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ASYMPTOTIC MODELS OF DILUTE COMPOSITES WITH IMPERFECTLY BONDED INCLUSIONS

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Abstract—The asymptotic scheme for the analysis of dilute elastic composites, which includes circular inclusions with imperfect bonding at the interface is presented. Interface is characterized by a discontinuous displacement field across it, linearly related to the tractions. The problem of a linear-elastic, circular inclusion with generic loading condition at infinity is solved, and used to analyze effective elastic moduli of composite materials. Effects due to the interaction of a small circular defect and a crack are investigated. It is shown that interfacial stiffness has a strong effect on the crack path and therefore may be an important design parameter for composites. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The description of the mechanical behaviour of fibrous and particulate-reinforced materials is crucial for design purposes. In these materials, the inclusions may often be imperfectly bonded to the matrix, owing to several reasons. In some cases, a thin layer, called interphase, is introduced to improve the performance of the composite. In other cases, the interphase may be the product of chemical interaction between phases or localized mechanical damage of one or both phases (see the detailed discussion by Aboudi, 1991). It is obvious that such interface conditions have a strong effect on the mechanical behaviour of the composite. In the last few years, strong research efforts have been devoted to analyze interfaces from the mechanical point of view. In particular, many different models of interfacial behaviour can be defined. The simplest of these models is the linear interface, in which a linear relationship holds at the interface between the traction vector and the displacement jump. This interfacial constitutive law has been formulated by Jones and Whittier (1967) and may be viewed as a simplification of the behaviour of a thin, soft elastic layer (Goland and Reissner, 1944; a rigorous proof of this in terms of asymptotic analysis has been found by Klarbring, 1991 and Geymonat and Krasuki, 1996 and is also given here in a different way). The model of linear interface has been employed by Mal and Bose (1974), Hashin (1990, 1991, 1992), Levy (1991), Qu (1993), Lipton and Vernescu (1995), and Lipton (1997) in homogenization problems, by Suo *et al.* (1992) and Bigoni *et al.* (1997) in determining the bifurcation loads of layered elastic structures subject to large strain, and by Tullini *et al.* (1997) in the Saint Venant analysis of layered elastic plates.

A special mention is deserved for the recent work of Gao (1995) who has solved the circular inclusion problem with linear interface subject to homogeneous remote strain. In the present article, this result is obtained with a different technique (the Kolosov–Muskhelishvili complex potentials) and generalized to generic conditions at infinity. In particular, the plane elastic problem of a circular inclusion connected with a linear interface to an infinite medium is considered. A generic loading at infinity is assumed, consisting in a displacement field representable as an analytic function of the position. This is a generalization of the Eshelby (1957) condition, where a homogeneous deformation is prescribed at infinity. The importance of this generalization, employed also in (Sendeckyj, 1970; Yu and Sendeckyj, 1974 and Gong and Meguid, 1993), may be related to the possibility of analyzing complex loading situations, such as the case of a bending stress field. As emphasized in (Achenbach and Zhu, 1989; Gao, 1995), the model of linear interface may predict (depending on the loading conditions, material and interfacial stiffness parameters) an unphysical overlapping between the two media in contact. This problem is not addressed here, except that we present the asymptotic derivation of the linear interface model, represented as an elastic layer of small stiffness and thickness. This analysis may shed light on the limits of the model.

In closure of this paper, the solution relative to the imperfectly bonded interface is employed in two problems, namely, determination of crack trajectory in a brittle elastic matrix with defects and determination of overall properties of a dilute-composite material. The latter problem may be analyzed in different ways: for instance using the variational technique of Hashin and Shtrikman (1963). Alternatively, following the approach of Movchan and Serkov (1997), the matrix of effective elastic moduli of dilute, periodic composites $\mathscr{H}^*$ may be written in the form

$$\mathscr{H}^* \sim \mathscr{H}^0 + \frac{f}{\pi}\mathscr{P},$$

where $\mathscr{H}^0$ corresponds to the matrix material, f is the volume fraction of inclusions and $\mathscr{P}$ is the Pólya–Szegö matrix, which depends on the elastic constants of matrix and inclusion, on the stiffness coefficients of the interface and on the radius of inclusion. Obviously, when $\mathscr{P}$ is zero, the effective elastic moduli coincide with elastic moduli of the matrix. One of the interesting results of this paper is that the Pólya–Szegö matrix may in fact become identically zero for inclusion stiffer than the matrix and an appropriate choice of the interface stiffness parameters. A similar result was obtained independently by Lipton and Vernescu (1995) for all volume-fractions, but for special interfacial and loading conditions. This and other results presented in this article show the strong effect of interfacial compliance. For instance, in the analysis of crack inclusion interaction it can be pointed out that interface stiffness practically "controls" the shape of crack trajectory, particularly for inclusions stiffer than the matrix. Therefore, when the interfacial mechanical characteristics can be used as design parameters for a composite, results of this paper can be applied to provide a best estimate for overall toughness of the material.

2. ASYMPTOTIC DERIVATION OF CONSTITUTIVE EQUATIONS OF LINEAR INTERFACE

In this Section an asymptotic expansion is presented for the solution of an isotropic–elastic circular inclusion in an isotropic–elastic plane, coated by a cylindrically–anisotropic, finite-thickness elastic layer. This asymptotic procedure yields the stiffness constants of the linear interface model, that will be used in the following sections. This model is here obtained as the behaviour of a thin, soft, elastic layer in the limit when the layer thickness and stiffness tend to zero. This limit can also be derived through a mixed variational principle in the case of isotropic interphase (Klarbring, 1991) and using the Γ-convergence technique for anisotropic interphase (Geymonat and Krasucki, 1996; see also Acerbi and Buttazzo, 1986a, b). We use the following notation for the geometry of the domain: Ξ_1 is the circular inclusion, Ξ_2 the interphase, and Ξ_3 the remaining part of infinite domain. The inclusion and the infinite plane are isotropic elastic solids, respectively characterised by the Lamé constants μ_1, λ_1, and μ_3, λ_3. The interface is assumed, for generality, to be cylindrically anisotropic (in the plane) and therefore characterized by the constitutive equation

$$\boldsymbol{\sigma} = \mathscr{H}\boldsymbol{\varepsilon},$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are vectors collecting the components of stress and strain tensors, i.e.

$$\boldsymbol{\sigma} = (\sigma_{rr}, \sigma_{\theta\theta}, \sqrt{2}\sigma_{r\theta})', \quad \boldsymbol{\varepsilon} = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \sqrt{2}\varepsilon_{r\theta})',$$

and

$$\mathscr{H} = \begin{pmatrix} C_{rr} & C_{r\theta} & 0 \\ C_{\theta r} & C_{\theta\theta} & 0 \\ 0 & 0 & 2G_{\theta r} \end{pmatrix}, \tag{1}$$

which depends on four stiffness coefficients (in the particular case of isotropic elasticity $C_{rr} = \lambda + 2\mu$, $C_{r\theta} = C_{\theta r} = \lambda$ and $G_{\theta r} = \mu$). Therefore, the elasticity problem in compound domain can be formulated as

$$\begin{aligned} L_{r\theta}(\mathbf{u}^{(1)}, \mu_1, \lambda_1) &= \mathbf{0}, \quad \mathbf{x} \in \Xi_1, \\ L_{r\theta}(\mathbf{u}^{(2)}, C_{ij}, G_{ij}) &= \mathbf{0}, \quad \mathbf{x} \in \Xi_2, \\ L_{r\theta}(\mathbf{u}^{(3)}, \mu_3, \lambda_3) &= \mathbf{0}, \quad \mathbf{x} \in \Xi_3, \end{aligned} \tag{2}$$

where $L_{r\theta}(\cdot)$ is the Navier operator written in the polar coordinate system. In matrix form it can be represented as

$$L_{r\theta}(\cdot) := \mathbf{D}_1 \mathscr{H} \mathbf{D}_2'(\cdot),$$

$$\mathbf{D}_1 = \begin{bmatrix} \dfrac{\partial}{\partial r} + \dfrac{1}{r} & -\dfrac{1}{r} & \dfrac{1}{\sqrt{2}} \dfrac{1}{r} \dfrac{\partial}{\partial \theta} \\ 0 & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial r} + \dfrac{2}{r}\right) \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} \dfrac{\partial}{\partial r} & \dfrac{1}{r} & \dfrac{1}{\sqrt{2}} \dfrac{1}{r} \dfrac{\partial}{\partial \theta} \\ 0 & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial r} - \dfrac{1}{r}\right) \end{bmatrix},$$

where $\mathscr{H}$ is given by eqn (1) for the interphase, whereas for the matrix and inclusion

$$\mathscr{H} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}.$$

The conditions at the outer and inner boundaries of the interphase correspond to a perfectly bonded interface:

$$\begin{aligned} \boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}^{(1)}) &= \boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}^{(2)}), \quad \mathbf{u}^{(1)} = \mathbf{u}^{(2)}, \quad \text{as } r = R \\ \boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}^{(3)}) &= \boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}^{(2)}), \quad \mathbf{u}^{(3)} = \mathbf{u}^{(2)}, \quad \text{as } r = R + \varepsilon, \end{aligned} \tag{3}$$

where ε is the thickness of the interphase, $\boldsymbol{\sigma}^{(\mathbf{n})}$ is the traction vector relative to the radial unit vector $\mathbf{n}$.

Solution of the above formulated problem is sought in the form of the asymptotic series:

$$\mathbf{u}^{(1)} = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}_i^{(1)}, \quad \mathbf{u}^{(2)} = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}_i^{(2)}, \quad \mathbf{u}^{(3)} = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}_i^{(3)}.$$

Let us introduce the scaled ("fast") variable ρ defined as

$$\rho = (r-R)\varepsilon^{-1},$$

where R is the radius of the inclusion, and let us apply this variable in the interphase (region 2) only. Then the equilibrium equations inside this region can be expanded via different powers of ε with the leading term

$$\frac{1}{\varepsilon^2}\begin{bmatrix} C_{rr}\dfrac{\partial^2 u_{r,0}^{(2)}}{\partial\rho^2} \\ G_{\theta r}\dfrac{\partial^2 u_{\theta,0}^{(2)}}{\partial\rho^2}\end{bmatrix} = 0. \tag{4}$$

The solution of system (4) can be found in the form

$$\begin{pmatrix} u_{r,0}^{(2)} \\ u_{\theta,0}^{(2)}\end{pmatrix} = \begin{pmatrix} A(\theta)\rho + B(\theta) \\ C(\theta)\rho + D(\theta)\end{pmatrix}. \tag{5}$$

The radial traction vector in any point of the interphase can therefore be rewritten, taking into account only the leading term, in the form:

$$\boldsymbol{\sigma}^{(\mathrm{n})}(\mathbf{u}_0^{(2)}) = \frac{1}{\varepsilon}\begin{pmatrix} C_{rr}A(\theta) \\ G_{\theta r}C(\theta)\end{pmatrix}. \tag{6}$$

From eqns (3) and (6) we immediately deduce that the leading terms of tractions at the outer and inner boundary of the interphase coincide:

$$\boldsymbol{\sigma}^{(\mathrm{n})}(\mathbf{u}_0^{(1)})\,|_{r=R} = \boldsymbol{\sigma}^{(\mathrm{n})}(\mathbf{u}_0^{(2)}) = \boldsymbol{\sigma}^{(\mathrm{n})}(\mathbf{u}_0^{(3)})\,|_{r=R+\varepsilon}. \tag{7}$$

Let us now analyze the condition of displacement continuity. Following eqn (5), the displacement jump between the outer boundary ($\rho = 1$) and inner boundary ($\rho = 0$) is specified by the functions $A(\theta)$ and $C(\theta)$. Employing eqn (6) we obtain

$$u_{r,0}^{(3)}\,|_{r=R+\varepsilon} - u_{r,0}^{(1)}\,|_{r=R} = \frac{1}{C_{rr}^*}\sigma_{rr}(u_0^{(1)})\Big|_{r=R}, \tag{8}$$

$$u_{\theta,0}^{(3)}\,|_{r=R+\varepsilon} - u_{\theta,0}^{(1)}\,|_{r=R} = \frac{1}{G_{\theta r}^*}\sigma_{r\theta}(u_0^{(1)})\Big|_{r=R}, \tag{9}$$

where we have assumed that the elasticity coefficients of the interphase go to zero as

$$C_{rr} = \varepsilon C_{rr}^*, \quad G_{\theta r} = \varepsilon G_{\theta r}^*. \tag{10}$$

Now we can expand first terms in eqns (8) and (9) in Taylor series (here we extend the functions $u_r^{(3)}$, $u_\theta^{(3)}$ smoothly in the thin interphase):

$$u_{(\cdot),0}^{(3)}\,|_{r=R+\varepsilon} = u_{(\cdot),0}^{(3)}\,|_{r=R} + \varepsilon\frac{\partial u_{(\cdot),0}^{(3)}}{\partial r}\Big|_{r=R} + O(\varepsilon^2). \tag{11}$$

Substituting the expansion (11) into (8) and (9), and considering only the coefficients of zero order in ε we obtain the interface conditions for a linear interface (viewed as a zero-thickness interphase)

$$u_{r,0}^{(3)}\,|_{r=R}-u_{r,0}^{(1)}\,|_{r=R}=\frac{1}{C_{rr}^{*}}\sigma_{rr}(u_0^{(1)})\Big|_{r=R}, \tag{12}$$

$$u_{\theta,0}^{(3)}\,|_{r=R}-u_{\theta,0}^{(1)}\,|_{r=R}=\frac{1}{G_{\theta r}^{*}}\sigma_{r\theta}(u_0^{(1)})\Big|_{r=R}. \tag{13}$$

It follows immediately from expressions (7), (12) and (13) that across the linear interface the tractions are continuous, but displacement jumps are possible. These jumps are related to the tractions at the interface with the stiffness coefficients

$$s_r=\frac{C_{rr}}{\varepsilon},\quad s_\theta=\frac{G_{\theta r}}{\varepsilon}. \tag{14}$$

If we assume instead of eqn (10) that the elasticity coefficients of the interphase go to zero as

$$C_{rr}=\varepsilon^{1-\beta}C_{rr}^{*},\quad G_{\theta r}=\varepsilon^{1-\beta}G_{\theta r}^{*},\quad 0<\beta<1, \tag{15}$$

the perfectly bonded interface conditions are obtained:

$$\boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}^{(1)})\,|_{r=R}=\boldsymbol{\sigma}^{(\mathbf{n})}(\mathbf{u}^{(3)})|_{r=R},\quad \mathbf{u}^{(1)}|_{r=R}=\mathbf{u}^{(3)}|_{r=R}\quad \text{as } \varepsilon\to 0. \tag{16}$$

In closure of this Section, we note that a counterpart to the mathematical simplicity of the linear interface model is the possibility of unphysical overlapping between matrix and inclusion. This issue may be interesting, but is not addressed here (see, e.g. Achenbach and Zhu, 1989; Gao, 1995).

3. CIRCULAR INCLUSION WITH IMPERFECT BONDING: LOADING AT INFINITY

An isotropic, linearly-elastic infinite plane is considered, which contains a circular, isotropic linearly-elastic inclusion of radius R. Quantities related to the plane and to the inclusion are denoted with indices + and −, respectively. According to this convention, λ_+, μ_+ and λ_-, μ_- are the Lamé constants of the matrix and of the inclusion. The circular inclusion is connected to the matrix with a linear interface, which, according to results of previous Section, is characterized (at $r=R$) by

$$\sigma_{rr}^{+}=\sigma_{rr}^{-},\quad \sigma_{r\theta}^{+}=\sigma_{r\theta}^{-}, \tag{17}$$

$$\sigma_{rr}^{+}=s_r[U_r^{+}(\mathbf{x}^{+})-U_r^{-}(\mathbf{x}^{-})],\quad \sigma_{r\theta}^{+}=s_\theta[U_\theta^{+}(\mathbf{x}^{+})-U_\theta^{-}(\mathbf{x}^{-})], \tag{18}$$

where $\mathbf{U}$ is the displacement vector, function of point $\mathbf{x}$, and s_r and s_θ are stiffness constants of the interface, assumed not negative. When both these coefficients tend to infinity, perfect bonding is recovered. In opposite, when s_r and s_θ are equal to zero, there is no connection between inclusion and matrix, and the problem reduces to that of a cavity in an infinite plane. Loading is prescribed at infinity by introducing a generic displacement field:

$$\mathbf{U}(\mathbf{x})\to\mathbf{U}_\infty(\mathbf{x}),\quad \text{as } \mathbf{x}\to\infty, \tag{19}$$

which satisfies the Navier equations and is assumed to be representable as

$$\mathbf{U}_\infty(\mathbf{x})=(p(x_1,x_2),q(x_1,x_2)), \tag{20}$$

where p and q are order N polynomial functions of the point (x_1,x_2).

Condition (20), employed among others by Sendeckyj (1970), Yu and Sendeckyj (1974) and Gong and Meguid (1993), is a generalization of the well-known condition of homogeneous remote strain. In summary, the problem considered in the present paper is to find a displacement field **U** satisfying the Navier equations in the matrix and in the inclusion

$$\mu_+ \Delta\mathbf{U}(\mathbf{x}) + (\lambda_+ + \mu_+)\nabla\nabla\cdot\mathbf{U}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x}\in\mathbf{R}^2\backslash G,$$
$$\mu_- \Delta\mathbf{U}(\mathbf{x}) + (\lambda_- + \mu_-)\nabla\nabla\cdot\mathbf{U}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x}\in G, \tag{21}$$

(where G is the region occupied by the inclusion), satisfying conditions (17) and (18) at the interface, and corresponding to condition (20) at infinity. The standard technique for solving two-dimensional problems reduces the boundary value problem to the calculation of the complex potentials $\varphi(z)$ and $\psi(z)$. These have direct connection with the components of the stress tensor and displacement (Muskhelishvili, 1953):

$$U_r + iU_\theta = \frac{1}{2\mu} e^{-i\theta}(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}),$$
$$\sigma_{rr} + \sigma_{\theta\theta} = 2(\varphi'(z) + \overline{\varphi'(z)}),$$
$$\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} = 2e^{2i\theta}(\bar{z}\varphi''(z) + \psi'(z)), \tag{22}$$

where $z = x_1 + ix_2$ and $\kappa = (\lambda+3\mu)(\lambda+\mu)^{-1}$ for plane strain, $\kappa = (5\lambda+6\mu)(3\lambda+2\mu)^{-1}$ for plane stress. The complex potentials φ and ψ are analytical functions in the region where they specify the solution of the elastostatic problem. As a result, the solution is sought employing the following representation of the complex potentials

$$\varphi_-(z) = \sum_{k=0}^{+\infty} c_k z^k, \quad \psi_-(z) = \sum_{k=0}^{+\infty} d_k z^k, \tag{23}$$

which are analytic inside the disk of radius R, and

$$\varphi_+(z) = \sum_{k=-\infty}^{+\infty} a_k z^k, \quad \psi_+(z) = \sum_{k=-\infty}^{+\infty} b_k z^k, \tag{24}$$

which are analytic in the outer region $\{\mathbf{x} : R < \sqrt{x_1^2 + x_2^2} < +\infty\}$ and have a pole at infinity. The coefficients a_k, b_k are defined from the conditions (20) at infinity for $0 \leqslant k \leqslant N$, are null for $k > N$ (i.e. $a_k = b_k = 0$, for $k > N$), and are unknown to be determined for N negative. These and coefficients c_k and d_k can be found from boundary conditions. In order to rewrite the interface boundary conditions in terms of complex potentials, we note that the continuity of traction at interface corresponds to the following condition:

$$\varphi_+(z) + z\overline{\varphi'_+(z)} + \overline{\psi_+(z)} = \varphi_-(z) + z\overline{\varphi'_-(z)} + \overline{\psi_-(z)}, \tag{25}$$

where $z = Re^{i\theta}$. Using the standard Muskhelishvili (1953) technique, the boundary integral equations holding in the whole plane are deduced

$$\oint_L \frac{[\varphi_+(t) + t\overline{\varphi'_+(t)} + \overline{\psi_+(t)}]\,\mathrm{d}t}{t-z} = \oint_L \frac{[\varphi_-(t) + t\overline{\varphi_-(t)} + \overline{\psi_-(t)}]\,\mathrm{d}t}{t-z},$$
$$\oint_L \frac{[\overline{\varphi_+(t)} + \bar{t}\varphi'_+(t) + \psi_+(t)]\,\mathrm{d}t}{t-z} = \oint_L \frac{[\overline{\varphi_-(t)} + \bar{t}\varphi'_-(t) + \psi_-(t)]\,\mathrm{d}t}{t-z}, \tag{26}$$

where L denotes the circular boundary of the inclusion.

Solving this system of equations via standard theorems on Cauchy-type integrals and employing the expansion for the complex potentials eqns (23)–(24), condition (26) can be transformed into

$$\sum_{k=-\infty}^{+\infty} a_k R^k z^k + \sum_{k=-\infty}^{+\infty} k\bar{a}_k R^k z^{2-k} + \sum_{k=-\infty}^{+\infty} \bar{b}_k R^k z^{-k}$$
$$= \sum_{k=0}^{+\infty} c_k R^k z^k + \sum_{k=0}^{+\infty} k\bar{c}_k R^k z^{2-k} + \sum_{k=0}^{+\infty} \bar{d}_k R^k z^{-k}, \quad |z| = 1. \quad (27)$$

Collecting coefficients near the same powers of z yields the system of linear equations for the unknown coefficients a_k, b_k, c_k, d_k:

$$a_n R^n + (2-n)\bar{a}_{2-n} R^{2-n} + \bar{b}_{-n} R^{-n} = c_n R^n + (2-n)\bar{c}_{2-n} R^{2-n} + \bar{d}_{-n} R^{-n}, \quad (28)$$

where $n \in \mathbf{Z}$ and coefficients a_n, b_n are known from conditions at infinity for $n > 0$, whereas coefficients c_n, d_n are equal to zero for $n < 0$. More precisely, index n ranges between $-N$ and N.

Let us consider now the second boundary condition (18). Using the complex potential representation, the radial and shearing components of the stress tensor can be rewritten in the following form:

$$\sigma_{rr} = \varphi'(z) + \overline{\varphi'(z)} - Re\,[e^{2i\theta}(\bar{z}\varphi''(z) + \psi'(z))],$$
$$\sigma_{r\theta} = Im\,[e^{2i\theta}(\bar{z}\varphi''(z) + \psi'(z))]. \quad (29)$$

Therefore, the boundary conditions (18) take the complex variable form:

$$s_r(U_r^+ - U_r^-) + is_\theta(U_\theta^+ - U_\theta^-) = \sigma_{rr}^+ + i\sigma_{r\theta}^+, \quad (30)$$

or, more explicitly

$$\frac{s_r+s_\theta}{2}\left[\frac{e^{-i\theta}}{2\mu_+}(\kappa_+\varphi_+(z) - z\overline{\varphi'_+(z)} - \overline{\psi_+(z)}) - \frac{e^{-i\theta}}{2\mu_-}(\kappa_-\varphi_-(z) - z\overline{\varphi'_-(z)} - \overline{\psi_-(z)})\right]$$
$$+ \frac{s_r-s_\theta}{2}\left[\frac{e^{i\theta}}{2\mu_+}(\kappa_+\overline{\varphi_+(z)} - \bar{z}\varphi'_+(z) - \psi_+(z)) - \frac{e^{i\theta}}{2\mu_-}(\kappa_-\overline{\varphi_-(z)} - \bar{z}\varphi'_-(z) - \psi_-(z))\right]$$
$$= \varphi'_+(z) + \overline{\varphi'_+(z)} - e^{-2i\theta}(\overline{z\varphi''_+(z)} + \overline{\psi'_+(z)}). \quad (31)$$

In terms of Cauchy integrals, system (31) can be written as:

$$\frac{s_r+s_\theta}{4\mu_+}\oint_L \frac{e^{-i\theta}(\kappa_+\varphi_+(t) - t\overline{\varphi'_+(t)} - \overline{\psi_+(t)})\,\mathrm{d}t}{t-z} - \frac{s_r+s_\theta}{4\mu_-}\oint_L \frac{e^{-i\theta}(\kappa_-\varphi_-(t) - t\overline{\varphi'_-(t)} - \overline{\psi_-(t)})\,\mathrm{d}t}{t-z}$$
$$+ \frac{s_r-s_\theta}{4\mu_+}\oint_L \frac{e^{i\theta}(\kappa_+\overline{\varphi_+(t)} - \bar{t}\varphi'_+(t) - \psi_+(t))\,\mathrm{d}t}{t-z}$$
$$- \frac{s_r-s_\theta}{4\mu_-}\oint_L \frac{e^{i\theta}(\kappa_-\overline{\varphi_-(t)} - \bar{t}\varphi'_-(t) - \psi_-(t))\,\mathrm{d}t}{t-z}$$
$$= \oint_L \frac{[\varphi'_+(t) + \overline{\varphi'_+(t)} - e^{-2i\theta}(\overline{t\varphi''_+(t)} + \overline{\psi'_+(t)})]\,\mathrm{d}t}{t-z},$$

$$\frac{s_r+s_\theta}{4\mu_+}\oint_L \frac{e^{i\theta}(\kappa_+\overline{\varphi_+(t)}-\bar{t}\varphi'_+(t)-\psi_+(t))\,dt}{t-z}$$
$$-\frac{s_r+s_\theta}{4\mu_-}\oint_L \frac{e^{i\theta}(\kappa_-\overline{\varphi_-(t)}-\bar{t}\varphi'_-(t)-\psi_-(t))\,dt}{t-z}$$
$$+\frac{s_r-s_\theta}{4\mu_+}\oint_L \frac{e^{-i\theta}(\kappa_+\varphi_+(t)-t\overline{\varphi'_+(t)}-\overline{\psi_+(t)})\,dt}{t-z}$$
$$-\frac{s_r-s_\theta}{4\mu_-}\oint_L \frac{e^{-i\theta}(\kappa_-\varphi_-(t)-t\overline{\varphi'_-(t)}-\overline{\psi_-(t)})\,dt}{t-z}$$
$$=\oint_L \frac{[\varphi'_+(t)+\overline{\varphi'_+(t)}-e^{2i\theta}(\bar{t}\varphi''_+(t)+\psi'_+(t))]\,dt}{t-z}. \qquad (32)$$

Taking the series expansion of eqn (32) and writing it on the circle of radius R, one gets:

$$\frac{s_r+s_\theta}{2}\left[\frac{1}{2\mu_+}\left(\kappa_+\sum_{j=-\infty}^{+\infty}a_jR^jz^{j-1}-\sum_{j=-\infty}^{+\infty}j\bar{a}_jR^jz^{1-j}-\sum_{j=-\infty}^{+\infty}\bar{b}_jR^jz^{-j-1}\right)\right.$$
$$\left.-\frac{1}{2\mu_-}\left(\kappa_-\sum_{j=0}^{+\infty}c_jR^jz^{j-1}-\sum_{j=0}^{+\infty}j\bar{c}_jR^jz^{1-j}-\sum_{j=0}^{+\infty}\bar{d}_jR^jz^{-j-1}\right)\right]$$
$$+\frac{s_r-s_\theta}{2}\left[\frac{1}{2\mu_+}\left(\kappa_+\sum_{j=-\infty}^{+\infty}\bar{a}_jR^jz^{1-j}-\sum_{j=-\infty}^{+\infty}ja_jR^jz^{j-1}-\sum_{j=-\infty}^{+\infty}b_jR^jz^{j+1}\right)\right.$$
$$\left.-\frac{1}{2\mu_-}\left(\kappa_-\sum_{j=0}^{+\infty}\bar{c}_jR^jz^{1-j}-\sum_{j=0}^{+\infty}jc_jR^jz^{j-1}-\sum_{j=0}^{+\infty}c_jR^jz^{j+1}\right)\right]$$
$$=\sum_{j=-\infty}^{+\infty}ja_jR^{j-1}z^{j-1}+\sum_{j=-\infty}^{+\infty}j\bar{a}_jR^{j-1}z^{1-j}$$
$$-\sum_{j=-\infty}^{+\infty}j(j-1)\bar{a}_jR^{j-1}z^{1-j}-\sum_{j=-\infty}^{+\infty}j\bar{b}_jR^{j-1}z^{-j-1},\quad |z|=1. \qquad (33)$$

After simplification and collection of coefficients near the same power of z in eqn (33), the following system of linear equations is obtained

$$\frac{s_r+s_\theta}{4\mu_+}(\kappa_+a_{n+1}R^{n+1}-(1-n)\bar{a}_{1-n}R^{1-n}-\bar{b}_{-n-1}R^{-n-1})$$
$$-\frac{s_r+s_\theta}{4\mu_-}(\kappa_-c_{n+1}R^{n+1}-(1-n)\bar{c}_{1-n}R^{1-n}-\bar{d}_{-n-1}R^{-n-1})$$
$$+\frac{s_r-s_\theta}{4\mu_+}(\kappa_+\bar{a}_{1-n}R^{1-n}-(n+1)a_{n+1}R^{n+1}-b_{n-1}R^{n-1})$$
$$-\frac{s_r-s_\theta}{4\mu_-}(\kappa_-\bar{c}_{1-n}R^{1-n}-(n+1)c_{n+1}R^{n+1}-d_{n-1}R^{n-1})$$
$$=(n+1)a_{n+1}R^n+(1-n^2)\bar{a}_{1-n}R^{-n}+(n+1)\bar{b}_{-n-1}R^{-n-2}, \qquad (34)$$

where coefficients a_n and b_n are known from condition at infinity for $n>0$, whereas $c_n=d_n=0$ for $n<0$. More precisely, index n ranges between $-N-1$ and $N-1$.

The linear system of eqns (28) and (34) can be solved, thus obtaining the complex potentials coefficients (see Appendix), which represent the solution of our problem:

$$a_{1-n} = \frac{\bar{b}_{n-1}R^{2n-2}}{D_n}\{s_r s_\theta R^2\Gamma^{\pm}_{-}(\mu_- - \mu_+) - 4(n^2-1)\mu_+^2\mu_-^2$$
$$+(s_r+s_\theta)R\mu_+\mu_-[(n+1)(\mu_- - \mu_+) - (n-1)\Gamma^{\pm}_{-}]\}$$
$$+\frac{\bar{a}_{n+1}R^{2n}(n+1)}{D_n}\{s_r s_\theta R^2\Gamma^{\pm}_{-}(\mu_- - \mu_+) - (s_r - s_\theta)R(\kappa_+ + 1)\mu_+\mu_-^2$$
$$+(s_r+s_\theta)R\mu_+\mu_-[(n+1)(\mu_- - \mu_+) - (n-1)\Gamma^{\pm}_{-}] - 4(n^2-1)\mu_+^2\mu_-^2\},$$

$$b_{-1} = 2\,Re(a_1)R^2\frac{s_r R(\Gamma^{-}_{+} - \Gamma^{\pm}_{-}) - 4\mu_+\mu_-}{s_r R(\Gamma^{\pm}_{-} + \mu_- - \mu_+) + 4\mu_+\mu_-},$$

$$b_{-2} = \bar{a}_2 R^4\frac{s_r s_\theta R(\kappa_+\mu_- - \kappa_-\mu_+) - 2(s_r+s_\theta)\mu_+\mu_-}{s_r s_\theta R\Gamma^{\pm}_{-} + 2(s_r+s_\theta)\mu_+\mu_-},$$

$$b_{-1-n} = \frac{\bar{b}_{n-1}R^{2n}(n-1)}{D_n}\{s_r s_\theta R^2\Gamma^{\pm}_{-}(\mu_- - \mu_+)$$
$$-(s_r+s_\theta)R\mu_+\mu_-[(\mu_+ - \mu_-)(n+1) + \Gamma^{\pm}_{-}(n-1)]$$
$$+(s_\theta - s_r)R\mu_+\mu_-(\kappa_+ + 1)\mu_- - 4(n^2-1)\mu_+^2\mu_-^2\}$$
$$+\frac{\bar{a}_{n+1}R^{2n+2}}{D_n}\{s_r s_\theta R^2[(\kappa_+ + 1)\mu_-(\Gamma^{-}_{+} - \Gamma^{\pm}_{-}) - n^2(\mu_+ - \mu_-)\Gamma^{\pm}_{-}]$$
$$-4n^2(n^2-1)\mu_+^2\mu_-^2 + 2(s_\theta - s_r)R(n^2-1)(\kappa_+ + 1)\mu_+\mu_-^2$$
$$-(s_r+s_\theta)R\mu_+\mu_-[2(\kappa_+ + 1)\mu_- + n^2[(n-1)\Gamma^{\pm}_{-} + (n+1)(\mu_+ - \mu_-)]]\},$$

$$c_0 = \frac{a_0\Gamma^{-}_{+} + \bar{b}_0(\mu_+ - \mu_-)}{(\kappa_- + 1)\mu_+} + \frac{2\bar{a}_2R^3 s_r s_\theta(\mu_+ - \mu_-)\Gamma^{\pm}_{-}}{(\kappa_- + 1)\mu_+[s_r s_\theta R\Gamma^{\pm}_{-} + 2(s_r+s_\theta)\mu_+\mu_-]}$$
$$+\frac{2\bar{a}_2R^2\mu_+\mu_-[2(s_r+s_\theta)(\mu_+ - \mu_-) + (s_r - s_\theta)(\kappa_+ + 1)\mu_-]}{(\kappa_- + 1)\mu_+[s_r s_\theta R\Gamma^{\pm}_{-} + 2(s_r+s_\theta)\mu_+\mu_-]},$$

$$c_1 = \frac{Re(a_1)Rs_r(\kappa_+ + 1)\mu_-}{s_r R(\Gamma^{\pm}_{-} + \mu_- - \mu_+) + 4\mu_+\mu_-} + i\frac{Im(a_1)(\kappa_+ + 1)\mu_-}{(\kappa_- + 1)\mu_+},$$

$$c_2 = \frac{a_2 R s_r s_\theta(\kappa_+ + 1)\mu_-}{s_r s_\theta R\Gamma^{\pm}_{-} + 2(s_r+s_\theta)\mu_+\mu_-},$$

$$c_{n+1} = \frac{b_{n-1}}{RD_n}(\kappa_+ + 1)\mu_+\mu_-^2(s_\theta - s_r)(n-1)$$
$$+\frac{a_{n+1}R(\kappa_+ + 1)\mu_-}{D_n}\{s_r s_\theta R\Gamma^{-}_{+} + \mu_+\mu_-[-(s_r+s_\theta)(n-1) + (s_r - s_\theta)(n^2-1)]\},$$

$$d_0 = \frac{b_0\Gamma^{\pm}_{-} - \bar{a}_0(\kappa_+\mu_- - \kappa_-\mu_+)}{(\kappa_- + 1)\mu_+}$$
$$-\frac{2a_2R^3 s_r s_\theta[(\kappa_- + 1)\mu_+(\kappa_+\mu_- - \kappa_-\mu_+) + (\mu_+ - \mu_-)\Gamma^{\pm}_{-}]}{(\kappa_- + 1)\mu_+[s_r s_\theta R\Gamma^{\pm}_{-} + 2(s_r+s_\theta)\mu_+\mu_-]}$$
$$+\frac{2a_2R^2\mu_+\mu_-[(s_r+s_\theta)2\Gamma^{\pm}_{-} - (s_r - s_\theta)(\kappa_+ + 1)\mu_-]}{(\kappa_- + 1)\mu_+[s_r s_\theta R\Gamma^{\pm}_{-} + 2(s_r+s_\theta)\mu_+\mu_-]},$$

$$d_{n-1} = \frac{b_{n-1}R(\kappa_+ + 1)\mu_-}{D_n}\{s_r s_\theta R\Gamma_-^+ + \mu_+\mu_-[(s_r - s_\theta)(n^2-1) + (s_r + s_\theta)(n+1)]\}$$

$$+ \frac{a_{n+1}R^3(n+1)(\kappa_+ + 1)\mu_-}{D_n}\{s_r s_\theta R(\Gamma_-^+ - \Gamma_+^-) + \mu_+\mu_-[(s_r - s_\theta)(n^2-2) + 2(s_r + s_\theta)]\}, \quad (35)$$

where $2 \leqslant n \leqslant N+1$, and

$$\Gamma_+^- = \kappa_+\mu_- + \mu_+, \quad \Gamma_-^+ = \kappa_-\mu_+ + \mu_-,$$

$$D_n = s_r s_\theta R^2 \Gamma_+^- \Gamma_-^+ + (s_r + s_\theta) R\mu_+\mu_-[\Gamma_+^-(n+1) + \Gamma_-^+(n-1)] + 4(n^2-1)\mu_+^2\mu_-^2.$$

3.1. *Homogeneous remote stress field*

The Eshelby condition of homogeneous stress at infinity, particularly relevant in view of applications (two of which, regarding crack propagation and homogenization of composite materials, are presented in Section 4) corresponds to a linearly varying displacement field. In this case, the general expression for the complex potentials becomes

$$\varphi_+(z) = a_1 z + a_{-1}z^{-1}, \quad \psi_+(z) = b_1 z + b_{-1}z^{-1} + b_{-3}z^{-3},$$
$$\varphi_-(z) = c_1 z + c_3 z^3, \quad \psi_-(z) = d_1 z, \quad (36)$$

where coefficients a_1 and b_1 are related to the conditions at infinity in the following way:

$$a_1 = \frac{\alpha\mu_+}{\kappa_+ - 1}, \quad b_1 = -\alpha\mu_+, \quad \text{when } \mathbf{U}_\infty = V^{(1)} = (\alpha x_1, 0),$$

$$a_1 = \frac{\beta\mu_+}{\kappa_+ - 1}, \quad b_1 = \beta\mu_+, \quad \text{when } \mathbf{U}_\infty = V^{(2)} = (0, \beta x_2),$$

$$a_1 = 0, \quad b_1 = i\sqrt{2}\gamma\mu_+, \quad \text{when } \mathbf{U}_\infty = V^{(3)} = \frac{\gamma}{\sqrt{2}}(x_2, x_1), \quad (37)$$

with $\alpha, \beta, \gamma \in \mathbf{R}$. Coefficients a_{-1}, b_{-1}, b_{-3}, c_1, c_3, d_1, result

$$a_{-1} = \frac{\bar{b}_1 R^2}{D_2}\{s_r s_\theta R^2 \Gamma_-^+(\mu_- - \mu_+) + (s_r + s_\theta)R\mu_+\mu_-(3(\mu_- - \mu_+) - \Gamma_-^+) - 12\mu_+^2\mu_-^2\},$$

$$b_{-1} = 2\,Re(a_1)R^2 \frac{s_r R(\mu_-(\kappa_+ - 1) - \mu_+(\kappa_- - 1)) - 4\mu_+\mu_-}{s_r R(\mu_+(\kappa_- - 1) + 2\mu_-) + 4\mu_+\mu_-},$$

$$b_{-3} = \frac{\bar{b}_1 R^4}{D_2}\{s_r s_\theta R^2 \Gamma_-^+(\mu_- - \mu_+) - R\mu_+\mu_-(s_r + s_\theta)(\Gamma_-^+ - 3\mu_- + 3\mu_+)$$
$$- R\mu_+\mu_-(s_r - s_\theta)(\Gamma_+^- + \mu_- - \mu_+) - 12\mu_+^2\mu_-^2\},$$

$$c_1 = \frac{Re(a_1)Rs_r(\kappa_+ + 1)\mu_-}{4\mu_+\mu_- + s_r R(\mu_+(\kappa_- - 1) + 2\mu_-)} + i\frac{Im(a_1)(\kappa_+ + 1)\mu_-}{(\kappa_- + 1)\mu_+},$$

$$c_3 = \frac{b_1(s_\theta - s_r)(\kappa_+ + 1)\mu_-^2\mu_+}{RD_2},$$

$$d_1 = \frac{b_1 R(\kappa_+ + 1)\mu_-(s_r s_\theta R\Gamma_-^+ + 6s_r\mu_+\mu_-)}{D_2}, \quad (38)$$

where

$$D_2 = s_r s_\theta \Gamma_+^- \Gamma_-^+ R^2 + (s_r + s_\theta) R (3\Gamma_+^- + \Gamma_-^+)\mu_+\mu_- + 12\mu_+^2\mu_-^2.$$

As noted by Gao (1995), the Eshelby (1957) theorem, stating that the deformation field is homogeneous inside the inclusion, holds if and only if $c_3 = 0$. Except for trivial cases, this occurs when the stiffness in radial and tangential directions are equal, i.e. $s_r = s_\theta = s$. In this case, the coefficients of complex potentials reduce to

$$\begin{aligned}
a_{-1} &= \bar{b}_1 R^2 \frac{sR(\mu_- - \mu_+) - 2\mu_+\mu_-}{s\Gamma_+^- R + 2\mu_+\mu_-},\\
b_{-1} &= 2\,Re(a_1) R^2 \frac{sR(\mu_-(\kappa_+ - 1) - \mu_+(\kappa_- - 1)) - 4\mu_+\mu_-}{sR(\mu_+(\kappa_- - 1) + 2\mu_-) + 4\mu_+\mu_-},\\
b_{-3} &= \bar{b}_1 R^4 \frac{sR(\mu_- - \mu_+) - 2\mu_+\mu_-}{s\Gamma_+^- R + 2\mu_+\mu_-},\\
c_1 &= \frac{Re(a_1) Rs(\kappa_+ + 1)\mu_-}{sR(\mu_+(\kappa_- - 1) + 2\mu_-) + 4\mu_+\mu_-} + i\frac{Im(a_1)(\kappa_+ + 1)\mu_-}{(\kappa_- + 1)\mu_+},\\
c_3 &= 0,\\
d_1 &= \frac{b_1 Rs(\kappa_+ + 1)\mu_-}{s\Gamma_+^- R + 2\mu_+\mu_-}.
\end{aligned} \tag{39}$$

4. ASYMPTOTIC MODELS OF CRACK TRAJECTORY AND EFFECTIVE PROPERTIES OF DILUTE COMPOSITES

Applications are presented of the solution obtained in Section 3 (in the specific case of homogeneous remote conditions) to problems of crack propagation and homogenization of a dilute suspension of imperfectly bonded, circular inclusions. These examples show the strong effect of the interface conditions. Roughly speaking, the presence of a finite interfacial compliance yields a solution which is "intermediate" between the limit cases of zero and infinite interfacial stiffness, corresponding to a circular cavity and a perfectly bonded elastic inclusion, respectively. Both the applications presented are based on asymptotic solutions bearing on the Pólya and Szegö (1951) dipole representation of a finite defect in an infinite plane (for a precise definition of this matrix, see Movchan and Movchan, 1995 and Movchan and Serkov, 1997).

The Pólya–Szegö matrix for circular inclusion with linear interface may be written as

$$\mathscr{P} = \frac{R^2}{4q}\begin{bmatrix} \xi + \dfrac{2\eta}{(\kappa_+ - 1)^2} & -\xi + \dfrac{2\eta}{(\kappa_+ - 1)^2} & 0 \\ -\xi + \dfrac{2\eta}{(\kappa_+ - 1)^2} & \xi + \dfrac{2\eta}{(\kappa_+ - 1)^2} & 0 \\ 0 & 0 & 2\xi \end{bmatrix}, \tag{40}$$

where

$$\eta = \frac{s_r R(\mu_-(\kappa_+ - 1) - \mu_+(\kappa_- - 1)) - 4\mu_+\mu_-}{s_r R(\mu_+(\kappa_- - 1) + 2\mu_-) + 4\mu_+\mu_-}, \tag{41}$$

$$\xi = \frac{s_r s_\theta R^2 \Gamma^{\pm}_{-}(\mu_- - \mu_+) + (s_r + s_\theta) R \mu_+ \mu_- (3(\mu_- - \mu_+) - \Gamma^{\pm}_{-}) - 12\mu_+^2 \mu_-^2}{s_r s_\theta \Gamma^{-}_{+} \Gamma^{\pm}_{-} R^2 + (s_r + s_\theta) R (3\Gamma^{-}_{+} + \Gamma^{\pm}_{-}) \mu_+ \mu_- + 12 \mu_+^2 \mu_-^2}, \tag{42}$$

and $q = (\lambda_+ + \mu_+)/(8\pi\mu_+(\lambda_+ + 2\mu_+))$. In the particular case of equal radial and tangential stiffness of the interface, $s_r = s_\theta = s$, the constant ξ reduces to

$$\xi = \frac{sR(\mu_- - \mu_+) - 2\mu_+\mu_-}{s\Gamma^{-}_{+} R + 2\mu_+\mu_-}.$$

It should be noted that $(R^2/2q)\xi$ (with multiplicity 2) and $(R^2/2q)\eta$ are the two distinct eigenvalues of $\mathscr{P}$. It can be therefore easily verified that when one of the eigenvalues of $\mathscr{P}$ is negative in the limit $s_r = s_\theta \to \infty$, i.e. for perfectly bonded interface, it remains negative for any finite (positive) value of interfacial parameters s_r and s_θ. On the other hand, when one of the eigenvalues of $\mathscr{P}$ is positive in the limit $s_r = s_\theta \to \infty$, it may always become negative for a set of values of s_r and s_θ.

When matrix $\mathscr{P}$ is positive definite in the limit case of perfectly bonded interface, it may happen that matrix $\mathscr{P}$ becomes identically zero for imperfectly bonded interface with a proper choice of parameters s_r and s_θ. The condition of vanishing of matrix $\mathscr{P}$ can be obtained imposing $\eta = \xi = 0$ and thus obtaining

$$s_r = \frac{4\mu_+\mu_-}{R[\mu_-(\kappa_+ - 1) - \mu_+(\kappa_- - 1)]}, \tag{43}$$

and

$$s_\theta = \frac{4\mu_+\mu_-\{3(\Gamma^{-}_{+} - \mu_- + \mu_+) - 2\Gamma^{\pm}_{-}\}}{R\{(\Gamma^{\pm}_{-} + 3\Gamma^{-}_{+})(\mu_- - \mu_+) + \Gamma^{\pm}_{-}(\Gamma^{\pm}_{-} - \Gamma^{-}_{+})\}}. \tag{44}$$

Conditions (43) and (44) can be satisfied for non negative values of s_r and s_θ, and in this case the inclusion does not affect the matrix (to the first order). To convince oneself of this fact, it suffices to consider the particular case

$$\kappa_+ = \kappa_- = \kappa = 1 + \varepsilon, \quad \rho = \frac{\mu_+}{\mu_-} < 1, \quad 0 < \varepsilon < 2.$$

Under these assumptions, both eqn (43) and the denominator of the left-hand-side of eqn (44) are positive. Moreover, it can be shown that

$$s_\theta > 0, \Leftrightarrow \rho > \frac{\varepsilon}{\varepsilon - 2} + \frac{1}{2},$$

and this provides an example that eqns (43) and (44) can be verified for positive values of interfacial stiffness parameters. It may be important to remark that Lipton and Vernescu (1995) have obtained a similar result for homogenization of an elastic material containing an arbitrary volume fraction of imperfectly bonded spherical inclusions, in the special case $s_r = s_\theta = s$. Their results can be recovered, in the present context, defining the "critical inclusion radii"

$$R_\xi = \frac{2\mu_+\mu_-}{s(\mu_- - \mu_+)}, \tag{45}$$

and

$$R_\eta = \frac{4\mu_+\mu_-}{s(\mu_-(\kappa_+-1)-\mu_+(\kappa_--1))}. \tag{46}$$

For given elastic properties of matrix and inclusion, and for given interfacial parameter s, if the radius of the inclusion equals one of the two values R_ξ and R_η, the Pólya and Szegö matrix $\mathscr{P}$ becomes singular [see eqns (41) and (42)]. In particular, when the radius equals R_ξ, we have $\xi = 0$. If the two critical radii are equal to the radius of inclusion, then the Pólya and Szegö matrix vanishes. Formulae (45) and (46) were obtained using variational methods by Lipton and Vernescu (1995) in the non-dilute case.

4.1. *Crack trajectory*

In a series of papers, Movchan and co-workers (see, Movchan and Movchan, 1995; Valentini *et al.*, 1997; and references cited therein), have given a framework for analyzing the trajectory described by a crack in a brittle material as influenced by the presence of generic defects. These defects deviate the crack path, which would be straight and under mode I conditions in the absence of any defect. The formulation is based on asymptotic analysis and is not repeated here (the interested reader is remanded to Movchan and Movchan, 1995). The crack is assumed to be at a sufficiently large distance from the defect (when compared to the defect size). To describe the generic crack tip position, a reference system is introduced having axis x_1 coincident with the unperturbed (i.e. rectilinear) mode I crack path. In this system, the position of the perturbed crack tip is singled out by coordinates $\{l, H(l)\}$, where H is a function of the unperturbed crack tip position l. In the specific case under consideration, i.e. for a defect consisting in a circular elastic inclusion with linear interface, this function is:

$$H(l) = \frac{R^2}{2x_2^0}\{\eta(x^2+x-2)+\xi(x-x^3)\}, \tag{47}$$

where R is the radius of the inclusion and

$$x = \frac{x_1^0-l}{\sqrt{(x_2^0)^2+(x_1^0-l)^2}},$$

with x_1^0, x_2^0 denoting the coordinates of the centre of the inclusion in the orthogonal reference system having the x_1 axis coincident with the unperturbed crack trajectory.

It may be worth mentioning that function $H(l)$ in eqn (47) reduces to that corresponding to a circular void, for $s_r = s_\theta = 0$, and to that corresponding to a perfectly bonded inclusion, for $s_r = s_\theta \to \infty$.

It should be noted that the crack may be attracted or repelled by the inhomogeneity. For instance, an inclusion "stiffer" than the matrix repels the crack and a void attracts it. More precisely, we say that a crack is attracted by an inclusion if its trajectory completely lies on the side of the inclusion (with respect to the unperturbed crack path). Conversely, the crack is repelled when its trajectory lies on the other side. Bearing this definition in mind, attraction and repulsion are related to the eigenvalues of the Pólya and Szegö matrix. Positive (or negative) definiteness of the Pólya and Szegö depends on inclusion radius R, and on material and interfacial parameters μ_+, μ_-, κ_+, κ_-, s_θ and s_r. For instance, when $s_r = s_\theta = s$ the Pólya and Szegö matrix is positive definite when

$$\frac{1}{\mu_+}-\frac{1}{\mu_-} > \frac{2}{sR}, \quad \frac{\kappa_+-1}{\mu_+}-\frac{\kappa_--1}{\mu_-} > \frac{4}{sR}, \tag{48}$$

and negative definite when the above expressions have reverse inequalities. Otherwise, the matrix $\mathscr{P}$ is indefinite. From the analysis of the eigenvalues of matrix $\mathscr{P}$, it may be concluded that:

—if the crack is attracted in the limit case of perfectly bonded interface, it will be not repelled for imperfectly bonded interface;
—if the crack deflection is partly positive and partly negative with matrix $\mathscr{P}$ indefinite, in the limit case of perfectly bonded interface, it will be not repelled for imperfectly bonded interface (i.e. matrix $\mathscr{P}$ will be indefinite or negative definite);
—if the crack is repelled in the limit case of perfectly bonded interface, the crack deflection may be positive, negative, indefinite or zero for imperfectly bonded interface.

The above statements may have implications in the design of brittle composites reinforced with tough inclusions. For instance, it may be concluded that imperfect interfacial bonding may drastically reduce the effects of inclusion stiffness. It may be curious to note that the condition for which the crack trajectory is zero at infinity, i.e. for $l \to \infty$, may be written as

$$H(\infty) = -\frac{R^2}{x_2^0}\eta = 0, \tag{49}$$

and therefore only depends on the radial stiffness parameter s_r as specified by eqn (43).

Crack trajectories $H(l)/x_2^0$ vs crack tip position l/x_2^0 are reported in Fig. 1 in a non-dimensional form (i.e. divided by the coordinate x_2^0 of the centre of the defect). These are the result of interaction with a circular elastic inclusion, positioned at (0, 1), and bonded with a linear interface. The stiffness of the interface enters in the non-dimensional form

$$\hat{s}_r = \frac{s_r x_2^0}{\mu_+}, \quad \hat{s}_\theta = \frac{s_\theta x_2^0}{\mu_+}. \tag{50}$$

For a better understanding of the figures, it may be useful to note that the crack trajectory would be coincident with the horizontal axis in the trivial case where the inclusion is suppressed. Various values of interfacial stiffness are considered, ranging between the extreme cases of perfectly bonded interface $s \to \infty$ and circular void $s = 0$. In the case of Fig. 1, equal radial and tangential stiffness of interface has been considered, i.e. $\hat{s}_r = \hat{s}_\theta = s$. Fig. 1(a) pertains to an inclusion stiffer than the matrix ($\mu_-/\mu_+ = 100$), whereas the opposite situation of matrix stiffer than the inclusion ($\mu_-/\mu_+ = 1/10$) is considered in Fig. 1(b). Moreover, $k_- = k_+ = 2$ has been considered in both figures (a) and (b). Note that the two rigidity ratios $\mu_-/\mu_+ = 100$, $1/10$ have been selected to give full evidence to results. A large ratio between defect centre ordinate and inclusion radius has been used, and therefore small crack deflections (divided by x_2^0) result. It can be observed that the crack trajectory changes even qualitatively, depending on the value of stiffness of the interface, for inclusion stiffer than the matrix (Fig. 1(a)). In particular, the crack is attracted for small interfacial stiffness and repelled when this is sufficiently high. In the opposite case of inclusion weaker than the

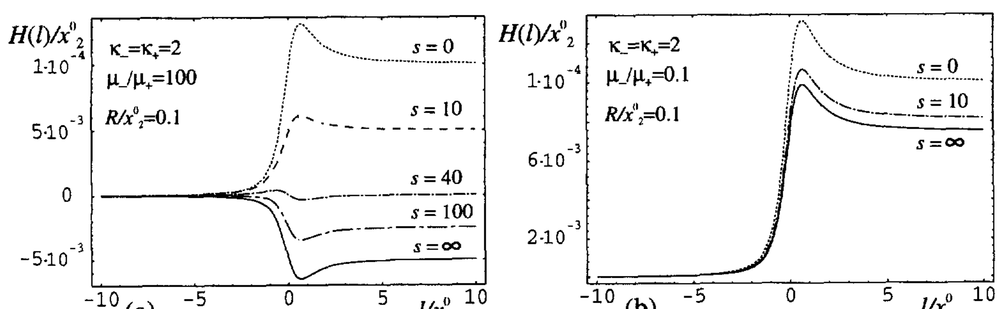

Fig. 1. Crack trajectories $H(l)/x_2^0$ vs crack tip position l/x_2^0, resulting from interaction with a circular elastic inclusion bonded with a linear interface with equal radial and tangential stiffness. Different values of interfacial stiffness are considered. (a) The inclusion is stiffer than the matrix ($\mu_-/\mu_+ = 100$), (b) the inclusion is weaker than the matrix ($\mu_-/\mu_+ = 1/10$).

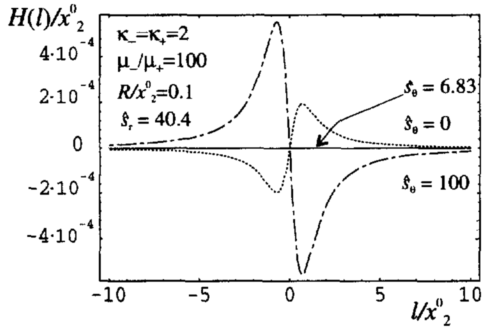

Fig. 2. Crack trajectories $H(l)/x_2^0$ vs crack tip position l/x_2^0, resulting from interaction with a circular elastic inclusion stiffer than the matrix and bonded with a linear interface. Different values of interfacial stiffness s_θ are considered, for the same value of parameter s_r, producing zero deflection of the crack path at infinity.

matrix (Fig. 1(b)), the effect of interfacial compliance is only quantitative, and the crack is attracted for all values of s.

Three values of tangential stiffness s_θ for the fixed value eqn (43) of radial stiffness corresponding to null crack deflection at infinity are considered in Fig. 2. The stiffness of the interface enters in the nondimensional form eqn (50) and the values $\hat{s}_\theta = 100$, $\hat{s}_\theta \approx 6.8345$, and $\hat{s}_\theta = 0$ are reported for $\hat{s}_r \approx 40.4040$. The Pólya–Szegö matrix is identically zero and crack remains straight when $\hat{s}_\theta \approx 6.8345$. In the cases $\hat{s}_\theta = 0$, 100, the Pólya–Szegö matrix is indefinite. The case $\mu_-/\mu_+ = 100$ and $k_- = k_+ = 2$ is considered. It can be noted that the tangential stiffness parameter s_θ strongly affects the crack path near the abscissa corresponding to the inclusion centre. Moreover, as an effect related to the tangential stiffness of interface, the curves may have reversed sign.

4.2. *Effective moduli of dilute composites*

In this Section the homogenization technique for composites including small inclusions with non-perfect interface is presented and discussed. A periodic distribution of isotropic–elastic defects in an isotropic–elastic matrix yields, in general, an anisotropic, elastic homogenized material (see e.g. Nemat-Nasser and Hori, 1993). Following Movchan and Serkov (1997), a periodic, dilute distribution of elastic defects may be described by a matrix $\mathscr{H}^*$ of effective moduli representable in the form

$$\mathscr{H}^* = \mathscr{H}^0 + \varepsilon^2 \mathscr{P} + O(\varepsilon^3), \quad \mathscr{H}^0 = \begin{pmatrix} \lambda_+ + 2\mu_+ & \lambda_+ & 0 \\ \lambda_+ & \lambda_+ + 2\mu_+ & 0 \\ 0 & 0 & 2\mu_+ \end{pmatrix} \tag{51}$$

where $\mathscr{H}^0$ is the matrix of elastic moduli of the matrix material, $\mathscr{P}$ is the Pólya–Szegö matrix, normalized with respect to R^2 (i.e. the matrix given by eqn (40) in which $R = 1$), ε is the ratio between the radius of inclusion and the characteristic size of the periodic cell. In terms of volume fraction f of the inclusions, defined as

$$f = \frac{\pi R^2}{C},$$

(where C is the area of elementary cell and R is the inclusion radius) eqn (51) can be written as

$$\mathscr{H}^* = \mathscr{H}^0 + \frac{f}{\pi}\mathscr{P} + O(f^2), \tag{52}$$

a formula which is valid even beyond the condition of periodicity, including the dilute, random, uniform distribution of iso-oriented defects.

For the given matrix $\mathscr{P}$ eqn (40), corresponding to circular defects, the homogenized material results to be isotropic and therefore defined by two independent elastic parameters. Moreover, in the limits $s_r = s_\theta = 0$ and $s_r = s_\theta \to \infty$, the effective moduli of a dilute distribution of circular voids and perfectly bonded inclusions are recovered, respectively (Nemat-Nasser and Hori, 1993, Sections 5.1.2 and 8.2.1). It can be interesting to analyze how the effective moduli change under different interface conditions, for fixed geometry and elastic properties of the composite. In other words, let us consider a given dilute, periodic distribution of circular inclusions with imperfect bonding and take the two interface stiffness parameters s_r and s_θ as two variables ranging between zero and infinity. We remark that this idea can be applied to inclusions of a general shape (yielding an anisotropic material) and a corresponding surface in 3-D space can be constructed, associated with the parameters describing effective moduli: the average Young's modulus E^*, the bulk modulus K^*, and the shear modulus μ^*:

$$E^* = E_+ + \varepsilon^2 \Delta E, \quad K^* = K_+ + \varepsilon^2 \Delta K, \quad \mu^* = \mu_+ + \varepsilon^2 \Delta\mu, \tag{53}$$

where, referring to the plain strain case, $E_+ = (4\mu_+(\mu_+ + \lambda_+))/(\lambda_+ + 2\mu_+)$ and $K_+ = \lambda_+ + \mu_+$. Moreover, ΔE, ΔK, and $\Delta\mu$ are functions of the interfacial stiffness. In our particular case of dilute distribution of circular inclusions, the material is isotropic, and only two of parameters (53) are independent.

In terms of Pólya–Szegö matrix (40), the perturbation in the elastic constants can be expressed as

$$\Delta E = 2(P_{11} + P_{22})\left(\frac{\kappa_+ - 1}{\kappa_+ + 1}\right)^2 + (P_{11} + P_{22} - 2P_{12})\frac{3 - \kappa_+}{\kappa_+ + 1},$$
$$\Delta K = \tfrac{1}{4}(P_{11} + P_{22} + 2P_{12}), \quad \Delta\mu = \tfrac{1}{2}P_{33}, \tag{54}$$

where P_{ij} denote the components of the Pólya–Szegö matrix (with $R = 1$).

The set of parameters $(\Delta E, \Delta K, \Delta\mu)$ describes the changes of the elastic moduli, and may be represented as a point in a 3-D space. When the interfacial stiffness parameters are changed, the point in 3-D space moves. Thus, all possible changes of the effective moduli tensor can be specified via some domain in 3-D space (Luri and Cherkaev, 1987), which in the specific elastic isotropic case under consideration may be anticipated to be a plane. Considering the representation (54) for ΔE, ΔK, $\Delta\mu$ and formula (40) for Pólya–Szegö matrix, we can note that

$$\Delta E = \alpha(\eta + 2\xi), \quad \Delta K = \beta\eta, \quad \Delta\mu = \gamma\xi,$$

where α, β and γ are constants only depending on the elastic coefficients of the matrix. Therefore, all points in the space $(\Delta E, \Delta K, \Delta\mu)$ can be characterized by the following mapping

$$\Upsilon : \{(s_r, s_\theta) \to (\alpha(\eta + 2\xi), \beta\eta, \gamma\xi), s_r \in [0, \infty), s_\theta \in [0, \infty)\}.$$

This is a one-to-one mapping having the following properties:

- Since the homogenized material is isotropic, all points $(\Delta E, \Delta K, \Delta\mu)$ lie on a plane having normal vector of components $(\beta\gamma, -\alpha\gamma, -2\alpha\beta)$. Thus the region $(s_r, s_\theta) : [0, \infty) \times [0, \infty)$ is mapped onto a 2-D domain. Note that the orientation of the plane only depends on the elastic properties of the matrix and is therefore independent of the elastic coefficients of the inclusion.

- The domain representing all possible variations of the set $(\Delta E, \Delta K, \Delta\mu)$ due to changes in the interface stiffness has vertices in the points where s_r and s_θ are equal to 0 or tend to infinity:

  $-s_r = s_\theta = 0$:

$$\Delta E = -\frac{6}{q(\kappa_+ + 1)^2}, \quad \Delta K = -\frac{1}{2q(\kappa_+ - 1)^2}, \quad \Delta\mu = -\frac{1}{4q};$$

  $-s_r = 0,\ s_\theta = \infty$:

$$\Delta E = \frac{2}{q(\kappa_+ + 1)^2}\left(\frac{6(\mu_- - \mu_+) - 2\Gamma^+_-}{3\Gamma^-_+ + \Gamma^+_-} - 1\right),$$

$$\Delta K = -\frac{1}{2q(\kappa_+ - 1)^2}, \quad \Delta\mu = \frac{3(\mu_- - \mu_+) - \Gamma^+_-}{4q(3\Gamma^-_+ + \Gamma^+_-)};$$

  $-s_r = \infty,\ s_\theta = 0$:

$$\Delta E = \frac{2}{q(\kappa_+ + 1)^2}\left(\frac{6(\mu_- - \mu_+) - 2\Gamma^+_-}{3\Gamma^-_+ + \Gamma^+_-} + \frac{\Gamma^-_+ - \Gamma^+_-}{\Gamma^+_- + \mu_- - \mu_+}\right),$$

$$\Delta K = \frac{\Gamma^-_+ - \Gamma^+_-}{2q(\Gamma^+_- + \mu_- - \mu_+)(\kappa_+ - 1)^2}, \quad \Delta\mu = \frac{3(\mu_- - \mu_+) - \Gamma^+_-}{4q(3\Gamma^-_+ + \Gamma^+_-)};$$

  $-s_r = \infty,\ s_\theta = \infty$:

$$\Delta E = \frac{2}{q(\kappa_+ + 1)^2}\left(\frac{2(\mu_- - \mu_+)}{\Gamma^-_+} + \frac{\Gamma^-_+ - \Gamma^+_-}{\Gamma^+_- + \mu_- - \mu_+}\right),$$

$$\Delta K = \frac{\Gamma^-_+ - \Gamma^+_-}{2q(\Gamma^+_- + \mu_- - \mu_+)(\kappa_+ - 1)^2}, \quad \Delta\mu = \frac{\mu_- - \mu_+}{4q\Gamma^-_+}.$$

- The domain representing all possible points $(\Delta E, \Delta K, \Delta\mu)$ is bounded by two parallel lines connecting the points corresponding to $s_r = s_\theta = 0$, $s_r = 0$, $s_\theta = \infty$ and $s_r = s_\theta = \infty$, $s_r = \infty$, $s_\theta = 0$. These lines are characterized by the condition $\eta = \text{const.}$ and have normal vector of components $(\alpha\gamma^2, 4\alpha^2\beta + \beta\gamma^2, -2\alpha^2\gamma)$. The other two boundaries of the area are connected by two curves. In particular, an analysis of these yields the following observations.

  —Curve of the boundary connecting points $s_r = s_\theta = 0$ and $s_r = \infty$, $s_\theta = 0$:

$$\eta(\xi) : \eta = \eta(s_r, s_\theta = 0), \quad \xi = \xi(s_r, s_\theta = 0),$$

$$\eta''(\xi) = \frac{\eta''\xi' - \eta'\xi''}{(\xi')^3} \geqslant 0, \quad \text{provided } \mu_+ - \mu_- \geqslant \frac{2}{3}\Gamma^+_- - \Gamma^-_+.$$

  —Curve of the boundary connecting points $s_r = s_\theta = \infty$ and $s_r = 0$, $s_\theta = \infty$:

$$\eta(\xi): \eta = \eta(s_r, s_\theta \to \infty), \quad \xi = \xi(s_r, s_\theta \to \infty),$$

$$\eta''(\xi) = \frac{\eta''\xi' - \eta'\xi''}{(\xi')^3} \leqslant 0, \quad \text{provided } \mu_+ - \mu_- \leqslant \frac{\Gamma_-^+(\Gamma_-^+ - \Gamma_+^-)}{3\Gamma_+^- + \Gamma_-^+}.$$

It can be therefore concluded that the set of points $(\Delta E, \Delta K, \Delta\mu)$ is convex when

$$\frac{2}{3}\Gamma_-^+ - \Gamma_+^- \leqslant \mu_+ - \mu_- \leqslant \frac{\Gamma_-^+(\Gamma_-^+ - \Gamma_+^-)}{3\Gamma_+^- - \Gamma_-^+}$$

holds.

In closure of this Section, it may be important to mention that when condition $\xi = 0$ eqn (42) is satisfied for positive values of s_r and s_θ, the homogenized material has the same shear modulus as the matrix. When $\eta = 0$ eqn (41) is verified for positive s_r and s_θ, the bulk modulus of the homogenized material and of the matrix are equal. Finally, when both eqns (43) and (44) are satisfied, i.e. when $\xi = \eta = 0$, for positive interfacial stiffness parameters, the inclusions do not have any influence on the properties of the composite (to first order in the volume fraction). These results, obtained here in the dilute approximation, where known for generic volume fraction of inclusions in the special case of equal radial and tangential interfacial stiffness—i.e. in terms of critical radii, see eqns (45) and (46)—(Lipton and Vernescu, 1995, Corollaries 4.3–4.6).

5. CONCLUSIONS

The problem of a circular elastic inclusion bonded with a linear interface to an infinite elastic matrix has been solved under general conditions at infinity. These remote conditions include, as special cases, the homogeneous and the bending stress fields. In order to give full evidence to the strong effect related to imperfect bonding, the obtained solution has been applied to problems of crack propagation and homogenization of dilute, periodic composites. Results obtained in this article clearly demonstrate a strong influence of interfacial conditions, particularly when the inclusions are stiffer than the matrix. The presented solutions for crack trajectory and homogenization may be employed to analyze toughness and stiffness of materials containing inclusions.

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APPENDIX

Equation (28) and eqn (34), holding for $n \in \mathbf{Z}$, can be rewritten as:

$$a_{1-n}R^{1-n}+(n+1)\bar{a}_{n+1}R^{n+1}+\bar{b}_{n-1}R^{n-1} = c_{1-n}R^{1-n}+(n+1)\bar{c}_{n+1}R^{n+1}+\bar{d}_{n-1}R^{n-1},$$
$$a_{1+n}R^{1+n}+(1-n)\bar{a}_{1-n}R^{1-n}+\bar{b}_{-n-1}R^{-n-1} = c_{1+n}R^{1+n}+(1-n)\bar{c}_{1-n}R^{1-n},$$

$$\frac{s_r+s_\theta}{4\mu}(\kappa_+ a_{n+1}R^{n+1}-(1-n)\bar{a}_{1-n}R^{1-n}-\bar{b}_{-n-1}R^{-n-1})-\frac{s_r+s_\theta}{4\mu_-}(\kappa_- c_{n+1}R^{n+1}-(1-n)\bar{c}_{1-n}R^{1-n})$$
$$+\frac{s_r-s_\theta}{4\mu_+}(\kappa_+\bar{a}_{1-n}R^{1-n}-(n+1)a_{n+1}R^{n+1}-b_{n-1}R^{n-1})-\frac{s_r-s_\theta}{4\mu_-}(\kappa_-\bar{c}_{1-n}R^{1-n}-(n+1)c_{n+1}R^{n+1}-d_{n-1}R^{n-1})$$
$$=(n+1)a_{n+1}R^n+(1-n^2)\bar{a}_{1-n}R^{-n}+(n+1)\bar{b}_{-n-1}R^{-n-2},$$

$$\frac{s_r+s_\theta}{4\mu_+}(\kappa_+ a_{-n+1}R^{-n+1}-(1+n)\bar{a}_{1+n}R^{1+n}-\bar{b}_{n-1}R^{n-1})-\frac{s_r+s_\theta}{4\mu_-}(\kappa_- c_{-n+1}R^{-n+1}-(1+n)\bar{c}_{1+n}R^{1+n}-\bar{d}_{n-1}R^{n-1})$$
$$+\frac{s_r-s_\theta}{4\mu_+}(\kappa_+\bar{a}_{1+n}R^{1+n}-(1-n)a_{-n+1}R^{-n+1}-b_{-n-1}R^{-n-1})-\frac{s_r-s_\theta}{4\mu_-}(\kappa_-\bar{c}_{1+n}R^{1+n}-(1-n)c_{-n+1}R^{-n+1})$$
$$=(1-n)a_{-n+1}R^{-n}+(1+n^2)\bar{a}_{1+n}R^n+(1-n)\bar{b}_{n-1}R^{n-2},$$
$$c_{-n}=d_{-n}=0,$$

where $n \in [-N-1, N-1]$.

Solving the system for $n = 0$ yields the coefficients c_1 and b_{-1}, whereas for $n = 1$ coefficients c_0, d_0, c_2, b_{-2} are obtained. Coefficients a_{1-n}, b_{-n-1}, c_{n+1}, d_{n-1} can be easily obtained for the appropriate values of $n \geqslant 2$. In particular, for $n = 2$, the coefficients a_{-1}, b_{-3}, c_3, d_1 are obtained:

$$a_{-1} = \frac{\bar{b}_1 R^2}{D_2}\{s_r s_\theta R^2 \Gamma^{+}_{-}(\mu_- - \mu_+)$$

$$+(s_r + s_\theta)R\mu_+\mu_-[3(\mu_- - \mu_+) - \Gamma^{+}_{-}] - 12\mu_+^2\mu_-^2\}$$

$$+\frac{3\bar{a}_3 R^4}{D_2}\{s_r s_\theta R^2 \Gamma^{+}_{-}(\mu_- - \mu_+) + R(s_\theta - s_r)(\kappa_+ + 1)\mu_+\mu_-^2$$

$$+R(s_r + s_\theta)\mu_+\mu_-[3(\mu_- - \mu_+) - \Gamma^{+}_{-}] - 12\mu_+^2\mu_-^2\},$$

$$b_{-3} = \frac{\bar{b}_1 R^4}{D_2}\{s_r s_\theta R^2 \Gamma^{+}_{-}(\mu_- - \mu_+) + s_r R\mu_+\mu_-(2(\mu_- - \mu_+) - \Gamma^{+}_{-} - \Gamma^{-}_{+})$$

$$+s_\theta R\mu_+\mu_-(4(\mu_- - \mu_+) - \Gamma^{+}_{-} + \Gamma^{-}_{+}) - 12\mu_+^2\mu_-^2\}$$

$$-\frac{\bar{a}_3 R^6}{D_2}\{s_r s_\theta R^2((\mu_+ - \mu_-)(3\Gamma^{+}_{-} + \Gamma^{-}_{+}) + \Gamma^{-}_{+}\Gamma^{+}_{-} - (\Gamma^{-}_{+})^2)$$

$$+4s_r R\mu_+\mu_-(\mu_+ - \mu_- + \Gamma^{+}_{-} + 2\Gamma^{-}_{+})$$

$$-4s_\theta R\mu_+\mu_-(4(\mu_+ - \mu_-) + \Gamma^{+}_{-} - \Gamma^{-}_{+}) + 48\mu_+^2\mu_-^2\},$$

$$c_3 = \frac{\mu_-(\kappa_+ + 1)}{RD_2}\{b_1\mu_+\mu_-(s_\theta - s_r) + a_3 R^2[s_r s_\theta R\Gamma^{-}_{+} + 2\mu_+\mu_-(2s_\theta - s_r)]\},$$

$$d_1 = \frac{R\mu_-(\kappa_+ + 1)}{D_2}\{b_1[s_r s_\theta R\Gamma^{+}_{-} + 6\mu_+\mu_- s_r] + 3a_3 R^2[s_r s_\theta R(\Gamma^{+}_{-} - \Gamma^{-}_{+}) + 4s_r\mu_+\mu_-]\},$$

where

$$D_2 = s_r s_\theta R^2 \Gamma^{-}_{+}\Gamma^{+}_{-} + (s_r + s_\theta)R(3\Gamma^{-}_{+} + \Gamma^{+}_{-})\mu_+\mu_- + 12\mu_+^2\mu_-^2.$$