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ON SMOOTH BIFURCATIONS IN NON-ASSOCIATIVE ELASTOPLASTICITY

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ABSTRACT

The second-order incremental constitutive equations proposed by Petryk and Thermann [(1985) Second-order bifurcation in elastic–plastic solids. *J. Mech. Phys. Solids* **33,** 577–593] are generalized to include non-associativity of the plastic flow rule. It is shown that the exclusion principle of Raniecki [(1979) Uniqueness criteria in solids with non-associated plastic flow laws at finite deformations. *Bull. Acad. Polon. Ser. Sci. Tech.* **XXVII**(8–9), 391–399] for first-order bifurcations is sufficient to exclude second-order bifurcations. The result holds true under specific regularity conditions and, accepting stronger regularity conditions, is extended to the case of *n*th-order bifurcations. Copyright © 1996 Elsevier Science Ltd

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1. INTRODUCTION

During quasistatic deformation of a solid body, the incremental response due to prescribed small perturbations of the boundary conditions may cease to be unique. This occurrence is indicated as a bifurcation of the equilibrium path. The usual incremental theories of deformation refer to the velocity–stress rate problem (first-order rate problem in the following). This problem corresponds to retaining only the first-order term in a series expansion of all quantities specifying the deformation of the body, which can be thought to depend on a time-like parameter (we refer here to rate independent materials). The general theoretical framework for stability and uniqueness of the first-order rate problem was stated by Hill (1958, 1959) with reference to associative elastoplasticity and later expressed through an alternative formulation by Nguyen (1987). In particular, Hill (1958) obtained an exclusion condition for first-order bifurcations (also called angular bifurcations). At least in principle, however, any incremental and sufficiently smooth function of the time-like parameter governing the deformation can be expanded into a series up to an arbitrary order and, therefore, bifurcations of higher order (called also smooth bifurcations) could occur even when first order bifurcations are excluded. This problem was analyzed by Triantafyllidis (1983), Petryk and Thermann (1985), Léger and Potier-Ferry (1988, 1993), Nguyen and Triantafyllidis (1989) and Cheng and Lu (1993). In particular, Triantafyllidis provided an example of smooth bifurcation occurring in a simple, and therefore significant, elastoplastic structure. This example was discussed and generalized by Léger and Potier-Ferry. Under appropriate regularity conditions of the fields, Petryk and Thermann proved that the Hill exclusion condition does in fact also exclude second-order bifurcations and this result was extended by Nguyen and Triantafyllidis, and later, using a different formalism, by Cheng and Lu, to *n*th-order bifurcations. It may be important to remark that all these proofs bear on regularity assumptions of the fields, that become more and more restrictive when the order of possible bifurcation becomes higher.

All of the above mentioned works refer to time independent elastoplastic solids with the associative flow rule. Non-associative flow rules were introduced by Mróz (1963, 1966) and Mandel (1966) and soon become the object of a very intensive research. This is due to two main reasons. First, the non-associative flow rules can represent the behaviour of many engineering materials more accurately than the associative flow rule [we can mention : single crystals (Qin and Bassani, 1992), porous metals and metals showing the SD effect (Spitzig *et al*., 1976), compacted powders (Bortzmeyer, 1992), polymers (Whitney and Andrews, 1967), structural ceramics (Chen and Reyes-Morel, 1986), concrete, rocks and soils (Lade and Kim, 1995)]. Moreover, many materials (e.g. rock-like materials) exhibit a coupling between elastic and plastic deformation, which has the effect of making the flow rule formally similar to the non-associative rule (Hueckel, 1976; Maier and Hueckel, 1979; Capurso, 1979). Secondly, the non-associativity of the flow rule produces a lowering in the threshold for elastoplastic bifurcations, which would be unrealistically high in the case of the associative flow rule (Rice, 1977; Needleman, 1979). Despite the importance of non-associative plasticity, the context of bifurcation and stability remains not fully understood for these solids [see the lucid discussion given by Needleman (1979)]. After pioneering work by Maier (1970), resting in the framework of the infinitesimal theory, Raniecki (1979) and Raniecki and Bruhns (1981) gave a generalization of the Hill exclusion principle, valid for non-associative plasticity with a smooth yield function and plastic potential. The Hill exclusion principle is based on the definition of a linear comparison solid, which in turn coincides with the plastic branch of the constitutive operator. Roughly speaking, the failure of the exclusion principle corresponds to an effective bifurcation of the underlying elastoplastic solid only when conditions of continuing yielding in the plastic zone occur for both the fundamental and the bifurcated solutions [Hutchinson (1973); see also the detailed discussion in Léger and Potier-Ferry (1988)]. Similarly, the Raniecki and Bruhns exclusion principle is based on the definition of a family of linear comparison solids, which are in some sense a symmetrization of the loading branch of the constitutive operator. As a consequence, the failure of the Raniecki and Bruhns exclusion principle is less critical for bifurcation in non-associative plasticity than is the failure of the Hill exclusion principle in associative plasticity. In other words, when the exclusion principle fails for non-associative flow rule, a bifurcation still may not be possible for the underlying elastoplastic solid. Applications of the Raniecki and Bruhns exclusion principle were given by Bruhns (1982), Tvergaard (1982) and Tomita *et al*. (1988), whereas investigations on strong ellipticity for the comparison solids can be found in Bigoni and Zaccaria (1992a, b).

In the present paper, we generalize the result obtained by Petryk and Thermann

(1985) to time independent, non-associative elastoplastic constitutive laws with smooth yield function and plastic potential. Therefore, second-order constitutive laws are introduced for non-associative elastoplastic materials and the second-order problem is posed. Under the same regularity assumptions of Petryk and Thermann, it is proved that the exclusion condition of Raniecki and Bruhns for first-order bifurcations to exclude second-order bifurcations is still valid. Accepting the restrictive regularity assumptions of Cheng and Lu, the result can be extended to nth-order bifurcations, as is shown in the last section of the paper.

2. FIRST- AND SECOND-ORDER RATE CONSTITUTIVE EQUATIONS

The first-order incremental constitutive equations, relating the material derivatives of two work-conjugate [in the Hill (1978) sense], symmetric, stress and strain measures can be written in the form

$$\dot{\mathbf{S}}_0 = \mathbb{L}\dot{\mathbf{E}}, \quad \mathbb{L} = \mathbb{E} - \frac{\alpha}{g}\mathbf{P} \otimes \mathbf{Q}, \tag{2.1}$$

where $\mathbb{E}$ is the incremental elastic fourth-order tensor having minor symmetries (its major symmetries are not needed for subsequent calculations), $\mathbf{Q}$ is the yield function gradient in strain space, $\mathbf{P}$ the plastic flow mode tensor in strain space (possibly the gradient of a sufficiently regular plastic potential function) and g the plastic modulus, which is assumed to be strictly positive (to exclude locking materials). The scalar α takes the two discrete values 0 and 1, to distinguish, respectively, between the elastic and plastic response.

Constitutive equation (2.1) is related to the existence of the yield function $f(\mathbf{E}, \mathbf{H})$ (defined here in the strain space), i.e. a sufficiently smooth scalar function of strain and internal variables $\mathbf{H}$ ($\mathbf{H}$ symbolizes a collection of scalars, vectors or tensors, depending on the complete history of inelastic straining), which associates negative values to elastic states and the value zero when plastic incremental deformations are possible. In the first-order rate theory, to which (2.1) is referred, elastic behaviour, plastic loading and elastic unloading are defined in the usual way†

$$\begin{aligned} \alpha = 0 \quad & \text{if } f(\mathbf{E}, \mathbf{H}) < 0 \quad \text{(elastic state)} \\ & \text{or } f(\mathbf{E}, \mathbf{H}) = 0 \quad \text{and} \quad \mathbf{Q} \cdot \dot{\mathbf{E}} < 0 \quad \text{(elastic unloading)} \\ \alpha = 1 \quad & \text{if } f(\mathbf{E}, \mathbf{H}) = 0 \quad \text{and} \quad \mathbf{Q} \cdot \dot{\mathbf{E}} > 0 \quad \text{(plastic loading)} \end{aligned} \tag{2.2}$$

and the problem caused by neutral loading, i.e. $f(\mathbf{E}, \mathbf{H}) = 0$ and $\mathbf{Q} \cdot \dot{\mathbf{E}} = 0$, where α is not well defined, is bypassed using the continuity of constitutive relation (2.1) between $\dot{\mathbf{S}}_0$ and $\dot{\mathbf{E}}$. In (2.1) and (2.2), $\mathbb{E}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{H}$ and g, as well as f, are all functionals of the entire path of deformation reckoned from some ground state. Their dependence on the deformation history is to some extent not essential in the following analysis, but has to be taken sufficiently smooth to make the series expansion in the time-like

† Throughout the paper, a dot over a symbol denotes the forward time rate, whereas $\mathbf{A} \cdot \mathbf{B}$ denotes, as in Gurtin (1981), the natural inner product of second-order tensors $\mathbf{A}$ and $\mathbf{B}$.

parameter possible. This assumption is in reality more restrictive than it may appear. It excludes in fact discontinuities in the tangent modulus with respect to the stress and/or the strain (except when this discontinuity occurs at the elastic–plastic transition and all the body is everywhere homogeneously deformed in the plastic range, a condition frequently met, for instance, in the Shanley column model). Anyway, a smooth dependence of the tangent modulus on strain is often postulated, e.g. accepting Ramberg–Osgood type hardening laws (Hutchinson, 1973; Tvergaard and Needleman, 1975).

It may be worth mentioning that non-associative flow theories of plasticity at finite strain proposed by Rudnicki and Rice (1975), Rice (1977), Raniecki and Bruhns (1981), as well as the infinitesimal flow theories of plasticity based on smooth plastic potential and yield function, can be expressed in the form (2.1), (2.2), with an appropriate choice of the stress and strain rates, of the elastic moduli tensor and of tensors **P** and **Q**. Moreover, constitutive equations (2.1) and (2.2) reduce to those analyzed by Petryk and Thermann (1985)—in turn proposed by Hill (1958)—in the particular case of associative flow rule: $\mathbf{P} = \mathbf{Q}$.

Before generalizing the constitutive equations to second-order rate, it may be convenient to express them in terms of the material derivative of the first Piola–Kirchhoff stress tensor† $\dot{\mathbf{S}}$, and of the material derivative of the deformation gradient $\dot{\mathbf{F}}$. This is obviously possible for every choice of the stress and strain measure in (2.1); we will present this derivation when $\mathbf{S}_0$ and $\mathbf{E}$ are identified with the second Piola–Kirchhoff stress tensor and with the Green–Lagrange strain tensor. Hence,

$$\dot{\mathbf{S}} = \mathbf{F}\dot{\mathbf{S}}_0 + \mathbf{LKF}^{-\mathrm{T}}, \quad \dot{\mathbf{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{DF}, \tag{2.3}$$

where **K** is the Kirchhoff stress, **L** is the velocity gradient and **D** its symmetric part, i.e. the velocity of deformation. Due to the symmetry of **E**, the yield function gradient $\mathbf{Q} = \partial f/\partial \mathbf{E}$, is also symmetric, and the following property holds true

$$\mathbf{Q}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{DF} = \tfrac{1}{2}\mathbf{Q}\cdot[\mathbf{F}^{\mathrm{T}}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{F}] = \tfrac{1}{2}(\mathbf{FQ}\cdot\mathbf{LF}+\mathbf{FQ}\cdot\mathbf{L}^{\mathrm{T}}\mathbf{F}) = \mathbf{FQ}\cdot\dot{\mathbf{F}}. \tag{2.4}$$

From (2.4) it can immediately be concluded that the loading and unloading conditions (2.2) can be expressed in terms of $\mathbf{FQ}\cdot\dot{\mathbf{F}}$, instead of $\mathbf{Q}\cdot\dot{\mathbf{E}}$. Note also that the yield condition can clearly be written as a function of the deformation gradient, i.e. in the form $f_{\mathrm{F}}(\mathbf{F},\mathbf{H})$, and that its gradient with respect to **F** is **FQ**. This follows in fact from the chain rule of differentiation and from the symmetry of **Q**

$$\frac{\partial f_{\mathrm{F}}}{\partial \mathbf{F}}\cdot\mathbf{U} = \frac{\partial f}{\partial \mathbf{E}}\cdot\frac{\partial \mathbf{E}}{\partial \mathbf{F}}[\mathbf{U}] = \mathbf{Q}\cdot[\tfrac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{U}+\mathbf{U}^{\mathrm{T}}\mathbf{F})] = \mathbf{FQ}\cdot\mathbf{U}, \tag{2.5}$$

where **U** is any second-order tensor.‡

A substitution of (2.1) and $(2.3)_2$ in $(2.3)_1$ gives

† The first Piola–Kirchhoff stress tensor used here is the transpose of the nominal stress tensor employed, among others, by Hill (1978).

‡ From the definition of derivative:

$$\frac{\partial \mathbf{E}}{\partial \mathbf{F}}[\mathbf{U}] = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}[(\mathbf{F}+\varepsilon\mathbf{U})^{\mathrm{T}}(\mathbf{F}+\varepsilon\mathbf{U}) - \mathbf{F}^{\mathrm{T}}\mathbf{F}] = \tfrac{1}{2}(\mathbf{U}^{\mathrm{T}}\mathbf{F}+\mathbf{F}^{\mathrm{T}}\mathbf{U}).$$

$$\dot{\mathbf{S}} = \mathbb{C}\dot{\mathbf{F}}, \quad \mathbb{C} = \mathbb{G} - \frac{\alpha}{g}\mathbf{M} \otimes \mathbf{N}, \quad \mathbb{G} = \mathbb{B} + \mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}, \tag{2.6}$$

where $\mathbf{N} = \mathbf{FQ}$, $\mathbf{M} = \mathbf{FP}$, and tensor $\mathbb{B}$ is defined as

$$\mathbb{B} = (\mathbf{F} \boxtimes \mathbf{I})\mathbb{E}(\mathbf{F} \boxtimes \mathbf{I})^{\mathrm{T}}. \tag{2.7}$$

The tensor product denoted in (2.6) and (2.7) by symbol $\boxtimes$ was introduced by Del Piero (1979), for any second-order tensors **A**, **B**, **C**, as†

$$(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{ACB}^{\mathrm{T}}. \tag{2.8}$$

Turning attention now to the second-order rate problem, we have to generalize the condition for elastic unloading and plastic loading. This is done in the way proposed by Petryk and Thermann: at a given material point the deformation is elastic (plastic) if any sufficiently short segment of the deformation path, starting from the current state $(\mathbf{F}, \mathbf{H})$, remains within (leaves) the current elastic domain

$$\begin{aligned} \alpha = 0 \quad & \text{if } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) < 0 \quad \text{(elastic state)} \\ & \text{or } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) = 0 \quad \text{and} \quad f_{\mathrm{F}}(\mathbf{F}+\delta\mathbf{F}, \mathbf{H}) < 0 \quad \text{(elastic unloading)} \\ \alpha = 1 \quad & \text{if } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) = 0 \quad \text{and} \quad f_{\mathrm{F}}(\mathbf{F}+\delta\mathbf{F}, \mathbf{H}) > 0 \quad \text{(plastic loading)}, \end{aligned} \tag{2.9}$$

which is the generalization of condition (2.2). Note that the generalization of Prager consistency is $f_{\mathrm{F}}(\mathbf{F}+\delta\mathbf{F}, \mathbf{H}+\delta\mathbf{H}) = 0$, during plastic loading. In order to specify criteria (2.9) for second-order rate, we note that **F** (and **H**) depends on a time-like parameter t, governing the loading. Therefore, assuming that function $f_{\mathrm{F}}(\mathbf{F}, \mathbf{H})$ can be expanded in a Taylor series (with **H** held fixed)

$$\begin{aligned} f_{\mathrm{F}}(\mathbf{F}+\delta\mathbf{F}, \mathbf{H}) - f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) &= f_{\mathrm{F}}(\mathbf{F}(t+\delta t), \mathbf{H}(t)) - f_{\mathrm{F}}(\mathbf{F}(t), \mathbf{H}(t)) \\ &= \left[\dot{\mathbf{F}} \cdot \left.\frac{\partial f_{\mathrm{F}}}{\partial \mathbf{F}}\right|_{(t)}\right]\delta t + \frac{1}{2}\left[\dot{\mathbf{F}} \cdot \left.\frac{\partial^2 f_{\mathrm{F}}}{\partial \mathbf{F}^2}\right|_{(t)} \dot{\mathbf{F}} + \ddot{\mathbf{F}} \cdot \left.\frac{\partial f_{\mathrm{F}}}{\partial \mathbf{F}}\right|_{(t)}\right]\delta t^2 + o(\delta t)^2. \end{aligned} \tag{2.10}$$

Noting from (2.6) that plastic first-order rate of deformation, and therefore also $\dot{\mathbf{H}}$, vanishes when $\mathbf{N} \cdot \dot{\mathbf{F}} = 0$, we arrive at the identity

$$\dot{\mathbf{F}} \cdot \frac{\partial^2 f_{\mathrm{F}}}{\partial \mathbf{F}^2} \dot{\mathbf{F}} + \ddot{\mathbf{F}} \cdot \frac{\partial f_{\mathrm{F}}}{\partial \mathbf{F}} \dot{\mathbf{F}} = (\mathbf{N} \cdot \dot{\mathbf{F}})^{\boldsymbol{\cdot}}, \tag{2.11}$$

(valid when $\mathbf{N} \cdot \dot{\mathbf{F}} \leqslant 0$) which allows us to write conditions (2.9) in the form

$$\begin{aligned} \alpha = 0 \quad & \text{if } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) < 0 \\ & \text{or } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) = 0 \quad \text{and} \quad \mathbf{N} \cdot \dot{\mathbf{F}} < 0 \\ & \text{or } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) = \mathbf{N} \cdot \dot{\mathbf{F}} = 0 \quad \text{and} \quad (\mathbf{N} \cdot \mathbf{F})^{\boldsymbol{\cdot}} < 0 \\ \alpha = 1 \quad & \text{if } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) = 0 \quad \text{and} \quad \mathbf{N} \cdot \dot{\mathbf{F}} > 0 \\ & \text{or } f_{\mathrm{F}}(\mathbf{F}, \mathbf{H}) = \mathbf{N} \cdot \dot{\mathbf{F}} = 0 \quad \text{and} \quad (\mathbf{N} \cdot \dot{\mathbf{F}})^{\boldsymbol{\cdot}} > 0. \end{aligned} \tag{2.12}$$

† The components of tensors $\mathbf{A} \boxtimes \mathbf{B}$ and $\mathbb{B}$ are defined as

$$(\mathbf{A} \boxtimes \mathbf{B})_{ijhk} = A_{ih}B_{jk}, \quad \mathbb{B}_{ijhk} = F_{is}\mathbb{E}_{sjtk}F_{ht}.$$

We are now in a position to formulate the second-order constitutive laws. To this end, we assume that the rate of change of tensor $\mathbb{C}$ be defined by the prior strain history and by the actual values of strain rate and parameter α, defined in turn by condition (2.12). Therefore,

$$\ddot{\mathbf{S}} = \dot{\mathbb{C}}\dot{\mathbf{F}} + \mathbb{C}\ddot{\mathbf{F}}, \quad \dot{\mathbb{C}} = \dot{\mathbb{G}} - \frac{\alpha}{g}\left[-\frac{\dot{g}}{g}\mathbf{M} \otimes \mathbf{N} + \dot{\mathbf{M}} \otimes \mathbf{N} + \mathbf{M} \otimes \dot{\mathbf{N}} \right]. \tag{2.13}$$

Parameter α is not well defined when $\mathbf{N}\cdot\dot{\mathbf{F}} = (\mathbf{N}\cdot\dot{\mathbf{F}})^{\cdot} = 0$, but the dependence of $\ddot{\mathbf{S}}$ on $\ddot{\mathbf{F}}$ still remains continuous. This can be easily proved by observing that the quantity

$$\begin{aligned} \ddot{\mathbf{S}}(\alpha = 1) - \ddot{\mathbf{S}}(\alpha = 0) &= -\frac{1}{g}\left[\mathbf{M}(\mathbf{N}\cdot\ddot{\mathbf{F}}) - \frac{\dot{g}}{g}\mathbf{M}(\mathbf{N}\cdot\dot{\mathbf{F}}) + \dot{\mathbf{M}}(\mathbf{N}\cdot\dot{\mathbf{F}}) + \mathbf{M}(\dot{\mathbf{N}}\cdot\dot{\mathbf{F}}) \right] \\ &= -\frac{1}{g}\left\{ \mathbf{M}(\mathbf{N}\cdot\dot{\mathbf{F}})^{\cdot} - \left[\frac{\dot{g}}{g}\mathbf{M} - \dot{\mathbf{M}} \right](\mathbf{N}\cdot\dot{\mathbf{F}}) \right\} \end{aligned} \tag{2.14}$$

vanishes when $\mathbf{N}\cdot\dot{\mathbf{F}} = (\mathbf{N}\cdot\dot{\mathbf{F}})^{\cdot} = 0$.

It should be noted that the second-order rate constitutive equation (2.13) is linear in $\ddot{\mathbf{F}}$ (but non-homogeneous) either when $\mathbf{N}\cdot\dot{\mathbf{F}} < 0$ or $\mathbf{N}\cdot\dot{\mathbf{F}} > 0$, and becomes piecewise-linear when $\mathbf{N}\cdot\dot{\mathbf{F}} = 0$.

3. UNIQUENESS FOR THE REGULAR SECOND-ORDER RATE PROBLEM

In order to formulate the second-order rate problem, we assume that the current geometry and the state of the body is known, as well as the solution of the first-order rate problem, namely, the velocity field and the related stress rate, both corresponding to a prescribed increment in the boundary conditions. In a second-order rate problem, the class of functions where solutions have to be found should be widened to include possibility of discontinuities of accelerations. As in Petryk and Thermann (1985), we restrict our study to the regular problem, namely, we assume stress, displacements and velocity to be continuously differentiable (i.e. smooth) functions of the place $\mathbf{X}$, whereas first-order stress rate and acceleration are assumed continuous, piecewise-smooth functions of the place. The second-order rate of stress, the acceleration gradient and the constitutive tensor $(2.13)_2$ can suffer jumps across regular surfaces but remain continuously differentiable in the rest of the body, shortly, they are piecewise smooth. We do not consider, for simplicity, body forces and therefore, the local conditions of equilibrium give, for first- and second-order rate†

† Differently from Hill (1978), the divergence operator of a smooth tensor field $\mathbf{S}$ (with respect to the material coordinates) is defined here as in Gurtin (1981):

$$(\mathrm{Div}\,\mathbf{S})\cdot\mathbf{a} = \mathrm{Div}(\mathbf{S}^{T}\mathbf{a}),$$

for every (constant) vector $\mathbf{a}$.

$$\mathrm{Div}\,\dot{\mathbf{S}} = \mathbf{0} \quad \text{in } \Omega, \tag{3.1}$$

$$\mathrm{Div}\,\ddot{\mathbf{S}} = \mathbf{0} \quad \text{in } \Omega \backslash \Sigma, \tag{3.2}$$

$$[\![\ddot{\mathbf{S}}]\!]\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma, \tag{3.3}$$

where Ω is the region occupied by the body in the reference configuration, having piecewise-smooth closure $\partial\Omega$. Σ is any surface of discontinuity of $\ddot{\mathbf{S}}$ in the reference configuration, $\mathbf{v}$ its unit normal vector, and the symbol $[\![\ ]\!]$ denotes the jump of the relevant argument.

We consider mixed boundary conditions where the displacement and the nominal tractions are prescribed, sufficiently smooth, functions of place and time over specific portions $\partial\Omega_\chi$ and $\partial\Omega_\sigma$ of the boundary in the reference configuration ($\partial\Omega_\chi \bigcup \partial\Omega_\sigma = \partial\Omega$). Differentiation of these functions with respect to time gives the first-order rate boundary conditions

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\xi}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\chi, \\ \dot{\mathbf{S}}\mathbf{n} &= \dot{\boldsymbol{\sigma}}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\sigma, \end{aligned} \tag{3.4}$$

and the second-order rate boundary conditions

$$\begin{aligned} \ddot{\mathbf{x}} &= \ddot{\xi}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\chi, \\ \ddot{\mathbf{S}}\mathbf{n} &= \ddot{\boldsymbol{\sigma}}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\sigma, \end{aligned} \tag{3.5}$$

where $\mathbf{n}$ denotes the outward unit vector (in the reference configuration) normal to that part of the boundary where tractions are prescribed.

The constitutive equations (2.6) and (2.13) can be employed to write $\dot{\mathbf{S}}$ and $\ddot{\mathbf{S}}$ in (3.1)–(3.3) and $(3.4)_2$ and $(3.5)_2$ in terms of velocity, acceleration and their gradients

$$\mathrm{Div}(\mathbb{C}\dot{\mathbf{F}}) = \mathbf{0} \quad \text{in } \Omega, \tag{3.6}$$

$$(\mathbb{C}\dot{\mathbf{F}})\mathbf{n} = \dot{\boldsymbol{\sigma}}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\sigma, \tag{3.7}$$

$$\mathrm{Div}(\dot{\mathbb{C}}\dot{\mathbf{F}} + \mathbb{C}\ddot{\mathbf{F}}) = \mathbf{0} \quad \text{in } \Omega \backslash \Sigma, \tag{3.8}$$

$$[\![\dot{\mathbb{C}}\dot{\mathbf{F}} + \mathbb{C}\ddot{\mathbf{F}}]\!]\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma, \tag{3.9}$$

$$(\dot{\mathbb{C}}\dot{\mathbf{F}} + \mathbb{C}\ddot{\mathbf{F}})\mathbf{n} = \ddot{\boldsymbol{\sigma}}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\sigma. \tag{3.10}$$

Therefore, the regular second-order rate problem can be stated as follows: given a solution of first-order rate problem, i.e. a continuously twice differentiable velocity field satisfying conditions $(3.4)_1$, (3.6) and (3.7), find a continuous and piecewise continuously twice differentiable (shortly, admissible) acceleration field, which satisfies conditions $(3.5)_1$ and (3.8)–(3.10).

If this problem admits two solutions, say $\ddot{\mathbf{x}}_1$ and $\ddot{\mathbf{x}}_2$, their difference $\Delta\ddot{\mathbf{x}}$ defines an admissible acceleration field with a gradient $\Delta\ddot{\mathbf{F}}$ satisfying (3.8) in Ω/Σ and (3.9) on Σ and corresponding to homogeneous conditions on $\partial\Omega$. On application of the divergence theorem it follows that

$$\int_{\Omega} \Delta\ddot{\mathbf{S}} \cdot \Delta\ddot{\mathbf{F}} = 0. \tag{3.12}$$

Therefore, a sufficient condition to exclude second-order rate bifurcations is

$$\int_{\Omega} \Delta\ddot{\mathbf{S}} \cdot \Delta\ddot{\mathbf{F}} > 0, \tag{3.13}$$

for all pairs of distinct, admissible acceleration fields taking the given values $(3.5)_1$ on $\partial\Omega_\chi$.

Note that the exclusion condition (3.13) would be true even replacing "$>$" with "$<$". As in Hill (1958), we do not consider this possibility here.

4. EXCLUSION CONDITION OF REGULAR SECOND-ORDER RATE BIFURCATIONS

Raniecki (1979) and Raniecki and Bruhns (1981) have introduced a family of linear comparison solids defined by the constitutive tensor (for every $\psi \in \mathbb{R}^+$)

$$\mathbb{C}^r = \mathbb{G} - \frac{1}{4\psi g}(\mathbf{M}+\psi\mathbf{N}) \otimes (\mathbf{M}+\psi\mathbf{N}), \tag{4.1}$$

such that the following comparison theorem holds true

$$\Delta\dot{\mathbf{S}} \cdot \Delta\dot{\mathbf{F}} \geqslant \Delta\dot{\mathbf{F}} \cdot (\mathbb{C}^r \Delta\dot{\mathbf{F}}), \tag{4.2}$$

for every difference of tensors $\Delta\mathbf{F} = \dot{\mathbf{F}}_1 - \dot{\mathbf{F}}_2$ and related difference $\Delta\dot{\mathbf{S}} = \mathbb{C}\dot{\mathbf{F}}_1 - \mathbb{C}\dot{\mathbf{F}}_2$. Therefore, the first-order exclusion condition for bifurcation

$$\int_{\Omega} \Delta\dot{\mathbf{S}} \cdot \Delta\dot{\mathbf{F}} > 0, \tag{4.3}$$

(for all pairs of distinct, continuous and piecewise continuously twice differentiable velocity fields, taking the given values on $\partial\Omega_\chi$) is necessary satisfied when the stronger condition holds true

$$\int_{\Omega} \dot{\mathbf{F}} \cdot (\mathbb{C}^c \dot{\mathbf{F}}) > 0, \tag{4.4}$$

for all continuous and piecewise continuously twice differentiable velocity fields, satisfying homogeneous conditions on $\partial\Omega_\chi$. The comparison solid $\mathbb{C}^c$ in (4.4) is equal, by definition, to $\mathbb{C}^r$ in the current plastic zone [i.e. where $f_F(\mathbf{F}, \mathbf{H}) = 0$] and to $\mathbb{G}$ in the current elastic zone [i.e. where $f_F(\mathbf{F}, \mathbf{H}) < 0$].

The advantage of condition (4.4) compared to (4.3) is in the linearity of the integrand. In particular, bifurcation is excluded if it is possible to find a positive defined tensor $\mathbb{C}^c$ at every point of Ω.

We show now that, under the field regularity conditions invoked in Section 3, condition (4.4) also rules out second-order bifurcations. By assumption, the first-order rate problem has unique solution, let us assume now the existence of two solutions of the acceleration problem which define $\Delta\ddot{\mathbf{F}}$. We prove now the second-order rate comparison theorem

$$\Delta\ddot{\mathbf{S}}\cdot\Delta\ddot{\mathbf{F}} \geqslant \Delta\ddot{\mathbf{F}}\cdot(\mathbb{C}^r\Delta\ddot{\mathbf{F}}), \tag{4.5}$$

for every difference of tensors $\Delta\ddot{\mathbf{F}} = \ddot{\mathbf{F}}_1 - \ddot{\mathbf{F}}_2$ and related difference $\Delta\ddot{\mathbf{S}} = (\mathbb{C}\dot{\mathbf{F}}_1)^{\cdot} - (\mathbb{C}\dot{\mathbf{F}}_2)^{\cdot}$, for which $\dot{\mathbf{F}}_1 = \dot{\mathbf{F}}_2 = \dot{\mathbf{F}}$ (the first-order solution is unique), but $\ddot{\mathbf{F}}_1 \neq \ddot{\mathbf{F}}_2$.

Proof:

In the two cases $\mathbf{N}\cdot\dot{\mathbf{F}} < 0$ or $\mathbf{N}\cdot\dot{\mathbf{F}} > 0$ (i.e. parameter α is zero or 1 for each solution, respectively), the constitutive equation (2.13) is linear in $\ddot{\mathbf{F}}$, hence $\Delta\ddot{\mathbf{S}} = \mathbb{C}\Delta\ddot{\mathbf{F}}$, and

$$\Delta\ddot{\mathbf{F}}\cdot(\mathbb{C}-\mathbb{C}^r)\,\Delta\ddot{\mathbf{F}} = \begin{cases} \dfrac{(\mathbf{M}\cdot\Delta\ddot{\mathbf{F}}+\psi\mathbf{N}\cdot\Delta\ddot{\mathbf{F}})^2}{4\psi g} \geqslant 0 & \text{if } \mathbf{N}\cdot\dot{\mathbf{F}} < \mathbf{0} \\ \dfrac{(\mathbf{M}\cdot\Delta\ddot{\mathbf{F}}-\psi\mathbf{N}\cdot\Delta\ddot{\mathbf{F}})^2}{4\psi g} \geqslant 0 & \text{if } \mathbf{N}\cdot\dot{\mathbf{F}} > 0 \end{cases}, \tag{4.6}$$

so that (4.5) is verified. The condition $\mathbf{N}\cdot\dot{\mathbf{F}} = 0$ only needs to be examined. In this case, we have

$$\Delta\ddot{\mathbf{S}} = \mathbb{G}\Delta\ddot{\mathbf{F}} - \frac{\alpha_1}{g}(\mathbf{N}\cdot\ddot{\mathbf{F}}_1 + \dot{\mathbf{N}}\cdot\dot{\mathbf{F}})\mathbf{M} + \frac{\alpha_2}{g}(\mathbf{N}\cdot\ddot{\mathbf{F}}_2 + \dot{\mathbf{N}}\cdot\dot{\mathbf{F}})\mathbf{M}, \tag{4.7}$$

where indices 1 and 2 refer to the two different solutions. The two cases $\alpha_1 = \alpha_2 = 0$ and $\alpha_1 = \alpha_2 = 1$ lead to the same condition (4.6), and so only the two cases $\alpha_1 = 0$ and $\alpha_2 = 1$, and $\alpha_1 = 1$ and $\alpha_2 = 0$ need to be analyzed. In particular, we have to prove that

$$4\psi\{\alpha_1[\mathbf{N}\cdot\ddot{\mathbf{F}}_1 + \dot{\mathbf{N}}\cdot\dot{\mathbf{F}}] - \alpha_2[\mathbf{N}\cdot\ddot{\mathbf{F}}_2 + \dot{\mathbf{N}}\cdot\dot{\mathbf{F}}]\}\mathbf{M}\cdot\Delta\ddot{\mathbf{F}} \leqslant (\mathbf{M}\cdot\Delta\ddot{\mathbf{F}} + \psi\mathbf{N}\cdot\Delta\ddot{\mathbf{F}})^2, \tag{4.8}$$

which, after algebraic manipulation, can be written as

$$\{(\alpha_2-\alpha_1)\mathbf{M}\cdot\Delta\ddot{\mathbf{F}} + \psi[\mathbf{N}\cdot\ddot{\mathbf{F}}_1 + \mathbf{N}\cdot\ddot{\mathbf{F}}_2] + 2\psi\dot{\mathbf{N}}\cdot\dot{\mathbf{F}}\}^2 - 4\psi^2(\mathbf{N}\cdot\dot{\mathbf{F}}_1)^{\cdot}(\mathbf{N}\cdot\dot{\mathbf{F}}_2)^{\cdot} \geqslant 0, \tag{4.9}$$

valid when $\alpha_1 = 0$ and $\alpha_2 = 1$, or $\alpha_1 = 1$ and $\alpha_2 = 0$. To conclude the proof it is sufficient to note from the loading–unloading second-order conditions (2.12) that the non-squared term on the left hand side of condition (4.9) is always positive.

In the elastic zone, the second-order constitutive relation (2.13), reduces to the elastic constitutive relation, which is linear in $\ddot{\mathbf{F}}$, so that, for $\dot{\mathbf{F}}_1 = \dot{\mathbf{F}}_2$, $\Delta\ddot{\mathbf{S}}\cdot\Delta\ddot{\mathbf{F}} = \Delta\ddot{\mathbf{F}}\cdot(\mathbb{G}\Delta\ddot{\mathbf{F}})$. Therefore, with the comparison theorem (4.5), the sufficient condition for uniqueness of second-order rate problem becomes

$$\int_\Omega \ddot{\mathbf{F}}\cdot(\mathbb{C}^c\ddot{\mathbf{F}}) > 0, \tag{4.10}$$

for all admissible acceleration fields satisfying homogeneous conditions on $\partial\Omega_\chi$.

Condition (4.10) is equivalent to condition (4.4). Therefore, the exclusion condition obtained by Raniecki and Bruhns for first-order rate bifurcations, is sufficient to exclude any second-order bifurcation of the regular second-order rate problem.

It should be noted finally that, the first-order solution being known, the parts of the plastic zone are known where $\mathbf{N}\cdot\dot{\mathbf{F}}$ is different from zero. In these zones, the value of α is known. Therefore, as in Cheng and Lu (1993), a "refined" definition of comparison solid $\mathbb{C}^c$ could be proposed, in which $\mathbb{C}^c = \mathbb{C}^r$ where $f_F(\mathbf{F},\mathbf{H}) = \mathbf{N}\cdot\dot{\mathbf{F}} = 0$, $\mathbb{C}^c = \mathbb{G}$ where $f_F(\mathbf{F},\mathbf{H}) < 0$ or $f_F(\mathbf{F},\mathbf{H}) = 0$ and $\mathbf{N}\cdot\dot{\mathbf{F}} < 0$, and, finally, $\mathbb{C}^c = \mathbb{C}$, with $\alpha = 1$, where $f_F(\mathbf{F},\mathbf{H}) = 0$ and $\mathbf{N}\cdot\dot{\mathbf{F}} > 0$. If we denote with $\mathbb{C}^*$ this solid, the property

$$\Delta\ddot{\mathbf{S}}\cdot\Delta\ddot{\mathbf{F}} \geqslant \Delta\ddot{\mathbf{F}}\cdot(\mathbb{C}^*\Delta\ddot{\mathbf{F}}), \tag{4.11}$$

holds true in every point of the body, for all $\Delta\ddot{\mathbf{F}} = \ddot{\mathbf{F}}_1 - \ddot{\mathbf{F}}_2$ and related difference $\Delta\ddot{\mathbf{S}} = (\mathbb{C}\dot{\mathbf{F}}_1)^\cdot - (\mathbb{C}\dot{\mathbf{F}}_2)^\cdot$, such that $\dot{\mathbf{F}}_1 = \dot{\mathbf{F}}_2$. The definition of this "refined" comparison solid, seems however to be useless in the present context. In fact, when the exclusion condition for first-order bifurcation fails, nothing is *a priori* known about uniqueness of first-order solution, so that the hypothesis $\dot{\mathbf{F}}_1 = \dot{\mathbf{F}}_2$ loses validity.

5. EXCLUSION CONDITION OF REGULAR *n*th-ORDER RATE BIFURCATIONS

The extension of results of Sections 2–4 to *n*th-order bifurcations can be obtained, when the strong regularity conditions introduced by Cheng and Lu (1993) are imposed. The result parallels the same arguments used previously, and therefore are exposed briefly in the section.

We refer to the regular *n*th-order rate problem, in which all $n-1$ rate fields are known. Furthermore, the stress, together with its $n-2$ rates, and the displacement, together with its $n-1$ rates, are assumed smooth functions of the place. The $n-1$ rates of stress and velocity are assumed continuous, piecewise-smooth functions of place. The *n*th-rates of stress and of displacement gradient are assumed piecewise smooth, and therefore they can suffer jumps across regular discontinuity surfaces. As remarked by Petryk (1993), it is important to note that the regularity assumption for the *n*th-order rate problem is much less satisfactory than is the regularity assumption for the second-order rate problem. High-order rate fields may in fact be discontinuous and have other singularities absent in low-order rate problems.

The loading–unloading criterion (2.9) can be expressed for *n*th-order rate as:

$$\begin{aligned}
\alpha = 0 \quad &\text{if} \quad f_F(\mathbf{F},\mathbf{H}) < 0\\
&\text{or} \quad f_F(\mathbf{F},\mathbf{H}) = 0 \quad \text{and} \quad \mathbf{N}\cdot\dot{\mathbf{F}} < 0\\
&\text{or} \quad f_F(\mathbf{F},\mathbf{H}) = (\mathbf{N}\cdot\dot{\mathbf{F}})^{(k)} = 0, \quad \text{and} \quad (\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} < 0,\\
&\qquad \text{for a given } i \in [0, n-1] \text{ and all } k \in [0, i-1].\\
\alpha = 1 \quad &\text{if} \quad f_F(\mathbf{F},\mathbf{H}) = 0 \quad \text{and} \quad \mathbf{N}\cdot\dot{\mathbf{F}} > 0\\
&\text{or} \quad f_F(\mathbf{F},\mathbf{H}) = (\mathbf{N}\cdot\dot{\mathbf{F}})^{(k)} = 0, \quad \text{and} \quad (\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} > 0,\\
&\qquad \text{for a given } i \in [0, n-1] \text{ and all } k \in [0, i-1].
\end{aligned} \tag{5.1}$$

where the superscript index $(\ )^{(i)}$ denotes the *i*th-derivative with respect to time.

The nth-order rate constitutive equations can be written as

$$\overset{n}{\mathbf{S}} = \sum_{i=0}^{n-1} \binom{n-1}{i} \overset{i}{\mathbb{C}} \overset{n-i}{\mathbf{F}}, \tag{5.2}$$

where the index over tensors denote time derivative, $\overset{\circ}{\mathbb{C}} = \mathbb{C}$ and the binomial coefficient is defined as

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}. \tag{5.3}$$

Equation (5.2) can be transformed into

$$\overset{n}{\mathbf{S}} = \sum_{i=0}^{n-1} \binom{n-1}{i} \overset{i}{\mathbb{G}} \overset{n-i}{\mathbf{F}} - \alpha \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{1}{g}\right)^{(i)} \sum_{j=0}^{n-1-i} \binom{n-1-i}{j} (\mathbf{N}\cdot\dot{\mathbf{F}})^{(j)} \overset{n-1-i-j}{\mathbf{M}}. \tag{5.4}$$

When $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} > 0$ or $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} < 0$, for $i < n-1$, the constitutive equation (5.2) is linear in $\overset{n}{\mathbf{F}}$. When $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} = 0$, for all $i \in [0, n-2]$, the constitutive equation (5.2) becomes piecewise-linear in $\overset{n}{\mathbf{F}}$. Finally, when $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} = 0$ for all $i \in [0, n-1]$, α is not well defined, but the constitutive response remains continuous. In fact, it can be immediately deduced from (5.4) that

$$\overset{n}{\mathbf{S}}(\alpha = 1) - \overset{n}{\mathbf{S}}(\alpha = 0) = 0, \tag{5.5}$$

where $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} = 0$, for all $i \in [0, n-1]$.

The regular nth-order problem with mixed boundary conditions, corresponds to finding a continuous piecewise twice differentiable field $\overset{n}{\mathbf{x}}$ and its constitutively related field $\overset{n}{\mathbf{S}}$ satisfying

$$\overset{n}{\mathbf{x}} = \overset{n}{\xi}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\chi,$$

$$\overset{n}{\mathbf{S}}\,\mathbf{n} = \overset{n}{\boldsymbol{\sigma}}(\mathbf{X}, t) \quad \text{on } \partial\Omega_\sigma, \tag{5.6}$$

$$\operatorname{Div} \overset{n}{\mathbf{S}} = \mathbf{0} \quad \text{in } \Omega\backslash\Sigma, \tag{5.7}$$

$$[\![\overset{n}{\mathbf{S}}]\!]\,\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma, \tag{5.8}$$

where Σ denotes a regular surface of discontinuity of $\overset{n}{\mathbf{S}}$, having unit normal $\mathbf{v}$, and $\mathbf{n}$ denotes the outward unit vector to $\partial\Omega_\sigma$.

If the nth-order problem admits a bifurcation, the difference between two solutions satisfies

$$\Delta\overset{n}{\mathbf{x}} = \mathbf{0} \quad \text{on } \partial\Omega_\chi,$$

$$\Delta\overset{n}{\mathbf{S}}\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega_\sigma, \tag{5.9}$$

$$\operatorname{Div} \Delta\overset{n}{\mathbf{S}} = \mathbf{0} \quad \text{in } \Omega\backslash\Sigma, \tag{5.10}$$

$$\left[\!\left[\overset{n}{\Delta\mathbf{S}} \right]\!\right] \mathbf{v} = \mathbf{0} \quad \text{on } \Sigma, \tag{5.11}$$

where, clearly, $\overset{n}{\Delta\mathbf{S}} = (\mathbb{C}\Delta\dot{\mathbf{F}})^{n-1}$. Therefore an exclusion condition for nth-order bifurcations is

$$\int_{\Omega} \overset{n}{\Delta\mathbf{S}} \cdot \overset{n}{\Delta\mathbf{F}} > 0, \tag{5.12}$$

for all pairs of distinct, continuous and piecewise continuously twice differentiable fields $\overset{n}{\mathbf{x}}$, taking the given values $(5.6)_1$ on $\partial\Omega_\chi$.

We prove now the nth-order rate comparison theorem

$$\overset{n}{\Delta\mathbf{S}} \cdot \overset{n}{\Delta\mathbf{F}} \geqslant \overset{n}{\Delta\mathbf{F}} \cdot (\mathbb{C}^{r} \overset{n}{\Delta\mathbf{F}}), \tag{5.13}$$

for every difference of tensors $\overset{n}{\Delta\mathbf{F}} = \overset{n}{\mathbf{F}}_1 - \overset{n}{\mathbf{F}}_2$ and related difference $\overset{n}{\Delta\mathbf{S}} = (\mathbb{C}\dot{\mathbf{F}}_1)^{n-1} - (\mathbb{C}\dot{\mathbf{F}}_2)^{n-1}$, for which $\overset{i}{\mathbf{F}}_1 = \overset{i}{\mathbf{F}}_2 = \overset{i}{\mathbf{F}}$ (for all $i \in [0, n-1]$), but $\overset{n}{\mathbf{F}}_1 \neq \overset{n}{\mathbf{F}}_2$ (representing two possible nth-order solutions).

When (5.13) holds, nth-order bifurcations are excluded when

$$\int_{\Omega} \overset{n}{\mathbf{F}} \cdot (\mathbb{C}^{c} \overset{n}{\mathbf{F}}) > 0, \tag{5.14}$$

for all continuous and piecewise continuously twice differentiable fields $\overset{n}{\mathbf{x}}$ satisfying homogeneous conditions on $\partial\Omega_\chi$. Therefore the Raniecki and Bruhns exclusion condition for first-order bifurcations is still able to exclude nth-order bifurcations. Note that the comparison solid $\mathbb{C}^{c}$ is defined as in (4.4) : equal to $\mathbb{G}$ and to $\mathbb{C}^{r}$ in the elastic and plastic zones of Ω, respectively.

Proof:

In the cases $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(k)} = 0$ and $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} \neq 0$, for a given $i \in [0, n-2]$ and all $k \in [0, i-1]$, (i.e. parameter α is zero or 1 for both solutions), the constitutive equation (5.4) is linear in $\overset{n}{\mathbf{F}}$, therefore, $\overset{n}{\Delta\mathbf{S}} = \mathbb{C}\overset{n}{\Delta\mathbf{F}}$ and

$$\overset{n}{\Delta\mathbf{F}} \cdot (\mathbb{C} - \mathbb{C}^{r}) \overset{n}{\Delta\mathbf{F}} = \begin{cases} \dfrac{\left(\mathbf{M}\cdot\overset{n}{\Delta\mathbf{F}} + \psi\mathbf{N}\cdot\overset{n}{\Delta\mathbf{F}}\right)^2}{4\psi g} \geqslant 0 & \text{if } (\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} < 0, \quad i < n-1 \\[2ex] \dfrac{\left(\mathbf{M}\cdot\overset{n}{\Delta\mathbf{F}} - \psi\mathbf{N}\cdot\overset{n}{\Delta\mathbf{F}}\right)^2}{4\psi g} \geqslant 0 & \text{if } (\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} > 0, \quad i < n-1 \end{cases}, \tag{5.15}$$

so that inequality (5.13) is satisfied. The condition $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)} = 0$ for all $i \in [0, n-2]$ only needs to be examined. In this case, condition (5.13) may be written

$$-\alpha_1(\mathbf{N}\cdot\dot{\mathbf{F}}_1)^{(n-1)}\mathbf{M}\cdot\Delta\overset{n}{\mathbf{F}}+\alpha_2(\mathbf{N}\cdot\dot{\mathbf{F}}_2)^{(n-1)}\mathbf{M}\cdot\Delta\overset{n}{\mathbf{F}}+\frac{1}{4\psi}\left[\mathbf{M}\cdot\Delta\overset{n}{\mathbf{F}}+\psi\mathbf{N}\cdot\Delta\overset{n}{\mathbf{F}}\right]^2\geqslant 0, \tag{5.16}$$

where indices 1 and 2 refer to the two different solutions. The two cases $\alpha_1=\alpha_2=0$ and $\alpha_1=\alpha_2=1$, lead to the same condition (5.15), therefore only the two cases $\alpha_1=0$ and $\alpha_2=1$, and $\alpha_1=1$ and $\alpha_2=0$ need to be analyzed. In particular, being the solution unique till order $n-1$, we can write

$$(\mathbf{N}\cdot\dot{\mathbf{F}}_\gamma)^{(n-1)}=\mathbf{N}\cdot\overset{n}{\mathbf{F}}_\gamma+\sum_{i=1}^{n-1}\binom{n-1}{i}\overset{i}{\mathbf{N}}\cdot\overset{n-i}{\mathbf{F}}, \tag{5.17}$$

where the index γ takes the two values 1 and 2, and, hence

$$(\mathbf{N}\cdot\dot{\mathbf{F}}_1)^{(n-1)}-(\mathbf{N}\cdot\dot{\mathbf{F}}_2)^{(n-1)}=\mathbf{N}\cdot\Delta\overset{n}{\mathbf{F}}. \tag{5.18}$$

Using (5.17) and (5.18), inequality (5.16) can be written in the form

$$\left\{(\alpha_2-\alpha_1)\mathbf{M}\cdot\Delta\overset{n}{\mathbf{F}}+\psi\left(\mathbf{N}\cdot\overset{n}{\mathbf{F}}_1+\mathbf{N}\cdot\overset{n}{\mathbf{F}}_2\right)+2\psi\sum_{i=1}^{n-1}\binom{n-1}{i}\overset{i}{\mathbf{N}}\cdot\overset{n-i}{\mathbf{F}}\right\}^2$$
$$-4\psi^2(\mathbf{N}\cdot\dot{\mathbf{F}}_1)^{(n-1)}(\mathbf{N}\cdot\dot{\mathbf{F}}_2)^{(n-1)}\geqslant 0, \tag{5.19}$$

valid when $\alpha_1=0$ and $\alpha_2=1$, or $\alpha_1=1$ and $\alpha_2=0$. To conclude the proof it is sufficient to note from the loading–unloading nth-order conditions (5.1) that the non-squared term on the left hand side of inequality (5.19) is always positive.

Therefore, the exclusion condition obtained by Raniecki and Bruhns for first-order rate bifurcations, is sufficient to exclude any nth-order bifurcation of the regular nth-order rate problem.

A particular consequence of the obtained results is that positive definiteness of the second order work (Mróz, 1963, 1966; Hueckel and Maier, 1977; Maier and Hueckel, 1979; Raniekci and Bruhns, 1981) in infinitesimal, non-associative elastoplasticity excludes both angular and smooth bifurcations of the regular infinitesimal rate problem.

Analogously to the case of the second-order rate problem, we can observe that a "refined" comparison solid could be defined replacing $\mathbb{C}^r$ in those parts of the plastic zone where all $i-1$ ($<n-3$) time derivatives of $\mathbf{N}\cdot\dot{\mathbf{F}}$ are null by $\mathbb{G}$, if $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)}<0$, or by $\mathbb{C}$ with $\alpha=1$, if $(\mathbf{N}\cdot\dot{\mathbf{F}})^{(i)}>0$. Analogously to the 2nd-order rate problem, this new comparison solid seems to be useless in the present context.

6. CONCLUSIONS

The results obtained in this paper extend to non-associative elastoplasticity findings by Petryk and Thermann (1985), Nguyen and Triantafyllidis (1989) and Cheng and Lu (1993). In particular, it has been shown that, under appropriate regularity conditions, the second- and, more generally, the nth-order rate problems for time independent, non-associative elastoplastic solids with smooth yield function and plastic potential admit a unique solution, when the Raniecki and Bruhns sufficient condition for uniqueness of the first-order rate problem holds true. However, it is well known that failure of this condition is in general not critical for bifurcation. This is a difference with respect to associative plasticity, where failure of the Hill exclusion condition is critical for bifurcation under broad hypotheses (Hutchinson, 1973). The counterpart of this in the case of non-associative flow rule was rarely investigated [(Bruhns, 1982; Tvergaard, 1982; Tomita *et al.*, 1988), see also the related works regarding bifurcations in the comparison solid "in loading": Needleman (1979), Vardoulakis (1981), Chau and Rudnicki (1990)]. Moreover, another unexplored, but important point is related to the effect of non-associativity on the post-critical behaviour.

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