mathbf{a}^1\cdot\mathbf{q})}{\mathbf{a}_1\cdot\mathbf{a}^1}-\mathbf{p}\cdot\mathbf{q}\right]>0. \qquad (57)$$

When condition (57) is satisfied, flutter occurs for values of the plastic modulus g within the following interval:

$$\left.\begin{matrix} g_1 \\ g_2 \end{matrix}\right\} = \frac{1}{(\alpha_1-\alpha_2)}\left[\sqrt{(\mathbf{p}\cdot\mathbf{a}_1)(\mathbf{q}\cdot\mathbf{a}^1)/(\mathbf{a}^1\cdot\mathbf{a}_1)}\pm\sqrt{(\mathbf{p}\cdot\mathbf{a}_1)(\mathbf{q}\cdot\mathbf{a}^1)/(\mathbf{a}^1\cdot\mathbf{a}_1)-\mathbf{p}\cdot\mathbf{q}}\right]^2. \qquad (58)$$

The results obtained in this section are a generalization of those obtained in (Bigoni and Zaccaria, 1994). The possibility of taking into account non-symmetric acoustic tensors of the unloading branch of constitutive eqn (13), allows us to solve the eigenvalue problem for a wide class of constitutive equations.

5. ON COMPLEX EIGENVALUES FOR TWO MODELS OF ELASTOPLASTICITY

In this section, the previously derived formulae will be applied to two different constitutive models of elastoplasticity at finite strains. In particular, the spectral analysis will be performed and the condition for flutter will be obtained and discussed. The examples show that, even in cases where flutter instability is excluded in the infinitesimal theory, this instability may indeed occur as a consequence of unsymmetrical geometrical terms in the constitutive equation, which can be regarded as perturbations to the infinitesimal theory of elastoplasticity.

First constitutive model

The first considered constitutive model is obtained by postulating the following relationship between the Oldroyd derivative of Cauchy stress $\mathbf{T}$ and the velocity of deformation $\mathbf{D}$:

$$\mathring{\mathbf{T}} = \lambda \operatorname{tr}\mathbf{D}+2\mu\mathbf{D}-\frac{1}{g}\langle\mathbf{D}\cdot\mathbf{Q}\rangle\mathbf{P}, \qquad (59)$$

where λ and μ are the Lamé constants. A constitutive equation similar to eqn (59) in which, however, the Cauchy stress is replaced by the Kirchhoff stress, has been used by Hill (1962) and Hutchinson and Miles (1974). It is well known that the differences between Cauchy and Kirchhoff stress measures vanish in the case of isochoric deformations (as, for instance, in Hill and Hutchinson, 1975). On the other hand, in the presence of volumetric deformations, the constitutive equation (59) yields an unsymmetrical acoustic tensor (even for associative flow rule during both loading and unloading). The same circumstance occurs in the case of the constitutive equation identical to eqn (59), with the Oldroyd derivative replaced by the Jaumann derivative (Rudnicki and Rice model). The constitutive equation (59) becomes (the relative Lagrangean description is assumed):

$$\dot{\mathbf{S}} = [\lambda\mathbf{I}\otimes\mathbf{I}+2\mu\mathcal{S}+\mathbf{I}\boxtimes\mathbf{T}+\mathbf{T}\otimes\mathbf{I}][\mathbf{L}]-\frac{1}{g}\langle\mathbf{Q}\cdot\mathbf{D}\rangle\mathbf{P}, \qquad (60)$$

Note that the *geometrical* terms at the right hand side of eqn (60) have neither the minor, nor the major symmetries and can be interpreted as perturbations to the infinitesimal theory of elastoplasticity. From eqn (60), the acoustic tensor corresponding to unloading and neutral loading takes the form:

$$\mathbf{A}_E(\mathbf{n}) = (\lambda+\mu)\mathbf{n}\otimes\mathbf{n}+\mathbf{Tn}\otimes\mathbf{n}+(\mu+\mathbf{n}\cdot\mathbf{Tn})\mathbf{I}, \tag{61}$$

and therefore flutter is excluded and non-propagation instability can occur for non small values of $\rho(\mathbf{T})/(\lambda+\mu)$ (see Section 3). A spectral analysis of $\mathbf{A}_E$ reveals the following eigenvalues and eigenvectors:

$$\alpha_1 = \lambda+2\mu+2\mathbf{n}\cdot\mathbf{Tn}, \quad \alpha_2 = \alpha_3 = \mu+\mathbf{n}\cdot\mathbf{Tn}, \tag{62}$$

$$\begin{cases} \mathbf{a}_1 = \mathbf{n}, & \mathbf{a}^1 = \dfrac{\lambda+\mu}{\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}}\mathbf{n}+\dfrac{1}{\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}}\mathbf{Tn}, \\ \mathbf{a}_2 = (\mathbf{a}^2\cdot\mathbf{Tn})\mathbf{n}-(\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn})\mathbf{a}^2, & \mathbf{a}^2 = \mathbf{n}\times\mathbf{s}, \\ \mathbf{a}_3 = (\mathbf{a}^3\cdot\mathbf{Tn})\mathbf{n}-(\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn})\mathbf{a}^3, & \mathbf{a}^3 = \mathbf{s}, \end{cases} \tag{63}$$

where $\mathbf{s}$ is any unit vector orthogonal to $\mathbf{n}$. Condition (51) is satisfied and the flutter condition (57) becomes ($\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn} > 0$ is assumed):

$$\begin{cases} (\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})+\dfrac{\mathbf{p}\cdot\mathbf{n}}{\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}}[\mathbf{q}\cdot\mathbf{Tn}-(\mathbf{q}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Tn})] > 0, \\ (\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})-\mathbf{p}\cdot\mathbf{q}+\dfrac{\mathbf{p}\cdot\mathbf{n}}{\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}}[\mathbf{q}\cdot\mathbf{Tn}-(\mathbf{q}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Tn})] > 0. \end{cases} \tag{64}$$

If **P**, **Q** and **T** have a common eigenvector, the left hand side of inequality (64_2) vanishes for $\mathbf{n}$ parallel to that eigenvector. Therefore, in this case, the condition of coalescence of two eigenvalues of the acoustic tensor can always be met at an appropriate value of the plastic modulus. Moreover, it is important to note that if the terms divided by $\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}$ are neglected, conditions (64) become the conditions for flutter of the infinitesimal theory (Loret *et al.*, 1990). The terms divided by $\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}$ are generally small with respect to the others and can be considered as a perturbation. Let us restrict our study to the case of *associative flow-law*: $\mathbf{p} = \mathbf{q}$. Inequality (64_1) is always satisfied if $\mathbf{q}\cdot\mathbf{n}$ is sufficiently far from zero. Inequality (64_2) becomes:

$$(\mathbf{q}\cdot\mathbf{n})^2-\mathbf{q}\cdot\mathbf{q}+\frac{\mathbf{q}\cdot\mathbf{n}}{\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}}[\mathbf{q}\cdot\mathbf{Tn}-(\mathbf{q}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Tn})] > 0. \tag{65}$$

Condition (65) can be satisfied when $\mathbf{q}$ and $\mathbf{n}$ tend to become parallel ($\mathbf{n}\times\mathbf{q}\to\mathbf{0}$), i.e. $\mathbf{n}$ tends to coincide with an eigenvector of **Q**. Condition (65) can be rewritten as:

$$[\mathbf{n}(\mathbf{n}\cdot\mathbf{Qn})-\mathbf{Qn}]\cdot\left[\mathbf{Qn}-\frac{\mathbf{n}\cdot\mathbf{Qn}}{\lambda+\mu+\mathbf{n}\cdot\mathbf{Tn}}\mathbf{Tn}\right] > 0, \tag{66}$$

from which it is evident that *when* **Q** *and* **T** *are not coaxial, flutter is always possible* for directions $\mathbf{n}$ close to the eigenvectors of **Q** [for values of the plastic modulus internal to interval (58)]. It is important to note that, in this case, flutter is always possible for every given **Q**, i.e. regardless to the form of the yield surface. By imposing that $\mathbf{n}$ be an eigenvalue of **Q**, it is possible to obtain, from eqn (58), the critical plastic modulus for flutter:

$$\varphi_{fl} = \max_{|\mathbf{n}_i|} \frac{|\mathbf{n}_i \cdot \mathbf{Q}\mathbf{n}_i|^2}{\lambda + \mu + \mathbf{n}_i \cdot \mathbf{T}\mathbf{n}_i}, \tag{67}$$

where $\mathbf{n}_i$ are the three (unit) eigenvectors of $\mathbf{Q}$. The critical plastic modulus (67) is generally higher than the critical plastic modulus for strain localization and may be higher than the critical plastic modulus for loss of second order work positive definiteness (Maier and Hueckel, 1979). Trying to clarify the present discussion with an example, let us consider the case of the J_2-flow theory, for the uniaxial compression stress state. We assume also $\lambda = 0$ and neglect the term $\mathbf{n} \cdot \mathbf{T}\mathbf{n}$ with respect to μ. In these conditions the eigenvectors of $\mathbf{Q}$ are $\{-\mu 2/\sqrt{3}, \mu/\sqrt{3}, \mu/\sqrt{3}\}$, therefore $\varphi_{fl} = \mu 4/3$. The critical hardening modulus for flutter turns out to be $h_{fl} = \mu/3$. Therefore, flutter occurs during positive hardening, considerably before strain localization (which occurs during strain softening, see Rudnicki and Rice, 1975) and before loss of second order work positive definiteness (which occurs at $h = 0$, see Fig. 2, where σ_{11} and ε_{11} are the infinitesimal measures of stress and strain in uniaxial compression).

Besides the case in which $\mathbf{Q}$ and $\mathbf{T}$ are not coaxial there is, however, another possibility for flutter. Under special circumstances, in fact, condition (65) can be satisfied even if $\mathbf{Q}$ *and* $\mathbf{T}$ *are coaxial.* To this purpose, let us assume the following representation of $\mathbf{Q}$:

$$\mathbf{Q} = \gamma \mathbf{I} + \varepsilon \mathbf{T}, \tag{68}$$

where γ and ε are generic isotropic scalar functions of the Cauchy stress $\mathbf{T}$. By substituting eqn (68) into condition (66), the following condition for flutter is obtained ($\lambda + \mu + \mathbf{n} \cdot \mathbf{T}\mathbf{n} > 0$ is assumed):

$$[\mathbf{T}\mathbf{n} \cdot \mathbf{T}\mathbf{n} - (\mathbf{n} \cdot \mathbf{T}\mathbf{n})^2]\left[-(\lambda + \mu)\frac{\varepsilon^2}{\gamma^2} + \frac{\varepsilon}{\gamma}\right] > 0. \tag{69}$$

Condition (69) is satisfied when $\varepsilon/\gamma \in (0, 1/(\lambda + \mu))$. The last condition can be satisfied when the yield function gradient tends to become isotropic and the stress tensor does not, i.e. when ε/γ approaches 0^+ or 0^- (with $\mathbf{n}$ not in an eigenspace of $\mathbf{T}$). This may occur, when the yield surface tends to become orthogonal to the hydrostatic axis. This circumstance occurs *independently of the propagation direction* $\mathbf{n}$. *Therefore, for special shapes of the yield surface and for certain stress states, flutter can occur for every direction of* $\mathbf{n}$ (except for $\mathbf{T}\mathbf{n} \times \mathbf{n} = \mathbf{0}$). This fact implies that in this case flutter instability *occurs before strain localization into planar bands.*

Moreover, flutter can also occur if two eigenvalues of $\mathbf{Q}$ tend to coalesce and this tendency is not followed by $\mathbf{T}$. This has been shown by Bigoni and Zaccaria (1994), by employing perturbations to $\mathbf{Q}$. Anyway, parallel argumentations can be used in the present context.

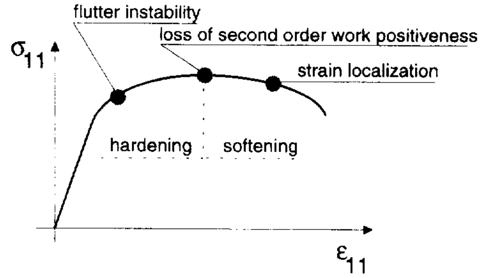

Fig. 2. Schematic representation of the occurrence of flutter instability, loss of second order work positive definiteness and strain localization. Reference is made to the uniaxial compression test for the J_2 flow theory (compressive strain and stress are taken as positive).

For the analyzed constitutive equation, flutter seems to be an important issue of the hypoelastic character of the unloading branch of the constitutive operator. In fact, it has been shown that flutter can always occur if the yield function gradient and the Cauchy stress are not coaxial, even in the case of associative flow rule. Finally, it may be important to note that we have focussed on the associative flow rule because we argue that non-associativity should make the material even more prone to flutter instability.

Second constitutive model

The first constitutive model was selected because it gives a simple example of unsymmetrical perturbation of the infinitesimal theory. That model is, however, not diffused in literature. A widely used model is that proposed by Rudnicki and Rice (1975):

$$\overset{\nabla}{\mathbf{T}} = \lambda \operatorname{tr} \mathbf{D} + 2\mu\mathbf{D} - \frac{1}{g}\langle \mathbf{D}\cdot\mathbf{Q}\rangle \mathbf{P}, \tag{70}$$

where $\overset{\nabla}{\mathbf{T}}$ denotes the Jaumann derivative of Cauchy stress. For this model, it was proved by An and Schaeffer (1992) that flutter instability may be caused, in plane strain, by noncoaxiality of yield function gradient and Cauchy stress. We show in the following that flutter may occur, for special shapes of the yield surface, even in the case of coaxiality. The constitutive equation (70) can be transformed in the form given by eqn (13) (the relative Lagrangean description is assumed):

$$\dot{\mathbf{S}} = [\lambda\mathbf{I}\otimes\mathbf{I} + 2\mu\mathscr{S} - (\mathbf{I}\boxtimes\mathbf{T} + \mathbf{T}\boxtimes\mathbf{I})\mathscr{S} + \mathbf{I}\boxtimes\mathbf{T} + \mathbf{T}\otimes\mathbf{I}][\mathbf{L}] - \frac{1}{g}\langle \mathbf{Q}\cdot\mathbf{D}\rangle\mathbf{P}. \tag{71}$$

The acoustic tensor for unloading or neutral loading is now given by:

$$\mathbf{A}_E(\mathbf{n}) = (\lambda+\mu)\mathbf{n}\otimes\mathbf{n} + \mu\mathbf{I} + \tfrac{1}{2}\{(\mathbf{n}\cdot\mathbf{T}\mathbf{n})\mathbf{I} + \mathbf{T}\mathbf{n}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{T}\mathbf{n} - \mathbf{T}\}, \tag{72}$$

and therefore flutter is excluded and non-propagation instability can occur for non small values of $\rho(\mathbf{T})/(\lambda+\mu)$ (see Section 3). The flutter analysis of model (71) will be restricted to the case in which tensor $\mathbf{T}$ is axially-symmetric, i.e.:

$$\mathbf{T} = (\tau_1 - \tau_2)\mathbf{t}\otimes\mathbf{t} + \tau_2\mathbf{I}, \tag{73}$$

where τ_i are the eigenvalues of $\mathbf{T}$, and $\mathbf{t}$ corresponds to the linear eigenspace of $\mathbf{T}$. Moreover, it is assumed that $\mathbf{Q}$ and $\mathbf{T}$ are coaxial and, in particular, that $\mathbf{Q}$ is an isotropic function of $\mathbf{T}$. Therefore:

$$\mathbf{q} = \xi_1\mathbf{n} + \xi_2(\mathbf{t}\cdot\mathbf{n})\mathbf{t}, \tag{74}$$

where ξ_1 and ξ_2 are generic isotropic scalar functions of the Cauchy stress $\mathbf{T}$. By using eqn (73), eqn (72) becomes:

$$\mathbf{A}_E(\mathbf{n}) = (\lambda+\mu)\mathbf{n}\otimes\mathbf{n} + \mu\mathbf{I} + \frac{\tau_1-\tau_2}{2}\{(\mathbf{t}\cdot\mathbf{n})^2\mathbf{I} + (\mathbf{t}\cdot\mathbf{n})(\mathbf{t}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{t}) - \mathbf{t}\otimes\mathbf{t}\}. \tag{75}$$

The eigenvalue analysis of tensor (75) gives (see Appendix B):

$$\begin{cases} \alpha_1 = \lambda + 2\mu, \\ \alpha_2 = \mu + (\tau_1-\tau_2)(\mathbf{t}\cdot\mathbf{n})^2 - \frac{1}{2}(\tau_1-\tau_2), \\ \alpha_3 = \mu + \frac{1}{2}(\tau_1-\tau_2)(\mathbf{t}\cdot\mathbf{n})^2 \end{cases} \tag{76}$$

Moreover, the following eigenvectors can be obtained by direct inspection of tensor (75):

$$\mathbf{a}^1 = \mathbf{n}, \quad \mathbf{a}_3 = \mathbf{a}^3 = \mathbf{n} \times \mathbf{t}. \tag{77}$$

From eqns (77) and (74) it is concluded that $\mathbf{q} \cdot \mathbf{a}^3 = 0$ and thus condition (51) is satisfied. In order to perform flutter analysis, the expression of $\mathbf{a}_1$ is needed. Such an expression is obtained in Appendix C in the form:

$$\mathbf{a}_1 = \mathbf{n} + \frac{(\tau_1 - \tau_2)}{\lambda + \mu^*}(\mathbf{t} \cdot \mathbf{n})[(\mathbf{t} \cdot \mathbf{n})\mathbf{n} - \mathbf{t}]. \tag{78}$$

where:

$$\mu^* = \mu - \frac{(\tau_1 - \tau_2)}{2}[2(\mathbf{t} \cdot \mathbf{n})^2 - 1]. \tag{79}$$

The necessary and sufficient conditions for flutter (57) become ($\alpha_1 - \alpha_2 > 0$ is assumed):

$$(\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) + \frac{(\tau_1 - \tau_2)}{\lambda + \bar{\mu}}(\mathbf{t} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n})[(\mathbf{t} \cdot \mathbf{n})(\mathbf{p} \cdot \mathbf{n}) - \mathbf{p} \cdot \mathbf{t}] > 0,$$

$$(\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) - \mathbf{p} \cdot \mathbf{q} + \frac{(\tau_1 - \tau_2)}{\lambda + \bar{\mu}}(\mathbf{t} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n})[(\mathbf{t} \cdot \mathbf{n})(\mathbf{p} \cdot \mathbf{n}) - \mathbf{p} \cdot \mathbf{t}] > 0. \tag{80}$$

The geometrical terms in conditions (80) are generally small with respect to the others and can be considered as a perturbation. In the particular case of *associative flow-law*, $\mathbf{p} = \mathbf{q}$, conditions (80) can be satisfied for $\mathbf{n}$ sufficiently close to an eigenvector of $\mathbf{Q}$. In fact, for these vectors $\mathbf{n}$, condition (80_1) is satisfied. Moreover, it is possible to transform condition (80_2), using eqn (74), in the form:

$$-(\mathbf{t} \cdot \mathbf{n})^2[1 - (\mathbf{t} \cdot \mathbf{n})^2]\left\{\frac{\xi_2^2}{\xi_1^2} + \frac{(\tau_1 - \tau_2)}{\lambda + \bar{\mu}}\left[\frac{\xi_2}{\xi_1} + \frac{\xi_2^2}{\xi_1^2}(\mathbf{t} \cdot \mathbf{n})^2\right]\right\} > 0. \tag{81}$$

Inequality (81) can be satisfied when ξ_2/ξ_1 approaches 0. This last possibility is analogous to that obtained for the previous constitutive model when $\mathbf{Q}$ tends to an isotropic tensor and $\mathbf{T}$ does not. This may be the case of a cap, a tension cut off or, in plane stress, the case of a yield surface which smoothly approaches the Hill criterion. An and Schaeffer (1992) have analyzed a model of the type given by eqn (70) in the case of a J_2-type, two-dimensional theory of plasticity. In that case, the circumstance that $\mathbf{Q}$ tends to become isotropic and $\mathbf{T}$ does not, is *a priori* excluded. Therefore, the present results are not in contrast with those of An and Schaeffer (1992).

6. ON "WAVE NON-PROPAGATION" INSTABILITY IN ELASTOPLASTICITY

Let us start by analyzing non-propagation instability for the loading branch of elastoplastic constitutive tensor at small strains. In this case, the acoustic tensor is:

$$\mathbf{A}(\mathbf{n}) = (\lambda + \mu)\mathbf{n} \otimes \mathbf{n} + \mu\mathbf{I} - \frac{1}{g}\mathbf{p} \otimes \mathbf{q}, \tag{82}$$

and the conditions for flutter are (Loret et al. 1990):

$$\begin{cases}(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) > 0 \\ (\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) - \mathbf{p}\cdot\mathbf{q} > 0,\end{cases} \tag{83}$$

with the plastic modulus belonging to the interval:

$$\left.\begin{matrix}g_1\\ g_2\end{matrix}\right\} = \frac{1}{\lambda+\mu}\left[\sqrt{(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})} \pm \sqrt{(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) - \mathbf{p}\cdot\mathbf{q}}\right]^2. \tag{84}$$

Let us examine wave non-propagation instability. The case when $\mathbf{n}$ is an eigenvector of both $\mathbf{P}$ and $\mathbf{Q}$ will be not considered: in fact $\mathbf{A}(\mathbf{n})$ becomes in this case symmetric. Moreover, it should be noted that the basis (48) can always be chosen to make $\mathbf{p}\cdot\mathbf{a}_3 = 0$. Therefore, if the acoustic tensor (82) has two eigenvalues equal to μ, non-propagation instability does not occur. Finally, when $\mathbf{n}$ is not an eigenvector of $\mathbf{Q}$, non-propagation instability occurs if matrix

$$\begin{bmatrix} \lambda+2\mu & -\dfrac{1}{g}\mathbf{p}\cdot\mathbf{n} \\ (\lambda+\mu)\mathbf{q}\cdot\mathbf{n} & \mu-\dfrac{1}{g}\mathbf{p}\cdot\mathbf{q} \end{bmatrix}, \tag{85}$$

is not diagonal and has two equal eigenvalues. If $\mathbf{n}$ is an eigenvector of $\mathbf{Q}$ (but not of $\mathbf{P}$), non-propagation instability occurs if matrix

$$\begin{bmatrix} \lambda+2\mu-\dfrac{1}{g}\mathbf{p}\cdot\mathbf{q} & 0 \\ -\dfrac{1}{g}(\mathbf{q}\cdot\mathbf{n})\mathbf{p}\cdot\dfrac{\mathbf{n}\times(\mathbf{n}\times\mathbf{p})}{|\mathbf{n}\times(\mathbf{n}\times\mathbf{p})|} & \mu \end{bmatrix}, \tag{86}$$

is not diagonal and has two equal eigenvalues. The coincidence of the eigenvalues of matrices (85) or (86) occurs when one of left hand sides of conditions (83) becomes zero. Using this condition, the critical values of the plastic modulus may be obtained from eqn (84). Therefore necessary and sufficient conditions for non-propagation instability are:

$$\begin{aligned} &\begin{pmatrix}(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) = \mathbf{p}\cdot\mathbf{q} \text{ and } g = \mathbf{p}\cdot\mathbf{q}/(\lambda+\mu) \text{ or} \\ (\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) = 0 \text{ and } g = -\mathbf{p}\cdot\mathbf{q}/(\lambda+\mu)\end{pmatrix} \text{ and } (\mathbf{p}\cdot\mathbf{n} \neq 0 \text{ or } \mathbf{q}\cdot\mathbf{n} \neq 0), && \text{if } \mathbf{n}\times\mathbf{q} \neq 0, \\ &g = \mathbf{p}\cdot\mathbf{q}/(\lambda+\mu) \text{ and } (\mathbf{p}\times\mathbf{n}) \neq 0. && \text{if } \mathbf{n}\times\mathbf{q} = 0. \end{aligned} \tag{87}$$

From condition (87) we can note that, for nonassociative plasticity when $\mathbf{Q}$ has at least an eigenvector which is not an eigenvector of $\mathbf{P}$, non-propagation instability occurs at the critical value of plastic modulus $g_{cr} = \mathbf{p}\cdot\mathbf{q}/(\lambda+\mu)$. This occurs when $\mathbf{P}$ and $\mathbf{Q}$ are not coaxial, but may occur even in the case of coaxiality. This is once more the case in which only one of tensors $\mathbf{P}$ and $\mathbf{Q}$ is an isotropic tensor. For instance, let us introduce tensors $\mathbf{P}$ and $\mathbf{Q}$ in the following way:

$$\mathbf{Q} = \zeta_1 \operatorname{dev}\mathbf{T} + \zeta_2\mathbf{I}, \quad \mathbf{P} = \zeta_3 \operatorname{dev}\mathbf{T} + \zeta_4\mathbf{I}, \tag{88}$$

where the ζ_is are scalar parameters, possibly depending on the invariants of $\mathbf{T}$. In these conditions flutter is *a priori* excluded when $\zeta_1\zeta_3 \geqslant 0$. However, if $\zeta_1 = 0$ and $\zeta_3 \neq 0$, at the critical value of the plastic modulus $g_{cr} = \zeta_2(\mathbf{n}\cdot\mathbf{Pn})/(\lambda+\mu)$, the acoustic tensor becomes:

$$\mathbf{A}(\mathbf{n}) = (\lambda+\mu)\mathbf{n}\otimes\mathbf{n}+\mu\mathbf{I}-\frac{\lambda+\mu}{\mathbf{n}\cdot\mathbf{P}\mathbf{n}}\mathbf{P}\mathbf{n}\otimes\mathbf{n}, \tag{89}$$

which has, for any $\mathbf{n}$ (not in an eigenspace of $\mathbf{P}$), one eigenvalue equal to μ with algebraic multiplicity 3 and geometric multiplicity 2. The fact that the non-propagation instability occurs at g_{cr} for every $\mathbf{n}$ (except those $\mathbf{n}$ for which $\mathbf{P}\mathbf{n}\times\mathbf{n}=0$) implies that this instability occurs before strain localization into planar bands. Therefore, the example shows that *non-propagation instability may occur before strain localization.* Moreover, the example confirms what we observed in the case of grade 1 hypoelasticity (Remark 2 of Section 3): *the non-propagation of the acceleration waves may become possible even if flutter is excluded.*

In the case of deviatoric associativity, condition $(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n}) = \mathbf{p}\cdot\mathbf{q}$ can be verified if and only if $\mathbf{n}$ is an eigenvector of both $\mathbf{P}$ and $\mathbf{Q}$. Moreover, if $\mathbf{p}\cdot\mathbf{n}=0$ (or $\mathbf{q}\cdot\mathbf{n}=0$), it is $\mathbf{p}\cdot\mathbf{q}\geqslant 0$, so that $g_{cr}\leqslant 0$. Thus, *in the case of the infinitesimal theory, non propagation instability is excluded for deviatoric associativity and positive values of the plastic modulus.* However, non-propagation instability becomes possible for deviatoric associativity if unsymmetric geometrical terms are taken into account in the constitutive law. This is the case of the two models presented in the previous section, even though we restrict ourselves to the associative flow-rule.

In the case of the first model, basis (48) can always be chosen to make $\mathbf{p}\cdot\mathbf{a}_3=0$. Therefore, as in the case of the infinitesimal theory, for $\mathbf{n}\times\mathbf{q}\neq 0$, non-propagation instability can occur when the left hand side of inequality (64_1), or inequality (64_2), becomes zero, at one of the two critical values of plastic modulus which may be obtained from eqn (58), by imposing $g_1 = g_2$. Let us analyze the case $\mathbf{n}\times\mathbf{q}=0$, for associative plasticity. In this case, $\mathbf{n}$ coincides with an eigenvector of $\mathbf{Q}$, the left hand side of inequality (64_2) vanishes and, therefore, at least two eigenvalues of the acoustic tensor coincide. In this condition, if $\mathbf{Q}$ has an eigenvector not in an eigenspace of $\mathbf{T}$ (as in the non-coaxial case), non-propagation instability may occur. In fact, the acoustic tensor, at $g_{cr} = \mathbf{q}\cdot\mathbf{q}/(\lambda+\mu+\mathbf{n}\cdot\mathbf{T}\mathbf{n})$, can be written in the form:

$$\mathbf{A}(\mathbf{n}) = -(\mathbf{n}\cdot\mathbf{T}\mathbf{n})\mathbf{n}\otimes\mathbf{n}+\mathbf{T}\mathbf{n}\otimes\mathbf{n}+(\mu+\mathbf{n}\cdot\mathbf{T}\mathbf{n})\mathbf{I}, \tag{90}$$

which has the eigenvalue $\mu+\mathbf{n}\cdot\mathbf{T}\mathbf{n}$ with algebraic multiplicity 3 and geometric multiplicity equal to 2 (unless $\mathbf{n}$ is in an eigenspace of $\mathbf{T}$). Therefore, when $\mathbf{T}$ and $\mathbf{Q}$ are not coaxial, wave non-propagation instability occurs in the model given by eqn (60) with associative flow rule, at $g_{cr} = \mathbf{q}\cdot\mathbf{q}/(\lambda+\mu+\mathbf{n}\cdot\mathbf{T}\mathbf{n})$, with $\mathbf{n}$ in an eigenspace of $\mathbf{Q}$.

In the case of the Rudnicki and Rice constitutive model [eqn (70)], a complete analysis of non-propagation instability may be complicated. Anyway, it is easy to show that this instability can occur. For nonassociative elastoplasticity, let $\mathbf{n}$ be in an eigenspace of both $\mathbf{P}$ and $\mathbf{Q}$, not coincident with an eigenspace of $\mathbf{T}$. The acoustic tensor can thus be written as ($\mathbf{Q}\mathbf{n} = q\mathbf{n}$ and $\mathbf{P}\mathbf{n} = p\mathbf{n}$)

$$\mathbf{A}_{\mathrm{E}}(\mathbf{n}) = \left(\lambda+\mu-\frac{pq}{g}\right)\mathbf{n}\otimes\mathbf{n}+\left(\mu+\frac{\mathbf{n}\cdot\mathbf{T}\mathbf{n}}{2}\right)\mathbf{I}+\tfrac{1}{2}\{\mathbf{T}\mathbf{n}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{T}\mathbf{n}-\mathbf{T}\}, \tag{91}$$

i.e. in the form (11) with $\lambda+\mu$ replaced by $\lambda+\mu-pq/g$, and where $\bar{\mu} = \mu+\mathbf{n}\cdot\mathbf{T}\mathbf{n}/2$, and $\gamma_1=\gamma_3=-\gamma_2=-1/2$. In these conditions flutter instability is excluded by Remark 4, as well as geometric multiplicity one for a triple eigenvalue (Remark 5)†. However, the critical plastic modulus for non-propagation instability, which corresponds to a double eigenvalue with single geometric multiplicity, can be directly obtained from representation (16), resulting in

† These instabilities could obviously occur for $\mathbf{n}$ in the neighbourhood of the eigenvectors of $\mathbf{P}$ and $\mathbf{Q}$.

$$\frac{1}{g_{cr}} = \frac{1}{pq}\left[\lambda+\mu-\frac{\mathbf{n}\cdot\mathbf{Tn}}{2}+\frac{\mathbf{l}\cdot\mathbf{Tl}+\mathbf{m}\cdot\mathbf{Tm}}{4}\pm\tfrac{1}{2}\sqrt{(\mathbf{l}\cdot\mathbf{Tl}-\mathbf{m}\cdot\mathbf{Tm})^2/4+(\mathbf{l}\cdot\mathbf{Tm})^2}\right]. \quad (92)$$

When $g = g_{cr}$ and excluding the very particular case‡

$$\mathbf{l}\cdot\mathbf{Tm} = \lambda+\mu-\frac{pq}{g}-\frac{\mathbf{n}\cdot\mathbf{Tn}+\mathbf{m}\cdot\mathbf{Tm}}{2} = 0 \quad \text{and} \quad \mathbf{m}\cdot\mathbf{Tm} \neq \mathbf{l}\cdot\mathbf{Tl},$$

non-propagation instability occurs if $\mathbf{n}\cdot\mathbf{Tl} \neq 0$ (see (15)), i.e. if $\mathbf{n}$ does not belong to an eigenspace of $\mathbf{T}$. Note that the term $\lambda+\mu$ is usually preponderant with respect to the others in brackets, thus the critical plastic modulus is positive for $pq > 0$, as in the case of associative flow rule.

We can conclude from the above reported examples that, similarly to flutter, non-propagation instability may be triggered by small geometrical terms added to the infinitesimal elastoplasticity. This is however not the case of infinitesimal elasticity.

In conclusion of this section, we will suggest the use of a general necessary and sufficient condition for wave non-propagation instability. In the case of one eigenvalue with algebraic multiplicity equal to three, non-propagation instability is possible if the acoustic tensor is not isotropic. In the case of a double eigenvalue, say, $\eta_I = \eta_{II} \neq \eta_{III}$, it can be noted that an endomorphism has a diagonal matrix if and only if its minimal polynomial can be factored into distinct factors all of the first degree (Bowen and Wang, 1976, Theorem 30.8). An immediate consequence is that the algebraic and the geometric multiplicities of $\eta_I = \eta_{II}$ coincide if and only if

$$[\mathbf{A}(\mathbf{n})-\eta_I\mathbf{I}][\mathbf{A}(\mathbf{n})-\eta_{III}\mathbf{I}] = 0, \quad (93)$$

which can be rewritten in the useful form

$$\mathbf{A}^2(\mathbf{n})-(\eta_I+\eta_{III})\mathbf{A}(\mathbf{n})+\eta_I\eta_{III}\mathbf{I} = 0. \quad (94)$$

Condition (93) or (94) is a necessary and sufficient condition to exclude non-propagation instabilities.

7. CONCLUSIONS

The loading and unloading branches of two elastoplastic constitutive models for finite strains have been analyzed from the viewpoint of flutter instability. A related wave non-propagation condition has been also analyzed, which corresponds to the occurrence of different geometric and algebraic multiplicity in the eigenvalues of the acoustic tensor.

Results of this study suggest that the occurrence of equal wave speeds may be viewed as a type of material instability in infinitesimal elastoplasticity and may not be considered an instability in other contexts (in particular we refer to infinitesimal, three-dimensional, isotropic elasticity). To this purpose we have defined types of perturbation to both infinitesimal elastoplasticity and isotropic elasticity. These perturbations have a clear physical meaning; namely, they may be identified with the so called *geometrical* (or *corotational*) terms in finite strains theories of elastoplasticity. Therefore, the considered perturbations can represent the effects of large deformations. These perturbations have been shown to have no effect on infinitesimal, three-dimensional isotropic theory of elasticity. In contrast, the same perturbations may yield flutter and non-propagation of wave modes in the loading branch of infinitesimal elastoplasticity.

As a final remark, it can be noted that the analyzed instabilities can occur for high values of plastic modulus, i.e. in the initial part of the plastic process. The instabilities can therefore occur prior to loss of second order work positive definiteness and loss of ellipticity.

‡ In this case, the algebraic and geometric multiplicity of the double eigenvalue of tensor (90) coincide.

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APPENDIX A

Vector $\mathbf{q}$ *is a left eigenvalue of* $\mathbf{A}_E(\mathbf{n})$ *in equation* (46)

It is assumed that $\mathbf{a}_1 \times \mathbf{q} = 0$. Projecting the eigenvalue problem (46) onto the dual bases $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ (note that the modulus of left and right eigenvectors $\mathbf{a}_i$ and $\mathbf{a}^j$ of $\mathbf{A}_E(\mathbf{n})$ will be taken in a proper way to satisfy: $\mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j$) the characteristic equation is readily obtained

$$\det \begin{bmatrix} \alpha_1 - \frac{1}{g}\mathbf{p}\cdot\mathbf{q} - \eta & 0 & 0 \\ -\frac{1}{g}(\mathbf{p}\cdot\mathbf{a}_2)(\mathbf{q}\cdot\mathbf{a}^1) & \alpha_2 - \eta & 0 \\ -\frac{1}{g}(\mathbf{p}\cdot\mathbf{a}_3)(\mathbf{q}\cdot\mathbf{a}^1) & 0 & \alpha_3 - \eta \end{bmatrix} = 0. \tag{A.1}$$

From (A.1) it can be concluded that flutter is not possible when $\mathbf{q}$ is a left eigenvector of $\mathbf{A}_E(\mathbf{n})$. Non-propagation instability can occur if $\mathbf{p}$ is not parallel to $\mathbf{a}_1$ and when the plastic modulus equals one of the two critical plastic moduli:

$$g_{cr} = \frac{\mathbf{p}\cdot\mathbf{q}}{\alpha_1 - \alpha_2}, \quad g_{cr} = \frac{\mathbf{p}\cdot\mathbf{q}}{\alpha_1 - \alpha_3}. \tag{A.2}$$

APPENDIX B

Determination of eigenvalues of the acoustic tensor (75)

The eigenvalues α_1 and α_3 can be obtained from tensor (75) by direct inspection. The eigenvalue α_2 can be obtained as follows. Let us search for the eigenvector $\mathbf{a}^2$ in the plane containing vectors $\mathbf{n}$ and $\mathbf{t}$; therefore:

$$\mathbf{a}^2 = \beta_1\mathbf{n} + \beta_2\mathbf{t}, \tag{B.1}$$

where coefficients β_i are the components of $\mathbf{a}^2$. By definition of the right eigenvector:

$$\mathbf{A}_E(\mathbf{n})(\beta_1\mathbf{n} + \beta_2\mathbf{t}) = \alpha_2(\beta_1\mathbf{n} + \beta_2\mathbf{t}). \tag{B.2}$$

By using the definition (75) of $\mathbf{A}_E(\mathbf{n})$ in eqn (B.2), the eigenvalue and the components of the eigenvector are obtained:

$$\alpha_2 = \mu + (\tau_1 - \tau_2)(\mathbf{t}\cdot\mathbf{n})^2 - \tfrac{1}{2}(\tau_1 - \tau_2), \tag{B.3}$$

$$\beta_1\{\lambda + \mu - (\tau_1 - \tau_2)[(\mathbf{t}\cdot\mathbf{n})^2 - \tfrac{1}{2}]\} = -\beta_2[\lambda + \mu - \tfrac{1}{2}(\tau_1 - \tau_2)](\mathbf{t}\cdot\mathbf{n}). \tag{B.4}$$

APPENDIX C

Determination of the left eigenvector (78)

The left eigenvector (78) is associated with the eigenvalue $\lambda + 2\mu$. Hence

$$\mathbf{A}_E^T\mathbf{a}_1 = (\lambda + 2\mu)\mathbf{a}_1. \tag{C.1}$$

For $\mathbf{t} \neq \mathbf{n}$, it is possible to represent $\mathbf{a}_1$ in the basis formed by the three orthogonal unit vectors

$$\mathbf{n}, \mathbf{l} = \frac{\mathbf{n}\times\mathbf{t}}{|\mathbf{n}\times\mathbf{t}|} \quad \text{and} \quad \mathbf{m} = \frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{t})}{|\mathbf{n}\times(\mathbf{n}\times\mathbf{t})|}, \quad \text{as:} \quad \mathbf{a}_1 = \beta_1\mathbf{n} + \beta_2\mathbf{l} + \beta_3\mathbf{m}, \tag{C.2}$$

where the β_i are the relevant (unknown) components. In order to satisfy the condition $\mathbf{a}_1 \cdot \mathbf{a}^1 = 1$, $\beta_1 = 1$ is selected. Moreover, a substitution of eqns (75) and (C.2) into eqn (C.1) yields a vectorial equation which may be projected onto the basis **n**, **l** and **m**. This projection gives two scalar equations in the unknowns β_2 and β_3. The solution to these equations is:

$$\beta_2 = 0, \quad \beta_3 = -\frac{(\tau_1 - \tau_2)}{\lambda + \mu^*}(\mathbf{t}\cdot\mathbf{n})(\mathbf{t}\cdot\mathbf{m}). \tag{C.3}$$

where μ^* is given by eqn (79). Keeping the following property into account:

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{t}) = (\mathbf{t}\cdot\mathbf{n})\mathbf{n} - (\mathbf{n}\cdot\mathbf{n})\mathbf{t}, \tag{C.4}$$

the eigenvector $\mathbf{a}_1$ is obtained, in the form (78).