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Effects of elastic anisotropy on strain localization and flutter instability in plastic solids

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Abstract

The influence of elastic anisotropy on elastic–plastic acceleration wave speeds is analyzed when the elastic anisotropy is described by a second-order fabric tensor. For a particular—and physically sound—structure of the elastic tensor, a correspondence principle is established that reduces the analysis of strain localization to a problem relative to elastic isotropy. Thus analytic solutions available in the framework of elastic–plastic constitutive laws with isotropic elasticity can be used. Applications show the strong effects of elastic anisotropy on all the localization characteristics.

For non-associative flow-laws, flutter instability, corresponding to the occurrence of complex conjugate squares of wave speeds, is considered. It is shown that, while this instability is precluded for elastic–plastic solids with elastic isotropy and obeying deviatoric associativity, this exclusion property is not preserved in the presence of anisotropic elasticity. In fact, we exhibit a particular structure of elasticity for which an infinitesimal amount of anisotropy is shown to give rise to flutter. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Mechanical anisotropy of engineering materials may result from oriented internal structures at different scales. Anisotropy may be inherent, due to the elaboration process where specific directional properties are designed or naturally produced, as for instance in the case of composites, foams, powders, multiphase solids, and most

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geomaterials. On the other hand, anisotropy may be induced by loadings, giving rise to various types of damage, as for instance in the case of concrete and highly deformed metals. However, invoking simplicity, elastic anisotropy is usually neglected in elastoplastic constitutive modeling. Indeed, embarking on a formulation that involves either inherent or induced elastic anisotropy is usually contradictory with tractability in algebraic manipulations. As an example, being solutions of a true cubic equation, the algebraic expressions of acceleration wave speeds are awkward, although accessible. Moreover, elastic anisotropy usually implies three distinct wave speeds and makes longitudinal and transversal waves possible only for special propagation directions. Crucially, elastic isotropy displays a shear wave speed of double multiplicity, which remains a neutral elastic–plastic wave speed of multiplicity at least one. This property is generally lost for anisotropic elasticity and the elastic–plastic acceleration wave speeds are usually distinct from their elastic counterparts.

However, for many engineering materials, elastic anisotropy is accessible to measurements and its effects can be quite pronounced (Boehler, 1987; Gibson and Ashby, 1988). For instance, strain localization is strongly affected, both in terms of critical hardening modulus and inclination of the shear bands. This was shown in experiments on sandstone by Millien (1993) and theoretically by Rizzi and Loret (1997). Nevertheless, even in the simplest contexts, an analytic solution for the onset of localization is hardly feasible, while the flatness of the objective functions hampers numerical maximization.

The aim of this paper is to analyze the effects of elastic anisotropy on the nature of elastic–plastic wave speeds. We are especially interested in the onset of strain localization, which corresponds to the first vanishing of a wave speed, and in flutter instability, where the squares of two wave speeds become complex conjugate. Elastic anisotropy is described in this article through a symmetric second-order fabric tensor which may be viewed as a macroscopic realization of microstructural features, associated with phase interfaces, void or microcrack patterns (Oda et al., 1982; Kanatani, 1984; Cowin, 1985). Alternatively, the considered anisotropy can be viewed as induced by deformation. The elastic tensor is obtained using representation theorems. Orthotropic elasticity results when the fabric tensor has distinct eigenvalues, while the existence of an eigenplane implies transverse isotropy.

We focus first on a particular model that has been used for the description of damage by Valanis (1990) and Zysset and Curnier (1995). Starting from a reference isotropic elasticity described by two Lame moduli, the model condenses all scalar and directional information pertaining to anisotropy in the fabric tensor. In addition to the directions of orthotropy, the model is defined by four scalar coefficients and it is believed to be able to describe with sufficient accuracy many features of inherent and induced anisotropy. It turns out that a kind of correspondence principle can be established for the strain localization analysis of elastic–plastic models based on this elastic anisotropy. Indeed, we show that strain localization of these models can be analyzed by considering corresponding elastic–plastic models based on the reference isotropic elasticity. Consequently, analytic results available in the latter framework can be used. As a simple application, we consider uniaxial traction of a von Mises associative model with transversely isotropic elasticity, making use of the results obtained for isotropic elasticity by Bigoni and Hueckel (1990, 1991). The solution reveals that anisotropy may strongly influence the localization threshold and dictate the band inclination. This is particularly evident at the boundary of the domain in which the elastic tensor is positive definite.

For non-associative flow rules, complex conjugate eigenvalues of the acoustic tensor may in principle occur, even prior to strain localization. This phenomenon, referred to as flutter instability, was investigated from different perspectives by An and Schaeffer (1990), Loret et al. (1990), Ottonsen and Runesson (1991), Loret (1992), Brannon and Drugan (1993), Bigoni and Willis (1994), Bigoni and Zaccaria (1994), Bigoni (1995), Loret et al. (1995), Simoes (1997) for single phase solids, and by Loret and Harireche (1991), Molenkamp (1991), Loret et al. (1997), Simoes et al. (1998) for fluid-saturated porous media. Loret et al. (1990) have shown that flutter instability is excluded in the important case of deviatoric associative flow rules. Even if complex eigenvalues of the acoustic tensor are excluded in this case, coalescence of two or three eigenvalues may occur. As pointed out by An and Schaeffer (1990), this circumstance is in some sense critical, because an appropriate and even infinitesimally small perturbation can then give rise to flutter. Specific perturbations in the constitutive equations have been investigated: deviation with respect to deviatoric associativity, Loret (1992), influence of large deformations, Bigoni (1995). Perturbations through boundary conditions have a similar destabilizing effect, Loret et al. (1995). The above works concern the onset of flutter. As for its mechanical interpretation, results are scant. Rice (1976) viewed the occurrence of complex conjugate wave speeds as giving rise to harmonic waves of exponentially growing amplitude in time and coined the terminology flutter instability, reminiscent of the physical phenomenon familiar in aeroelasticity. On the other hand, from the analysis of propagation of harmonic waves in fluid-saturated porous media (Simoes et al., 1998), the occurrence of complex conjugate wave speeds appears to lead to ill-posedness, that is the growth coefficient becomes unbounded in the limit of infinitely small wavelengths. Bigoni and Willis (1994) have shown that flutter ill-posedness precludes the existence of solutions for particular boundary conditions. In any case, these interpretations are not yet firmly established since they disregard elastic unloading.

All the above works on flutter refer to isotropic elasticity. We will prove that, in the presence of elastic anisotropy, flutter instability may occur even when the plastic equations obey deviatoric associativity. This fact is first observed in the model used for strain localization analysis. However, due to algebraic difficulties to manipulate the acceleration wave speeds, we develop another elastic–plastic model which displays an elastic wave speed with multiplicity two. This property facilitates the analysis of the nature of elastic–plastic wave speeds. For this model, an infinitesimal amount of anisotropy is shown to give rise to flutter when the axes of elastic and plastic orthotropy are not coincident. Otherwise, a finite amount of anisotropy is usually necessary to induce flutter.

The notation employed in this paper is mainly that of Gurtin (1981), where *Sym* denotes the set of symmetric second-order tensors. Bold roman letters denote vectors and second-order tensors, capital letters are used for the latter. The scalar product and the tensorial product of two vectors, or second-order tensors, **a** and **b** are designated by

$\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}$, respectively. Two second-order, symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ are defined coaxial if they commute, $\mathbf{AB}=\mathbf{BA}$. When coaxiality holds, the two tensors share at least one principal reference system. The Euclidean norm of tensors and vectors is denoted by the symbol $\|\cdot\|$ and tr is the trace operator. Finally, $\mathbf{I}$ indicates the second-order identity tensor.

2. Constitutive equations

2.1. *Elastic behavior*

Instead of the familiar structural anisotropy defined in terms of planes and axes of symmetry, the anisotropic character of elasticity is assumed here to be induced by a traceless symmetric second-order tensor $\mathbf{G}$. This is a fabric tensor, representing a macroscopic description of microstructural features, such as the directional distribution of contact forces in granular materials (Oda et al., 1982), or the spatial distribution of damage in the form of continuously distributed voids or micro-cracks (Valanis, 1990). In a stereological analysis of a scalar quantity $D(\mathbf{n})$, directionally dependent but symmetric with respect to the origin, the fabric tensor defines the spherical harmonics of degree 2:

$$D(\mathbf{n}) = D_0+\mathbf{n}\cdot\mathbf{G}\mathbf{n},$$

where D_0 is the unweighted average of D. Finer descriptions would involve higher harmonics of even order, yielding additional traceless tensors of order higher than two (Kanatani, 1984). These are not considered here.

There are several ways to introduce fabric into the elastic properties. One way is to assume that the free energy Ψ depends on both the infinitesimal strain tensor $\mathbf{E}$ and $\mathbf{G}$: $\Psi=\Psi(\mathbf{E},\mathbf{G})$ (Cowin, 1985). The second step consists in applying representation theorems for scalar-valued isotropic functions (Truesdell and Noll, 1965; Wang, 1970; Boehler, 1987). For linear elasticity, the free energy involves nine coefficients c_i, $i\in[1,9]$, which depend on the isotropic scalar invariants of $\mathbf{G}$, namely:

$$\Psi=\frac{c_1}{2}(\mathrm{tr}\,\mathbf{E})^2+\frac{c_2}{2}\mathrm{tr}\,\mathbf{E}^2+\frac{c_3}{2}(\mathrm{tr}\,\mathbf{EG})^2+c_4\,\mathrm{tr}\,\mathbf{E}^2\mathbf{G}+\frac{c_5}{2}(\mathrm{tr}\,\mathbf{EG}^2)^2+\frac{c_6}{2}\mathrm{tr}\,(\mathbf{EG})^2$$
$$+c_7\,\mathrm{tr}\,\mathbf{E}\,\mathrm{tr}\,\mathbf{EG}+c_8\,\mathrm{tr}\,\mathbf{EG}\,\mathrm{tr}\,\mathbf{EG}^2+c_9\,\mathrm{tr}\,\mathbf{E}\,\mathrm{tr}\,\mathbf{EG}^2. \quad (1)$$

The resulting fourth-order elastic tensor $\mathscr{E}=\partial^2\Psi/\partial\mathbf{E}\,\partial\mathbf{E}$, endowed with minor and major symmetries, takes the form:

$$\mathscr{E}=c_1\mathbf{I}\otimes\mathbf{I}+c_2\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}+c_3\mathbf{G}\otimes\mathbf{G}+c_4(\mathbf{G}\,\overline{\underline{\otimes}}\,\mathbf{I}+\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{G})+c_5\mathbf{G}^2\otimes\mathbf{G}^2+c_6\mathbf{G}\,\overline{\underline{\otimes}}\,\mathbf{G}$$
$$+c_7(\mathbf{I}\otimes\mathbf{G}+\mathbf{G}\otimes\mathbf{I})+c_8(\mathbf{G}\otimes\mathbf{G}^2+\mathbf{G}^2\otimes\mathbf{G})+c_9(\mathbf{I}\otimes\mathbf{G}^2+\mathbf{G}^2\otimes\mathbf{I}), \quad (2)$$

where, given three arbitrary second-order tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the product $\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B}$ defines the fourth-order tensor that assigns to $\mathbf{C}$ the tensor

$$(\mathbf{A}\,\underline{\overline{\otimes}}\,\mathbf{B})[\mathbf{C}] = \tfrac{1}{2}(\mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}} + \mathbf{A}\mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}). \tag{3}$$

The fact that $\mathbf{G}$ is traceless affects the formal representation (2) only through the functional dependence of the scalar functions c_i upon the invariants of $\mathbf{G}$.

Following Valanis (1990) and Zysset and Curnier (1995), there is an alternative, more heuristic way to define anisotropic elasticity. To this purpose, let us first record the elastic isotropic tensor $\mathscr{E}^{\mathrm{iso}}$ which will be viewed as a reference:

$$\mathscr{E}^{\mathrm{iso}} = \lambda\,\mathbf{I}\otimes\mathbf{I} + 2\mu\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}, \tag{4}$$

where λ and μ are the usual Lamé constants. Given a tensor $\mathbf{B}\in Sym$, a homogeneous de visu modification of $\mathscr{E}^{\mathrm{iso}}$ is

$$\mathscr{E} = \lambda\,\mathbf{B}\otimes\mathbf{B} + 2\mu\,\mathbf{B}\,\underline{\overline{\otimes}}\,\mathbf{B}, \tag{5}$$

which to a strain $\mathbf{E}$ associates the stress $\mathbf{T}$,

$$\mathbf{T} = \mathscr{E}[\mathbf{E}] \equiv \lambda(\mathbf{B}\cdot\mathbf{E})\mathbf{B} + 2\mu\,\mathbf{B}\mathbf{E}\mathbf{B}. \tag{6}$$

Clearly, for this transformation to be meaningful, $\mathbf{B}$ should satisfy some restrictions. For the elastic tensor to be positive definite, $\mathbf{B}$ should be definite, and, without loss of generality, we shall assume $\mathbf{B}$ to be positive definite. Consequently, $\mathbf{B}$ cannot be traceless. In addition, there is no loss of generality either in normalizing $\mathbf{B}$ to facilitate comparison with respect to the isotropic reference $\mathscr{E}^{\mathrm{iso}}$. Therefore, $\mathbf{B}$ is built from the fabric tensor $\mathbf{G}$ and satisfies the following conditions:

$$\mathbf{B} = g\mathbf{I} + \mathbf{G} \text{ Positive Definite}, \quad \operatorname{tr}\mathbf{B}^2 = 3. \tag{7}$$

As a consequence of the positive definiteness and normalization of $\mathbf{B}$, g belongs to the interval $]0,1]$. Given the positive definiteness of $\mathbf{B}$, the necessary and sufficient conditions for positive definiteness and strong ellipticity of $\mathscr{E}$ turn out to be the same as for the isotropic reference $\mathscr{E}^{\mathrm{iso}}$:

$$\mathscr{E} \text{ Positive Definite} \Leftrightarrow 3\lambda + 2\mu > 0 \quad \text{and} \quad \mu > 0, \tag{8}$$

$$\mathscr{E} \text{ Strongly Elliptic} \Leftrightarrow \lambda + 2\mu > 0 \quad \text{and} \quad \mu > 0. \tag{9}$$

Positive definiteness results from the relation $\mathbf{X}\cdot\mathscr{E}[\mathbf{X}] = \tilde{\mathbf{X}}\cdot\mathscr{E}^{\mathrm{iso}}[\tilde{\mathbf{X}}]$ that holds for any $\mathbf{X}\in Sym$, $\mathbf{X}$ being defined as $\mathbf{B}^{1/2}\mathbf{X}\mathbf{B}^{1/2}$. Analogously, the strong ellipticity property results from eqn (30) that will be shown later.

Moreover, the elastic compliance tensor is

$$\mathscr{E}^{-1} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\mathbf{B}^{-1}\otimes\mathbf{B}^{-1} + \frac{1}{2\mu}\mathbf{B}^{-1}\,\underline{\overline{\otimes}}\,\mathbf{B}^{-1}. \tag{10}$$

The above-defined elastic anisotropy involves four scalar parameters, namely λ, μ, and two of the three eigenvalues of $\mathbf{B}$, in addition to the eigenvectors $\mathbf{b}_i$, $i\in[1,3]$, of $\mathbf{B}$ which play the role of orthotropy directions. Now, two questions may arise:

(1) Is this anisotropy a particular case of the nine parameter family defined earlier, eqn (2)?

(2) Is this anisotropy a particular case of structural anisotropy?

Zysset and Curnier (1995) have given an affirmative answer to the first of the above questions and identified the coefficients c_i, $i \in [1, 9]$, in relation (2):

$$c_1 = \lambda g^2, \quad c_2 = 2\mu g^2, \quad c_3 = \lambda, \quad c_4 = 2\mu g, \qquad (11)$$
$$c_6 = 2\mu, \quad c_7 = \lambda g, \quad c_5 = c_8 = c_9 = 0.$$

The answer to the second question is affirmative as well. When the spectrum of **B** is separate, the material defined by (5) is structurally orthotropic and, when two of the eigenvalues of **B** are equal, the material is structurally transversely isotropic. Otherwise, due to the normalization eqn (7), $\mathbf{B} = \mathbf{I}$ and the material is isotropic. The proof is reported in Appendix A.

In the following, the expressions of various acoustic tensors will be needed. For a given unit direction of propagation **n**, the acoustic tensor associated with a constitutive tensor, say $\mathscr{E}$, is defined by the application which to a propagation vector **g** associates the vector $\rho^{-1}\mathscr{E}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}$, ρ being the mass density. Thus, the acoustic tensor $\mathbf{A}_e(\mathbf{n})$ associated with $\mathscr{E}$ is

$$\mathbf{A}_e(\mathbf{n}) = \frac{\lambda + \mu}{\rho} \mathbf{Bn} \otimes \mathbf{Bn} + \frac{\mu}{\rho} (\mathbf{n} \cdot \mathbf{Bn}) \mathbf{B}, \qquad (12)$$

which, for $\mathbf{B} = \mathbf{I}$, specializes to $\mathbf{A}_e^{iso}(\mathbf{n})$, associated with $\mathscr{E}^{iso}$,

$$\mathbf{A}_e^{iso}(\mathbf{n}) = \frac{\lambda + \mu}{\rho} \mathbf{n} \otimes \mathbf{n} + \frac{\mu}{\rho} \mathbf{I}. \qquad (13)$$

Since the elastic tensors $\mathscr{E}^{iso}$ and $\mathscr{E}$ are assumed to be positive definite, they are strongly elliptic and therefore the acoustic tensors are positive definite. Thus, they are invertible, indeed

$$\det \mathbf{A}_e(\mathbf{n}) = \rho^{-3} \det \mathbf{B} (\mathbf{n} \cdot \mathbf{Bn})^3 \mu^2 (\lambda + 2\mu) \qquad (14)$$

and

$$\mathbf{A}_e^{-1}(\mathbf{n}) = -\frac{\rho}{\mu} \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\mathbf{n} \otimes \mathbf{n}}{(\mathbf{n} \cdot \mathbf{Bn})^2} + \frac{\rho}{\mu} \frac{\mathbf{B}^{-1}}{\mathbf{n} \cdot \mathbf{Bn}}. \qquad (15)$$

Finally, using the spectral decomposition of **B** and its positive definiteness,

$$\mathbf{B} = \sum_{i=1}^{3} b_i \mathbf{b}_i \otimes \mathbf{b}_i, \quad b_i > 0, \quad i \in [1, 3], \qquad (16)$$

its square root $\mathbf{B}^{1/2}$ can be given an explicit expression:

$$\mathbf{B} = \mathbf{B}^{1/2}\mathbf{B}^{1/2}, \quad \mathbf{B} = \sum_{i=1}^{3} \sqrt{b_i}\mathbf{b}_i \otimes \mathbf{b}_i. \tag{17}$$

Remark 2.1. A connection with deformation induced anisotropy.[1]

So far we have viewed the tensor **B** as a fabric tensor and the resulting anisotropy has been referred to as structural anisotropy. But eqn (5) is also appropriate for deformation-induced anisotropy with a properly defined **B**. For example, in a formalism using intermediate configurations, the elastic tensor $\mathscr{E}^{\mathrm{ref}}$, relating the Kirchhoff stress and elastic strain $(\Phi^{\mathrm{T}}\Phi-\mathbf{I})/2$ in the intermediate configurations, transforms to $\mathscr{E}$ in the current configurations according to the operation involving the elastic deformation gradient Φ,

$$\mathscr{E}^{\mathrm{ref}}_{IJKL} \overset{\Phi}{\mapsto} \mathscr{E}_{ijkl} = \Phi_{iI}\Phi_{jJ}\Phi_{kK}\Phi_{lL}\mathscr{E}^{\mathrm{ref}}_{IJKL}. \tag{18}$$

In a Lagrangian formalism, $\mathscr{E}^{\mathrm{ref}}$ is the elastic tensor in the initial configuration and Φ the deformation gradient. If $\mathscr{E}^{\mathrm{ref}}$ is isotropic, $\mathscr{E}$ takes the form (5) with $\mathbf{B} = \Phi\Phi^{\mathrm{T}}$, that is $\mathbf{B} = \mathbf{V}^2$ in terms of the left stretch tensor **V** of Φ. Notice that the fact that this **B** does not obey the natural normalization akin to structural anisotropy eqn (7) is inconsequential.

2.2. *Elastic–plastic behavior*

The plastic constitutive equations we consider may embody any kind of anisotropy through the presence of internal variables $\mathscr{K}$ of arbitrary tensorial nature in the yield function f. The rate constitutive equations—valid in the small strain range—relate the stress and strain rates $\dot{\mathbf{T}}$ and $\dot{\mathbf{E}}$ in the following piecewise linear way

$$\dot{\mathbf{T}} = \begin{cases} \mathscr{E}[\dot{\mathbf{E}}] - \dfrac{1}{H} < \mathbf{Q}\cdot\mathscr{E}[\dot{\mathbf{E}}] > \mathscr{E}[\mathbf{P}] & \text{if } f(\mathbf{T},\mathscr{K}) = 0 \\ \mathscr{E}[\dot{\mathbf{E}}] & \text{if } f(\mathbf{T},\mathscr{K}) < 0 \end{cases}, \tag{19}$$

where both the yield function gradient $\mathbf{Q} \in Sym$ and the flow mode tensor $\mathbf{P} \in Sym$ are of unit norm. The plastic modulus $H > 0$ is assumed to be strictly positive and it is related to the hardening modulus h through

$$H = h + h_e \quad \text{with} \quad h_e = \mathbf{Q}\cdot\mathscr{E}[\mathbf{P}]. \tag{20}$$

In eqn (19), the operator $\langle\cdot\rangle$ denotes the Macaulay brackets, i.e. $\forall\alpha \in \Re$, $\langle\alpha\rangle = (\alpha + |\alpha|)/2$. All quantities appearing in these rate equations fully depend on the entire path of deformation reckoned from some ground state. In other words, **Q**, **P**, f, h, and H are functions of the state, represented through stress, plastic strain and, in general, internal variables.

[1] This remark has been suggested to us by an anonymous reviewer.

The special case of deviatoric associativity, where the deviatoric parts of **P** and **Q** are aligned, may be defined as

$$\mathbf{P} = \cos\chi\,\hat{\mathbf{S}} + \frac{\sin\chi}{\sqrt{3}}\mathbf{I}, \quad \mathbf{Q} = \cos\psi\,\hat{\mathbf{S}} + \frac{\sin\psi}{\sqrt{3}}\mathbf{I}, \tag{21}$$

where $\hat{\mathbf{S}} \in Sym$ is traceless and of unit norm. The angular parameters ψ and χ describe the pressure-sensitivity and the dilatancy of the material, respectively; they will be assumed to range in the interval $[0^\circ, 90^\circ[$. Therefore, purely isotropic plastic flow and plastic contraction are not considered here.

2.3. *Elastic–plastic acoustic tensor*

Let us consider an acceleration wave that propagates through an elastic–plastic material in a direction **n**. If both sides of the wave front follow the loading branch $(19)_1$ of the constitutive operator, the propagation is governed by the following eigenvalue problem involving the elastic–plastic acoustic tensor $\mathbf{A}_{ep}(\mathbf{n})$ (Hill, 1962; Mandel, 1962),

$$[\mathbf{A}_{ep}(\mathbf{n}) - c^2\mathbf{I}]\mathbf{g} = \mathbf{0}, \tag{22}$$

where c is the normal wave speed and **g** the mode of propagation. From the constitutive assumptions $(19)_1$, $\mathbf{A}_{ep}(\mathbf{n})$ can be cast in the format,

$$\mathbf{A}_{ep}(\mathbf{n}) = \mathbf{A}_e(\mathbf{n}) - \frac{1}{\rho H}\mathbf{p}(\mathbf{n}) \otimes \mathbf{q}(\mathbf{n}), \tag{23}$$

where **p** and **q** are two vectors, functions of **n**:

$$\mathbf{p} \equiv \mathscr{E}[\mathbf{P}]\mathbf{n} = \lambda(\mathbf{B} \cdot \mathbf{P})\mathbf{B}\mathbf{n} + 2\mu\mathbf{B}\mathbf{P}\mathbf{B}\mathbf{n}, \quad \mathbf{q} \equiv \mathscr{E}[\mathbf{Q}]\mathbf{n} = \lambda(\mathbf{B} \cdot \mathbf{Q})\mathbf{B}\mathbf{n} + 2\mu\mathbf{B}\mathbf{Q}\mathbf{B}\mathbf{n}. \tag{24}$$

3. Strain localization analysis

Strain localization is defined by the vanishing of a wave speed for at least one propagation direction **n**, or equivalently by the singularity of the acoustic tensor (Rice, 1976),

$$\mathbf{A}_{ep}(\mathbf{n})\mathbf{g} = \mathbf{0} \quad \text{for } \mathbf{g} \neq \mathbf{0} \Leftrightarrow \det \mathbf{A}_{ep}(\mathbf{n}) = 0. \tag{25}$$

During a loading process for which the hardening modulus continuously decreases, the onset of strain localization occurs when the hardening modulus reaches the largest value that makes $\mathbf{A}_{ep}(\mathbf{n})$ singular. Thus, the critical hardening modulus defining the onset of strain localization is the solution of the constrained maximization problem:

$$h^{crit} = \max_{\mathbf{n}, \|\mathbf{n}\|=1} \{\mathbf{q}(\mathbf{n}) \cdot \mathbf{A}_e^{-1}(\mathbf{n})\mathbf{p}(\mathbf{n})\} - h_e. \tag{26}$$

For isotropic elasticity, $\mathbf{B} = \mathbf{I}$, the maximization problem (26) was solved by Rudnicki and Rice (1975), Needleman and Ortiz (1991) and Ottosen and Runesson (1991) for special assumptions on **P** and **Q**, and by Bigoni and Hueckel (1990, 1991) under the sole hypothesis of coaxiality of **P** and **Q**. For anisotropic elasticity, the maximization problem has not been explored so far in a general context. Even in the simplest settings, a complete analytical solution is very difficult to obtain (Rizzi and Loret, 1997). Strikingly, it turns out that the form of elastic anisotropy defined by eqn (5) makes it possible to transform the maximization problem into a problem involving the elastic reference isotropic tensor only. Therefore, it becomes possible to use the above-quoted results concerning elastic isotropy.

3.1. *Explicit solution for strain localization*

Let us first define the transformed normals to the plastic potential $\tilde{\mathbf{P}}$ and to the yield surface $\tilde{\mathbf{Q}}$, the transformed propagation direction $\tilde{\mathbf{n}}$ and mode $\tilde{\mathbf{g}}$:

$$\tilde{\mathbf{P}} = \mathbf{B}^{1/2}\mathbf{P}\mathbf{B}^{1/2}, \quad \tilde{\mathbf{Q}} = \mathbf{B}^{1/2}\mathbf{Q}\mathbf{B}^{1/2}, \quad \tilde{\mathbf{n}} = \frac{\mathbf{B}^{1/2}\mathbf{n}}{\|\mathbf{B}^{1/2}\mathbf{n}\|}, \quad \tilde{\mathbf{g}} = \mathbf{B}^{1/2}\mathbf{g}. \tag{27}$$

The vector **p**, eqn (24), can be factored in the form:

$$\mathbf{p} = \|\mathbf{B}^{1/2}\mathbf{n}\|\mathbf{B}^{1/2}\tilde{\mathbf{p}} \quad \text{with } \tilde{\mathbf{p}} = \mathscr{E}^{\text{iso}}[\tilde{\mathbf{P}}]\tilde{\mathbf{n}}, \tag{28}$$

and similarly for **q**

$$\mathbf{q} = \|\mathbf{B}^{1/2}\mathbf{n}\|\mathbf{B}^{1/2}\tilde{\mathbf{q}} \quad \text{with } \tilde{\mathbf{q}} = \mathscr{E}^{\text{iso}}[\tilde{\mathbf{Q}}]\tilde{\mathbf{n}}. \tag{29}$$

For the elastic acoustic tensor, the following relation holds

$$\mathbf{A}_{\text{e}}(\mathbf{n}) = (\mathbf{n}\cdot\mathbf{B}\mathbf{n})\mathbf{B}^{1/2}\mathbf{A}_{\text{e}}^{\text{iso}}(\tilde{\mathbf{n}})\mathbf{B}^{1/2}. \tag{30}$$

In addition, observe that the modulus H, eqn (20), is form-invariant

$$H = h + \mathbf{Q}\cdot\mathscr{E}[\mathbf{P}] = h + \tilde{\mathbf{Q}}\cdot\mathscr{E}^{\text{iso}}[\tilde{\mathbf{P}}]. \tag{31}$$

Now, a glance at the elastic–plastic acoustic tensor eqn (23) shows that a relation similar to eqn (30) holds for the elastic–plastic tensor itself, namely

$$\mathbf{A}_{\text{ep}}(\mathbf{n}) = (\mathbf{n}\cdot\mathbf{B}\mathbf{n})\mathbf{B}^{1/2}\mathbf{A}_{\text{ep}}^{\text{iso}}(\tilde{\mathbf{n}})\mathbf{B}^{1/2} \tag{32}$$

where

$$\mathbf{A}_{\text{ep}}^{\text{iso}}(\tilde{\mathbf{n}}) = \mathbf{A}_{\text{e}}^{\text{iso}}(\tilde{\mathbf{n}}) - \frac{1}{\rho H}\tilde{\mathbf{p}}(\tilde{\mathbf{n}}) \otimes \tilde{\mathbf{q}}(\tilde{\mathbf{n}}). \tag{33}$$

Therefore the eigenvalue problem for the wave speeds eqn (22) can be transformed into

$$\left[\mathbf{A}_{\text{ep}}^{\text{iso}}(\tilde{\mathbf{n}}) - c^2\frac{\mathbf{B}^{-1}}{\mathbf{n}\cdot\mathbf{B}\mathbf{n}}\right]\tilde{\mathbf{g}} = \mathbf{0}. \tag{34}$$

The key point is to realize that, due to eqns (28)–(31), $\mathbf{A}_{\text{ep}}^{\text{iso}}$ is indeed the acoustic tensor associated with an elastic–plastic material defined by the reference isotropic elastic tensor $\mathscr{E}^{\text{iso}}$ and the transformed plastic characteristics $\mathbf{P}$, $\mathbf{Q}$ and h.

Still, the generalized eigenvalue problem eqn (34) has not the familiar structure due to the explicit presence of $\mathbf{B}$ instead of the identity. Nevertheless, the strain localization condition

$$\mathbf{A}_{\text{ep}}^{\text{iso}}(\tilde{\mathbf{n}})\tilde{\mathbf{g}} = \mathbf{0}, \tag{35}$$

has clearly the usual elastic–plastic structure akin to isotropic elasticity: hence, we have recovered the familiar framework of strain localization, for which analytical results are available. Therefore, the practical procedure to detect the onset of strain localization needs the following three steps:

(1) calculate the transformed quantities $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$, eqns $(27)_{1,2}$;
(2) solve the maximization problem in the transformed space, thus obtaining the critical hardening modulus h^{crit}, the critical directions $\tilde{\mathbf{n}}$ and eigenmodes $\tilde{\mathbf{g}}$;
(3) retrieve the critical directions $\mathbf{n}$ and $\mathbf{g}$ through eqns $(27)_{3,4}$.

Some words of caution may be necessary as to the use of the available analytical solutions in the framework of isotropic elasticity. First notice that the symmetric second-order tensors $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are usually no longer of unit norm. Second, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are in general not coaxial (i.e. they do not commute) even when $\mathbf{P}$ and $\mathbf{Q}$ are (a trivial but noticeable exception occurs when $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{B}$ are all coaxial). Therefore, the explicit solution developed by Bigoni and Hueckel (1990, 1991) cannot always be used. It can however be used for associative flow rules. The case where $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}$ are not coaxial may be solved in plane strain/stress but it requires further investigation in a general three-dimensional context. Hence, the full exploitation of the correspondence principle obtained here asks for new analytical solutions for elastic–plastic materials with isotropic elasticity.

Finally, let us conclude this section with a comment concerning the situation in which an infinite number of shear bands—forming a cone—are possible. For simplicity, let us consider associative plasticity only. For a critical cone to exist, the transformed normal $\tilde{\mathbf{Q}}$ must possess two equal eigenvalues (see Appendix B). Now, this circumstance may well occur even when $\mathbf{Q}$ has a separate spectrum. To convince oneself of this fact, it suffices to consider the case where $\mathbf{Q}$ and $\mathbf{B}$ are coaxial: indeed, from the spectral representation of $\mathbf{Q} = \Sigma_{i=1}^{3} Q_i \mathbf{b}_i \otimes \mathbf{b}_i$, $\tilde{\mathbf{Q}}$ is equal to $\Sigma_{i=1}^{3} Q_i b_i \mathbf{b}_i \otimes \mathbf{b}_i$, from which the point follows.

3.2. *Application to uniaxial traction*

The foregoing analysis is now applied to the simplest possible context, namely uniaxial traction along the axis $\mathbf{t}_\sigma$ of an associative elastic–plastic von Mises material which displays transverse isotropic elasticity about the axis $\mathbf{b}$.

The unit outward normals to the plastic potential and yield surface coincide and can be written as

$$\mathbf{P} = \mathbf{Q} = \sqrt{\tfrac{3}{2}}\,(\mathbf{t}_\sigma \otimes \mathbf{t}_\sigma - \tfrac{1}{3}\mathbf{I}). \tag{36}$$

The fabric tensor **B** takes the form

$$\mathbf{B} = b_1 \mathbf{b} \otimes \mathbf{b} + b_2 (\mathbf{I} - \mathbf{b} \otimes \mathbf{b}), \tag{37}$$

and, due to the normalization eqn (7), the eigenvalues b_1 and b_2 depend on a single angular parameter $\hat{b}$ whose range is limited to $]0^\circ, 90^\circ[$ by the positive definiteness of **B**:

$$b_1 = \sqrt{3}\cos\hat{b}, \quad b_2 = \sqrt{\tfrac{3}{2}}\sin\hat{b}. \tag{38}$$

Isotropic elasticity corresponds to $b_2 = b_1 = 1$ or $\hat{b} = \hat{b}_{\text{iso}} \approx 54.74^\circ$.

When the traction axis $\mathbf{t}_\sigma$ and orthotropy axis $\mathbf{b}$ are aligned, i.e. $\theta_\sigma \equiv \cos^{-1}\mathbf{t}_\sigma \cdot \mathbf{b} = 0^\circ$, there is complete rotational symmetry about this common direction. When these axes are distinct, that is $\theta_\sigma \neq 0^\circ$, they define a symmetry plane, referred to as the loading plane. This prompts the definition of two direct orthonormal triads $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ with $\mathbf{b}_1 = \mathbf{b}$, $\mathbf{t}_1 = \mathbf{t}_\sigma$ and a common out-of-plane axis $\mathbf{b}_3 = \mathbf{t}_3$ (Fig. 1).

In view of (36) and (37), the transformed tensor $\tilde{\mathbf{Q}}$, eqn (27), has a simple form when written in the $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$-triad:

$$[\tilde{\mathbf{Q}}] = \sqrt{\frac{3}{2}} \begin{bmatrix} b_1\left(\cos^2\theta_\sigma - \dfrac{1}{3}\right) & \sqrt{b_1 b_2}\sin\theta_\sigma\cos\theta_\sigma & 0 \\ \sqrt{b_1 b_2}\sin\theta_\sigma\cos\theta_\sigma & b_2\left(\sin^2\theta_\sigma - \dfrac{1}{3}\right) & 0 \\ 0 & 0 & -\dfrac{b_2}{3} \end{bmatrix}. \tag{39}$$

Its eigenvalues $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$,

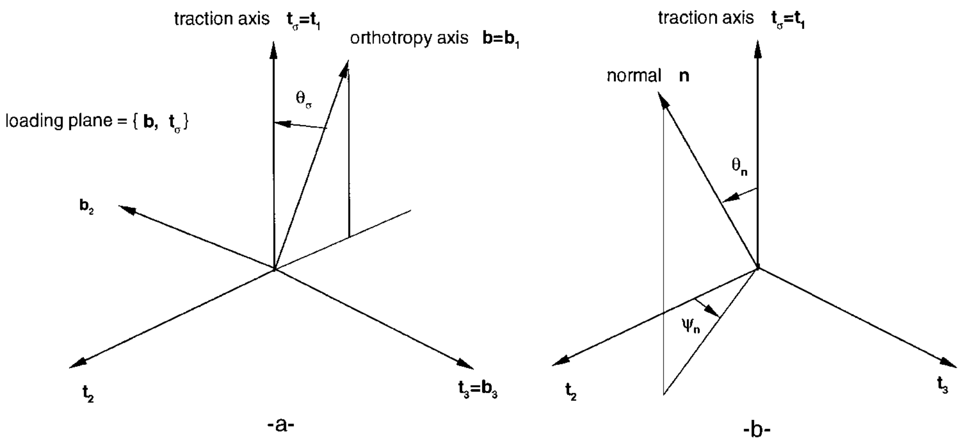

Fig. 1. Axis of elastic symmetry **b**, traction axis $\mathbf{t}_\sigma$, material axes $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, loading axes $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ with $\mathbf{b}_1 = \mathbf{b}$ and $\mathbf{t}_1 = \mathbf{t}_\sigma$ and loading plane $[\mathbf{b}, \mathbf{t}_\sigma]$.

$$\tilde{Q}_1 = \frac{1}{2}\sqrt{\frac{3}{2}}(S+\sqrt{\Delta}), \quad \tilde{Q}_2 = \frac{1}{2}\sqrt{\frac{3}{2}}(S-\sqrt{\Delta}), \quad \tilde{Q}_3 = -\frac{b_2}{\sqrt{6}}, \tag{40}$$

with

$$S = b_1(\cos^2\theta_\sigma - \tfrac{1}{3}) + b_2(\sin^2\theta_\sigma - \tfrac{1}{3}), \quad \Delta = S^2 + \tfrac{8}{9}b_1 b_2 > 0, \tag{41}$$

are distinct, except when the traction and orthotropy axes are aligned, in which case $\tilde{Q}_1 = \tilde{Q}_3 = -b_2/\sqrt{6} < \tilde{Q}_2 = b_1\sqrt{2/3}$. Notice that the eigenvectors associated with the eigenvalues $\tilde{Q}_1$ and $\tilde{Q}_2$ lie in the loading plane while the third eigenvector associated with $\tilde{Q}_3$ is $\mathbf{b}_3 = \mathbf{t}_3$. Using the above spectral analysis of $\mathbf{Q}$, the analytical solution of Bigoni and Hueckel (1990, 1991), which is reported for completeness in Appendix B, yields the critical hardening modulus and critical directions $\mathbf{n}$ at the onset of strain localization. The critical directions $\mathbf{n}$ are retrieved by the simple transformation eqn $(27)_3$. For plotting purposes, these directions are defined by two spherical coordinates (θ_n, ψ_n) associated with the triad $\{\mathbf{t}_1 = \mathbf{t}_\sigma, \mathbf{t}_2, \mathbf{t}_3\}$ (Fig. 1). Since the loading plane is a symmetry plane, to any solution $\mathbf{n}$ strictly on one side of this plane a mirror solution on the other side is associated. Therefore we content ourselves to collect solutions defined by a second angle ψ_n in $[0^\circ, 180^\circ]$. In other words, except when a critical cone exists, we find two solutions either belonging to the loading plane, or in symmetric position with respect to it. For the latter circumstance, we report in the figures only the solution with ψ_n belonging to $[0^\circ, 180^\circ]$.

There are three parameters that affect the normalized critical modulus h^{crit}/μ and the critical directions, namely, Poisson's ratio of the isotropic reference solid $v = \lambda/(2\lambda+2\mu)$, the angle b defining the fabric tensor, and the inclination $\theta_\sigma \in [0^\circ, 90^\circ]$ of the traction axis with respect to the orthotropy axis. The normalized critical hardening modulus h^{crit}/μ and the critical angles versus the fabric angle b are shown in Fig. 2, for $v = 0$ and $v = 1/3$ and for four loading directions θ_σ. Other values of v do not show qualitatively distinct features and therefore are not reported.

Some comments may be appropriate to highlight the qualitative and quantitative influence of elastic anisotropy on the strain localization characteristics.

Let us first recall that for elastic isotropy a cone of critical band normals exists, for which ψ_n remains arbitrary, while (Rudnicki and Rice, 1975)[2]

$$\frac{h^{\text{crit}}}{\mu} = -\frac{1+v}{3}, \quad \tilde{\theta}_n = \tan^{-1}\sqrt{\frac{1+v}{2-v}}. \tag{42}$$

Since the eigenvalues of $\mathbf{Q}$ are distinct except when the traction and orthotropy axes are aligned, in this particular case only a cone of critical solutions is displayed; however the angle of this cone is generally different from that corresponding to the isotropic case.

[2] The critical modulus h^{crit}/μ reported in Rudnicki and Rice (1975) differs from that given here because their $\mathbf{P}$ and $\mathbf{Q}$ are not of unit norm.

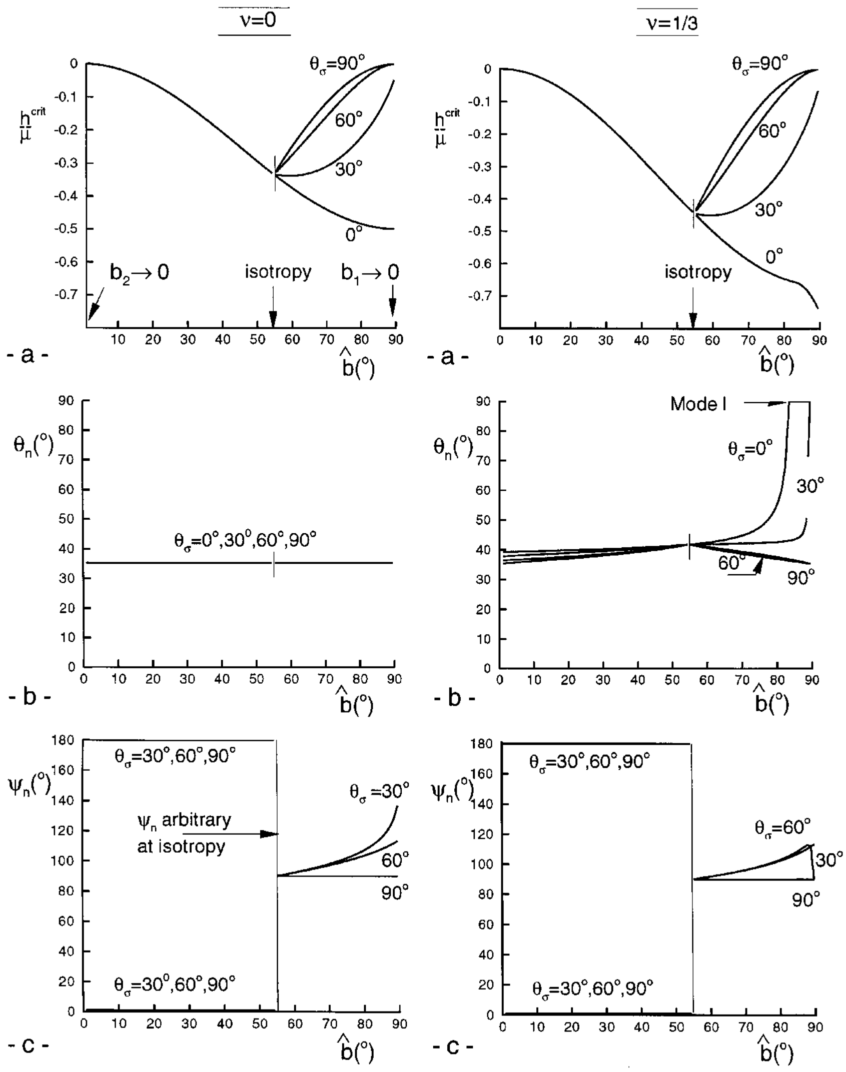

Fig. 2. Strain localization for an associated von Mises solid with transverse isotropy described by the angle b and subjected to uniaxial traction. Results are reported for $v = 0$ and $v = 1/3$ and different inclinations $\theta_\sigma = (\mathbf{b}, \mathbf{t}_\sigma)$. (a) Normalized critical hardening modulus; (b) angle $\theta_n = (\mathbf{t}_\sigma, \mathbf{n})$; (c) angle ψ_n. Note: when $\theta_\sigma = 0°$, a cone of critical normals exists about the traction axis and ψ_n is arbitrary.

The solutions for $v = 0$ and $v = 1/3$ are roughly similar, but there are several quantitative differences. In particular, as a general trend, the critical hardening modulus decreases when v increases and this effect has been observed for the whole admissible range of v, namely $]-1, 1/2[$. As a consequence of plastic associativity, these moduli are negative, corresponding to strain localization in the softening range. The case $v = 0$ is peculiar because the angles θ_n remain equal to the corresponding isotropic value, eqn (42), independently of the direction of traction and of elastic anisotropy.

Results reported in the figures show clearly that elastic anisotropy strongly influences all characteristics of strain localization. Actually, the angle $b = b_{\text{iso}}$ corresponding to isotropy separates two types of solutions. For $b < b_{\text{iso}}$, the critical normals lie in the loading plane, i.e. $\psi_n = 0^\circ$ and 180°, the angle θ_n varies slightly with respect to the isotropic value, and the normalized critical modulus h^{crit}/μ is equal to

$$h^{\text{crit}}/\mu = -(1+v)b_2^2/3. \tag{43}$$

Therefore, the critical modulus tends to its upper limit 0 at the boundary of the domain corresponding to the positive definiteness of the elastic tensor, i.e. when $b \to 0^\circ$. In addition, surprisingly, the critical angles are virtually independent of the loading direction.

A quite different picture emerges for $\hat{b} > \hat{b}_{\text{iso}}$. The band normals no longer belong to the loading plane, $\psi_n \neq 0^\circ$, 180°. In this case, the loading direction plays an important role, and the critical modulus does not necessarily tend to zero at the boundary of the domain of positive definiteness of the elastic tensor. Indeed, for $b \to 90^\circ$, the solution type strongly depends on the value of θ_σ with respect to $\sin^{-1}(1/\sqrt{3}) \approx 35.26^\circ$:

$$\theta_\sigma \geqslant 35.26^\circ, \quad \frac{h^{\text{crit}}}{\mu} = 0,$$

$$\theta_\sigma \leqslant 35.26^\circ, \quad \frac{h^{\text{crit}}}{\mu} = \begin{cases} -\dfrac{2}{1-v}\tilde{Q}_1^2, & \text{if } v \geqslant 0, \\ -2(1+v)\tilde{Q}_1^2, & \text{if } v \leqslant 0. \end{cases} \tag{44}$$

When $\hat{b} \to 90^\circ$, $\tilde{Q}_1$ tends to $3/2(\sin^2\theta_\sigma - 1/3)$, and the critical modulus decreases monotonically from 0 as θ_σ varies from 35.26° to 0° (Fig. 2(a)).

As already mentioned, when the traction and orthotropy axes are aligned, complete rotational symmetry makes the angle ψ_n arbitrary and

$$v \leqslant 2\frac{b_1}{b_2}, \quad \frac{h^{\text{crit}}}{\mu} = -\frac{1+v}{3}b_2^2, \quad \theta_n = \tan^{-1}\sqrt{\frac{(1+v)b_1}{2b_1 - vb_2}},$$

$$v \geqslant 2\frac{b_1}{b_2}, \quad \frac{h^{\text{crit}}}{\mu} = -\frac{1+v}{3}b_2^2 - \frac{1}{3(1-v)}(2b_1 - vb_2)^2, \quad \theta_n = 90^\circ. \tag{45}$$

When $\theta_n = 90^\circ$, which occurs for $\hat{b} \geqslant 83.28^\circ$ and $v = 1/3$ (Fig. 2(b)), the localization mode is a pure Mode I, i.e. axial splitting. This corresponds to **g**, eqn (25), parallel to **n**, which is orthogonal to the traction direction. It is clear from eqn (45) that the solution $\theta_n = 90^\circ$ is not possible for $\nu \leqslant 0$. Mode I also occurs for all inclinations $\mathbf{t}_\sigma \in [0^\circ, 35.26^\circ]$ at large values of angle b.

When the traction axis is orthogonal to the orthotropy axis, $\theta_\sigma = 90^\circ$, the solution is either in the loading plane, $\psi_n = 0^\circ$, 180°, for $b < b_{\rm iso}$, or in the plane $\mathbf{t}_1$–$\mathbf{t}_3$, corresponding to $\psi_n = 90^\circ$, for $b > b_{\rm iso}$ (Figs 2(c)). The normalized critical modulus is equal to $-(1+\nu)b_2^2/3$ in the former case, and to $-(1+\nu)b_1^2/3$ in the latter.

It is worth mentioning that many of the above features typical of elastic anisotropy have been observed for different types of structural transverse orthotropy in recent works (Loret and Rizzi, 1997a, b; Rizzi and Loret, 1997). In particular, Loret and Rizzi (1997a) have given arguments that explain the trend of the critical hardening modulus to reach its upper value 0 at the boundary of the domain defining the positive definiteness of the elastic tensor. The continuity in the neighborhood of isotropy of the critical modulus and of the angle θ_n and the discontinuity of the angle ψ_n are considered in Rizzi and Loret (1997) and Loret and Rizzi (1997b). There, the band normals were also observed to be independent of the traction direction for some ranges of the anisotropy parameters. However, the number of possible shear bands on each side of the loading plane were sometimes found to be equal to two.

4. Flutter instability analysis

For elastic–plastic models based on elastic isotropy and obeying deviatoric associativity, complex conjugate eigenvalues of the acoustic tensor are excluded (Loret et al. 1990). One is then naturally led to analyze the consequences of the deviations with respect to the two above assumptions. In fact, infinitesimal noncoaxial deviations with respect to deviatoric associativity give rise to flutter while coaxial deviations may require a finite amount of deviation (Loret, 1992). Coaxiality refers there to **P** and **Q**. We examine now the effects of deviation with respect to elastic isotropy for elastic–plastic models that obey deviatoric associativity, eqn (21). We will show that the two types of deviations share some similarity. Indeed, the analysis will highlight another coaxiality property, now between the fabric tensors **B** or **G** and **S**, which condenses the plastic directional properties. The pervasive influence of noncoaxiality on the onset of flutter was also noted in the analysis of the effects of a traction free boundary not coaxial with the axes of material orthotropy (Loret et al., 1995).

We begin by observing that flutter may well occur for the type of elastic anisotropy described by eqn (5). Then, we build another elastic model that both facilitates algebraic manipulations and is prone to flutter.

4.1. *A numerical example*

For the anisotropic elastic tensor defined by eqn (5), numerical calculations of acceleration waves, eqns (22) and (23), have shown that complex squares of wave speeds may occur in some directions of propagation and for some range of parameters. However, the fact that the elastic wave speeds are in general distinct makes an explicit analysis intractable. Therefore, to show with an example that anisotropy may actually trigger flutter, we consider the special situation where **B** and **S**, and therefore **Q** and **P**, have a common eigenvector, say $\mathbf{b}_3$, and we restrict our attention to waves propagating in the plane orthogonal to this direction, i.e., $\mathbf{n} \cdot \mathbf{b}_3 = 0$. Then the out-of-plane elastic wave speed $c_{e,3}$ (corresponding to propagation in the $\mathbf{b}_3$ direction) is equal to $c^{\text{iso}}_{e,s}\sqrt{(\mathbf{n} \cdot \mathbf{Bn})b_3}$, with $c^{\text{iso}}_{e,s} = \sqrt{\mu/\rho}$. The in-plane elastic wave speeds $c_{e,1} > c_{e,2}$ are obtained from the in-plane components of the elastic acoustic tensor (12), expressed in the orthogonal axes $\{\mathbf{n}, \mathbf{s}\}$,

$$A^e_{nn} = \frac{\lambda+2\mu}{\rho}(\mathbf{n} \cdot \mathbf{Bn})^2, \quad A^e_{ss} = \frac{\lambda+\mu}{\rho}(\mathbf{s} \cdot \mathbf{Bn})^2 + \frac{\mu}{\rho}(\mathbf{n} \cdot \mathbf{Bn})(\mathbf{s} \cdot \mathbf{Bs}),$$

$$A^e_{ns} = \frac{\lambda+2\mu}{\rho}(\mathbf{n} \cdot \mathbf{Bn})(\mathbf{s} \cdot \mathbf{Bn}), \tag{46}$$

from which it is concluded that waves not traveling in the eigendirections of **B** are neither longitudinal nor transverse.

Taking the trace and determinant of eqn (23), the squares of the elastic–plastic wave speeds are implicitly given by their sum and product

$$c_1^2 + c_2^2 = c_{e,1}^2 + c_{e,2}^2 + \frac{\mu^2}{\rho H}(\mathrm{f}_2 - \mathrm{f}_1), \quad c_1^2 c_2^2 = c_{e,1}^2 c_{e,2}^2 + \frac{\mu^2}{\rho H}(A^e_{nn}\mathrm{f}_2 - A^e_{ss}\mathrm{f}_1 + A^e_{ns}\mathrm{f}_3), \tag{47}$$

where

$$\mu^2 \mathrm{f}_1 = (\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}), \quad \mu^2 \mathrm{f}_2 = (\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{n}) - \mathbf{p} \cdot \mathbf{q},$$

$$\mu^2 \mathrm{f}_3 = (\mathbf{p} \cdot \mathbf{n})(\mathbf{q} \cdot \mathbf{s}) + (\mathbf{p} \cdot \mathbf{s})(\mathbf{q} \cdot \mathbf{n}), \tag{48}$$

with $\mathbf{p} = \mathbf{p}(\mathbf{n})$ and $\mathbf{q} = \mathbf{q}(\mathbf{n})$ defined by eqn (24). The squares of the elastic–plastic wave speeds are complex if

$$\mathrm{f}_4 \equiv 4\mathrm{f}_1\mathrm{f}_2(1 - ez(\mathbf{p}))(1 - ez(\mathbf{q})) > 0, \quad \mathrm{f}_5 \equiv \mathrm{f}_1 + \mathrm{f}_2 + 2e\mathrm{f}_3 > 0, \tag{49}$$

and if the positive plastic modulus belongs to the interval $]H_-, H_+[$

$$\frac{H_\pm}{\mu} = \frac{A^e_{nn} - A^e_{ss}}{c_{e,1}^2 - c_{e,2}^2}(\mathrm{f}_5 \pm \sqrt{\mathrm{f}_4}). \tag{50}$$

In the above formulas (49), the following definitions have been introduced

$$e = \frac{A^e_{ns}}{A^e_{nn} - A^e_{ss}}, \quad z(\mathbf{a}) = \frac{(\mathbf{a} \cdot \mathbf{n})^2 - (\mathbf{a} \cdot \mathbf{s})^2}{(\mathbf{a} \cdot \mathbf{n})(\mathbf{a} \cdot \mathbf{s})} \quad \text{for } \mathbf{a} = \mathbf{p}, \mathbf{q}. \tag{51}$$

For elastic isotropy, the off-diagonal coefficient A^e_{ns}, eqn (46), and hence e, is zero, and f_2 cannot be strictly positive for deviatoric associativity, which rules out satisfaction of the sign conditions (49). This is no longer true for anisotropic elasticity, where it becomes possible that the product of the two last factors in f_4 has a negative sign for some directions **n** not aligned with the eigendirections of **B**. As an example, let us consider transverse isotropic elasticity with $\hat{b} = 80^\circ$, eqn (38), $\lambda/\mu = 1$, together with the following plastic properties: friction coefficient $\psi = 30^\circ$, isochoric plastic flow, $\chi = 0^\circ$, and $H/\mu = 0.5$, corresponding to $h/\mu \approx 0.326$, Drucker–Prager yield condition, i.e. $\mathbf{S}$ equal to the stress deviator direction. The material is subjected to uniaxial traction along a direction $\mathbf{t}_\sigma$ inclined of $\theta_\sigma = 15^\circ$ with respect to the orthotropy axis $\mathbf{b}_1$. The squares of the normalized elastic–plastic wave speeds w_i^2, with $w_i = c_i/c_{\mathrm{e,s}}^{\mathrm{iso}}$ for $i \in [1, 3]$, are shown in Fig. 3(a), for all in-plane propagation directions singled out by the inclination angle $\theta = (\mathbf{t}_\sigma, \mathbf{n})$. The elastic–plastic tensor can be checked to be in this case positive definite and thus strong ellipticity is preserved and strain localization is excluded. However, there is a 6° fan centered about $\theta \approx -14^\circ$ where complex conjugate squared wave speeds appear (Fig. 3(b)).

It could be argued that the large anisotropy involved in this example is necessary to induce flutter. This is not true, as evidenced through a more general result presented in the next subsection. Starting from the usual elastic–plastic framework with elastic isotropy and deviatoric associativity, we consider elastic anisotropy as a perturbation. We show that when the fabric tensor and the tensor $\mathbf{S}$ characterizing the plastic state are not coaxial, an infinitesimal perturbation gives rise to flutter.

4.2. *A model with elastic anisotropy prone to flutter*

When the elasticity is isotropic, a key property that facilitates the analysis of elastic–plastic wave speeds is the fact that the elastic shear wave speed has double multiplicity; since the elastic–plastic acoustic tensor is a rank-one modification of its elastic counterpart, the elastic shear wave speed is a neutral elastic–plastic wave speed. The elastic anisotropy developed below is tailored to display the double multiplicity property.

In the framework of elastic anisotropy described by (2), we assume

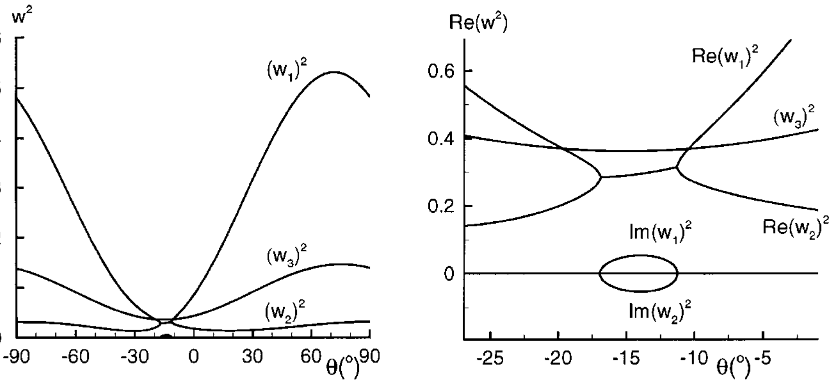

Fig. 3. Normalized wave speeds as a function of the angle $\theta = (\mathbf{t}_\sigma, \mathbf{n})$ for a non-associated Drucker–Prager solid with transverse isotropy subjected to uniaxial traction. Material parameters: $\nu = 0.25$, $b = 80^\circ$, $\theta_\sigma = 15^\circ$, $H/\mu = 0.5$, $\psi = 30^\circ$, $\chi = 0^\circ$. (a) Full range of propagation directions; (b) particular of the range of propagation directions where flutter occurs.

$$c_1 = \lambda, \quad c_2 = 2\mu, \quad c_3 = (\lambda+\mu)\omega^2, \quad c_7 = (\lambda+\mu)\omega, \qquad (52)$$
$$c_4 = c_5 = c_6 = c_8 = c_9 = 0,$$

where ω is an arbitrary dimensionless material parameter. The elastic tensor becomes

$$\mathcal{E} = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} + (\lambda+\mu)(\mathbf{N} \otimes \mathbf{N} - \mathbf{I} \otimes \mathbf{I}), \qquad (53)$$

where

$$\mathbf{N} = \mathbf{I} + \omega \mathbf{G}. \qquad (54)$$

In order to analyze positive definiteness of the elastic tensor (53), the Cauchy–Schwarz inequality is used, yielding, for every $\mathbf{X} \in Sym$,

$$\mathbf{X} \cdot \mathcal{E}[\mathbf{X}] \geqslant \lambda(\mathrm{tr}\,\mathbf{X})^2 + 2(\mu - \sqrt{3}|\lambda+\mu|\,\|\omega\mathbf{G}\|)\mathbf{X} \cdot \mathbf{X} + (\lambda+\mu)(\omega\mathbf{G} \cdot \mathbf{X})^2, \qquad (55)$$

from which the following sufficient condition for positive definiteness, and thus strong ellipticity, of $\mathcal{E}$ is obtained:

$$\lambda + \frac{2}{3}\mu > \frac{2}{\sqrt{3}}|\lambda+\mu|\,\|\omega\mathbf{G}\|, \quad \mu > \sqrt{3}|\lambda+\mu|\,\|\omega\mathbf{G}\| \Rightarrow \mathcal{E} \text{ Positive Definite.} \qquad (56)$$

Therefore, $\lambda+\mu$ and μ are positive and positive definiteness of $\mathcal{E}$ bounds the anisotropic perturbation,

$$\|\omega\mathbf{G}\| < \frac{\min\{\frac{3}{2}\lambda+\mu, \mu\}}{\sqrt{3}(\lambda+\mu)}. \qquad (57)$$

The elastic acoustic tensor takes the form

$$\mathbf{A}_e(\mathbf{n}) = \frac{\lambda+\mu}{\rho}\tilde{n}^2 \tilde{\mathbf{n}} \otimes \tilde{\mathbf{n}} + \frac{\mu}{\rho}\mathbf{I}, \qquad (58)$$

where $\tilde{\mathbf{n}}$ is the unit vector defined as

$$\tilde{n}\tilde{\mathbf{n}} = \mathbf{N}\mathbf{n}, \quad \tilde{n} = \|\mathbf{N}\mathbf{n}\|. \qquad (59)$$

Thus a quasi-longitudinal mode of propagation along $\tilde{\mathbf{n}}$ with speed $c_{e,l}$ exists, while all directions in the plane orthogonal to $\mathbf{n}$ are quasi-transverse modes with speed $c_{e,s}$,

$$\rho c_{e,l}^2 = (\lambda+\mu)\tilde{n}^2 + \mu > \rho c_{e,s}^2 = \mu. \qquad (60)$$

When the elastic tensor (53) is used to build an elastic–plastic model, the resulting elastic–plastic acoustic tensor is still given in the format eqn (23) with $\mathbf{A}_e(\mathbf{n})$ defined by eqn (58), $\mathbf{p}(\mathbf{n}) = \mathcal{E}[\mathbf{P}]\mathbf{n}$, $\mathbf{q}(\mathbf{n}) = \mathcal{E}[\mathbf{Q}]\mathbf{n}$ and $H = h + \mathbf{P} \cdot \mathcal{E}[\mathbf{Q}]$. Consequently, the eigenmode of propagation in the plane orthogonal to $\mathbf{n}$ and $\mathbf{q}$ is still associated with the quasi-transversal elastic wave speed $c_{e,s}$, and the other two elastic–plastic wave speeds c_1 and c_2 are solutions of the equations

$$c_1^2+c_2^2 = c_{e,1}^2+c_{e,s}^2+\frac{\mu^2}{\rho H}(f_2-f_1), \quad c_1^2c_2^2 = c_{e,1}^2c_{e,s}^2+\frac{\mu^2}{\rho H}(c_{e,1}^2f_2-c_{e,s}^2f_1), \tag{61}$$

with

$$\mu^2 f_1 = (\mathbf{p}(\mathbf{n})\cdot\tilde{\mathbf{n}})(\mathbf{q}(\mathbf{n})\cdot\tilde{\mathbf{n}}), \quad \mu^2 f_2 = (\mathbf{p}(\mathbf{n})\cdot\tilde{\mathbf{n}})(\mathbf{q}(\mathbf{n})\cdot\tilde{\mathbf{n}})-\mathbf{p}(\mathbf{n})\cdot\mathbf{q}(\mathbf{n}). \tag{62}$$

Necessary and sufficient conditions for the squares of the elastic–plastic wave speeds to be complex conjugate are that both f_1 and f_2 be strictly positive and that the modulus H belongs to the interval $]H_-, H_+[$,

$$\frac{H_\pm}{\mu} = \frac{1}{\tau}(\sqrt{f_1}\pm\sqrt{f_2})^2, \tag{63}$$

with $\tau = c_{e,1}^2/c_{e,s}^2-1 > 0$.

The analysis of the sign conditions of f_1 and f_2 delineates three cases which highlight the importance of the noncoaxiality of **G** and **S**.

4.2.1. *Non coaxial fabric*

Let us first consider the sign of f_2. To this purpose, it is useful to introduce the projection operator **J** onto the plane orthogonal to **n**,

$$\mathbf{J} = \mathbf{I}-\tilde{\mathbf{n}}\otimes\tilde{\mathbf{n}}. \tag{64}$$

Therefore, f_2 may be recast in a compact form, namely

$$\mu^2 f_2 = -\mathbf{p}(\mathbf{n})\cdot\mathbf{J}\mathbf{q}(\mathbf{n}) = -\mathbf{J}\mathbf{p}(\mathbf{n})\cdot\mathbf{J}\mathbf{q}(\mathbf{n}). \tag{65}$$

Since $\mathbf{JNn} = \mathbf{0}$, $\mathbf{Jp}(\mathbf{n})$ may be simplified to

$$\mathbf{Jp}(\mathbf{n}) = 2\mu z(\chi)\cos\chi\,\mathbf{J}(\mathbf{Z}(\chi)\mathbf{n}), \tag{66}$$

where $\mathbf{Z}(\chi)\in Sym$ has unit norm,

$$z(\chi)\mathbf{Z}(\chi) = \hat{\mathbf{S}}-\frac{1}{2\sqrt{3}}\tan\chi\mathbf{I}, \quad z(\chi) = \|\hat{\mathbf{S}}-\frac{1}{2\sqrt{3}}\tan\chi\mathbf{I}\|. \tag{67}$$

Expressions similar to (66) and (67) hold for **q** with χ replaced by ψ. Since we assume non-associativity, $\chi \neq \psi$, we can introduce the angle $\phi\in]0^\circ, 90^\circ[$ and the two tensors **X** and **Y**,

$$\mathbf{X} = \frac{\mathbf{Z}(\chi)+\mathbf{Z}(\psi)}{2\cos\phi}, \quad \mathbf{Y} = \frac{\mathbf{Z}(\chi)-\mathbf{Z}(\psi)}{2\sin\phi}, \quad \text{with}\cos 2\phi = \mathbf{Z}(\chi)\cdot\mathbf{Z}(\psi). \tag{68}$$

Notice that **X** and **Y** are orthogonal and of unit norm,

$$\mathbf{X}\cdot\mathbf{Y} = 0, \quad \|\mathbf{X}\| = 1, \quad \|\mathbf{Y}\| = 1. \tag{69}$$

Expressing $\mathbf{Z}(\chi)$ and $\mathbf{Z}(\psi)$ in terms of **X**, **Y** and ϕ, the quantity f_2, eqn (65), is found to be proportional, through the positive proportionality coefficient $4z(\chi)z(\psi)\cos\chi\cos\psi$, to

$$-\cos^2\phi\,\|\mathbf{J}(\mathbf{Xn})\|^2+\sin^2\phi\,\|\mathbf{J}(\mathbf{Yn})\|^2. \tag{70}$$

Finally, let us denote by $\mathbf{n}_{X*}$ and $\mathbf{n}_{Y*}$ the respective eigenvectors of the second-order tensors $\mathbf{X}^*$ and $\mathbf{Y}^*$,

$$\mathbf{X}^* = \mathbf{N}^{-1/2}\mathbf{X}\mathbf{N}^{-1/2}, \quad \mathbf{Y}^* = \mathbf{N}^{-1/2}\mathbf{Y}\mathbf{N}^{-1/2}. \tag{71}$$

$\mathbf{J}(\mathbf{Xn})$ and $\mathbf{J}(\mathbf{Yn})$ vanish when $\mathbf{n}$ is parallel to $\mathbf{N}^{-1/2}\mathbf{n}_{X*}$ and to $\mathbf{N}^{-1/2}\mathbf{n}_{Y*}$, respectively. For $\mathbf{J}(\mathbf{Xn})$ and $\mathbf{J}(\mathbf{Xn})$ to vanish for the same direction $\mathbf{n}$, this $\mathbf{n}$ must be an eigenvector of both $\mathbf{G}$ and $\mathbf{S}$. Obviously, such an $\mathbf{n}$ does not exist if $\mathbf{G}$ and $\mathbf{S}$, and consequently $\mathbf{G}$, $\mathbf{P}$ and $\mathbf{Q}$, do not have any common eigendirection. Then f_2 is strictly positive along and around the critical directions $\mathbf{n}^{flu}$ parallel to $\mathbf{N}^{-1/2}\mathbf{n}_{X*}$.

We can now consider the sign condition on f_1. First, observe that, when elastic anisotropy decreases, corresponding to small but non zero ω, the three critical directions $\mathbf{n}^{flu}$ tend to the eigendirections of $\mathbf{S}$. Now, it is well-established that for isotropic elasticity at least one eigendirection of $\mathbf{S}$ exists in the neighborhood of which f_1 is strictly positive [Loret, 1992, eqns (3.5) and (3.6)]. Using eqns (53) and $(62)_1$, it can also be shown that the property still applies for small ω. Hence, for sufficiently small ω, at least one direction $\mathbf{n}^{flu}$ exists in the neighborhood of which both f_1 and f_2 are strictly positive and flutter occurs.

In the above deductions, the spectra of $\mathbf{X}^*$ and $\mathbf{Y}^*$ are assumed to be separate. However, it may be shown that flutter can also occur when this is not the case.

4.2.2. *G and S share one common eigendirection*

Assume that $\mathbf{G}$ and $\mathbf{S}$ have one common eigendirection, say $\mathbf{v}$, corresponding to the intermediate or minimum eigenvalue of $\mathbf{S}$. Then $\mathbf{v}$ is also a common eigendirection of $\mathbf{X}$ and $\mathbf{Y}$. In the neighborhood of the directions $\mathbf{N}^{-1/2}\mathbf{n}_{X*}$, with $\mathbf{n}_{X*}$ eigenvector of $\mathbf{X}^*$ orthogonal to $\mathbf{v}$, f_2 is strictly positive. Repeating the argumentation employed in the previous subsection, it may be concluded that f_1 is also strictly positive in the neighborhood of at least one of these directions, and flutter occurs.

4.2.3. *Coaxial fabric*

In order to analyze the case of coaxial fabric, we write

$$\mathbf{P} = \xi_1\mathbf{Q}+\xi_2\mathbf{I}, \tag{72}$$

where

$$\xi_1 = \frac{\cos\chi}{\cos\psi}, \quad \xi_2 = -\frac{\cos\chi\tan\psi}{\sqrt{3}}+\frac{\sin\chi}{\sqrt{3}}. \tag{73}$$

It should be noted that ξ_1 is always positive (under the hypotheses assumed here) and that ξ_2 becomes zero in the limit case of the associative flow rule. We introduce the tensor $\mathbf{K}$

$$\mathbf{q} = \mathbf{Kn}, \quad \mathbf{p} = \xi_1\mathbf{Kn}+\xi_2[3(\lambda+\mu)\mathbf{Nn}-\mu\mathbf{n}], \tag{74}$$

which allows us to rewrite quantity f_2 $(62)_2$ as

$$\tilde{n}^2\mu^2 \mathrm{f}_2 = \xi_1[(\mathbf{Kn}\cdot\mathbf{Nn})^2-(\mathbf{Nn}\cdot\mathbf{Nn})(\mathbf{Kn}\cdot\mathbf{Kn})] \\ -\mu\xi_2[(\mathbf{Kn}\cdot\mathbf{Nn})(\mathbf{n}\cdot\mathbf{Nn})-(\mathbf{Nn}\cdot\mathbf{Nn})(\mathbf{n}\cdot\mathbf{Kn})]. \quad (75)$$

The coefficient of ξ_1 in eqn (75) is always negative, thus flutter is excluded for an associative flow rule, where $\xi_2 = 0$. We show now that f_2, eqn (75), may become positive for any given ξ_2, under the hypothesis that the elastic anisotropy tensor **G** and yield function gradient **Q** have a common eigenvector. This requires some algebraic transformation of (75). First, we represent **Kn** in the form

$$\mathbf{Kn} = K_{nn}\mathbf{n}+K_{tn}\mathbf{t}, \quad (76)$$

where **t** is a unit vector orthogonal to **n**. Therefore, (75) becomes

$$\frac{\tilde{n}^2\mu^2\mathrm{f}_2}{\xi_1} = [(\mathbf{Nn}\cdot\mathbf{Nn})-N_{nn}^2](-K_{nn}^2+\mu\xi K_{nn}) \\ +K_{tn}N_{nn}N_{tn}(2K_{nn}-\mu\xi)-K_{tn}^2[(\mathbf{Nn}\cdot\mathbf{Nn})-N_{tn}^2], \quad (77)$$

where $\xi = \xi_2/\xi_1$. Second, we assume that **G** and **Q**, and thus **K**, have a common eigenvector. This implies that for **n** orthogonal to this eigenvector, the following relation holds true

$$N_{tn}^2 = \mathbf{Nn}\cdot\mathbf{Nn}-N_{nn}^2, \quad (78)$$

and condition (77) becomes (the identity $N_{tn} = \omega G_{tn}$ has been used)

$$\frac{\tilde{n}^2\mu^2\mathrm{f}_2}{\xi_1} = -(\omega G_{tn})^2\left[\frac{K_{tn}N_{nn}}{\omega G_{tn}}-K_{nn}\right]\left[\frac{K_{tn}N_{nn}}{\omega G_{tn}}-K_{nn}+\mu\xi\right]. \quad (79)$$

Using (53) and (54), simple calculations yield

$$K_{tn}N_{nn}-\omega G_{tn}K_{nn} = 2\mu Q_{tn}+\omega\mu[(\mathrm{tr}\,\mathbf{Q}-2Q_{nn})G_{tn}+2Q_{tn}G_{nn}]. \quad (80)$$

It is clear from (80) and (79) that ω can always be chosen to satisfy positiveness of f_2, for any given ξ and even in the case when **G** and **Q** are coaxial. In the noncoaxial case, the component Q_{tn} can be made as small as desired when **n** approaches an eigenvector of **Q**. In this case, the minimal value of ω for positiveness of f_2 tends to zero, therefore flutter can be induced by a small-as-needed anisotropy and results relative to the previous subsection are recovered. Let us discuss the coaxial case in more detail. Denoting with indices *i* and *j* two eigenvalues of **G** and **Q**, eqn (79) can be rewritten as

$$\frac{\tilde{n}^2\mathrm{f}_2}{\xi_1} = -[\omega(G_i-G_j)\sin\theta\cos\theta]^2 d(d+\xi), \quad (81)$$

where θ is the angle between **n** and the *i*th-eigenvector of **G** and **Q** and

$$d = \frac{2(Q_i-Q_j)+\omega[\mathrm{tr}\,\mathbf{Q}(G_i-G_j)-2(Q_jG_i-Q_iG_j)]}{\omega(G_i-G_j)}. \quad (82)$$

It can be concluded from (81) that, depending on the current state, represented by **Q**, f_2 can be even positive for a vanishing small ω. This is for instance the case when $Q_i \to Q_j$ and G_i remains different from G_j. In this case f_1 is also surely positive when Q_i and Q_j are greater than the other eigenvalue. In general, positiveness of f_2 in the coaxial case may correspond to a finite value of ω, which can yield a not positive definite elastic tensor [i.e. restriction (57) is violated, but $\mathscr{E}$ still remains strong elliptic]. This represents the main difference between coaxial and noncoaxial perturbations, a circumstance also highlighted in slightly different context (Loret, 1992).

4.2.4. *A discussion on the onset of flutter*

Since the onset of flutter instability corresponds to a change in the nature of the squares of the wave speeds, it requires coalescence of at least two eigenvalues of the acoustic tensor. Moreover, for elastic–plastic materials with elastic isotropy and obeying deviatoric associativity, we know that an eigendirection of **S** always exists along which the three elastic–plastic wave speeds coalesce and for which f_1, eqn (62), is strictly positive while f_2 is zero [Loret, 1992, eqn (3.4)]. If one perturbs the constitutive equations through a noncoaxial elastic anisotropy of the form (53), flutter will occur in this situation as we have shown in Section 4.2.1.

It may be interesting to remark that coalescence of eigenvalues of the acoustic tensor has practically no relation with the occurrence of other, more familiar, local instabilities. In order to give full evidence to this point, let us consider uniaxial traction of an elastic–plastic Drucker–Prager material with isotropic elasticity and with yield function and plastic potential gradients defined by (21), with **S** equal to the stress deviator. Normalized critical hardening moduli corresponding to different values of χ and ψ, ranging between 0° and 30° are reported in Table 1 for loss of positive definiteness (PD), strong ellipticity (SE) and ellipticity (E) of the constitutive operator, and for coalescence of the three eigenvalues of the acoustic tensor (F). Table 1 displays well-known features. Loss of (PD) always occurs before softening and not later than (SE). Loss of (SE) cannot follow loss of (E) and both can occur either in the softening or hardening regime (Bigoni and Zaccaria, 1992). It may also happen that (SE) is lost in the hardening regime while (E) is lost during the softening regime. On the other hand, coalescence of eigenvalues of the acoustic tensor, and hence possibility of flutter, depends on the stress state, plasticity parameters χ and ψ, and strongly on Poisson's ratio v. Table 1 shows clearly that the occurrence of flutter relative to the other local instabilities is hardly typical. In fact, depending on Poisson's ratio, (F) may occur well before or well after any other instability.

The indications yielded by Table 1 are sketched in Fig. 4. The thresholds (PD), (SE), (E) and (F) are qualitatively marked on the curve. Five coalescence thresholds (F) are reported, corresponding to different values of Poisson's ratio. While the position of the (PD), (SE) and (E) thresholds changes as well with Poisson's ratio as indicated by Table 1, these changes are small compared to those in the (F) threshold, and they are neglected for illustrative purposes. When coalescence of eigenvalues occurs, flutter may be induced by a perturbation in terms of a small elastic anisotropy: this is the reason for the use of the acronym (F).

Table 1
Normalized critical hardening moduli h/μ defining loss of positive definiteness (PD), strong ellipticity (SE) and ellipticity (E) of the elastic–plastic constitutive operator and coalescence of three eigenvalues of the acoustic tensor (F), for several values of Poisson's ratio ν, pressure-sensitivity angle ψ, and plastic dilatancy angle χ.

ψ (°)	χ (°)	ν	P.D.	S.E.	E	F
0	0	0	0	−0.333	−0.333	0.667
		0.3	0	−0.433	−0.433	−0.933
15	15	0	0	−0.120	−0.120	1.520
		0.3	0	−0.156	−0.156	0.297
15	0	0	0.034	−0.181	−0.188	1.132
		0.23	0.076	−0.192	−0.223	0.059
		0.25	0.083	−0.190	−0.226	−0.034
		0.29	0.101	−0.182	−0.231	−0.220
		0.3	0.170	−0.179	−0.232	−0.267
30	30	0	0	−0.008	−0.008	1.966
		0.3	0	−0.011	−0.011	1.206
30	15	0	0.034	−0.014	−0.018	1.805
		0.3	0.084	0.015	−0.009	0.840
30	0	0	0.134	0.018	−0.005	1.520
		0.3	0.384	0.204	0.047	0.417

5. Concluding remarks

Two aspects of the influence of elastic anisotropy on the nature of elastic–plastic wave speeds have been analyzed. In particular, the two non-propagation criteria corresponding to strain localization and flutter instability have been investigated.

For the particularly appealing structure of anisotropy described by eqn (5), a correspondence principle has been established, that links strain localization of the actual material to strain localization of a corresponding elastic–plastic material, endowed with elastic isotropy and transformed plastic properties. Thus, available analytical solutions in the framework of elastic isotropy can be used for associative flow rules. In this context, a simple example regarding uniaxial traction of a von Mises material with transverse anisotropic elasticity, shows important features of strain localization. These are mainly:

- Elastic anisotropy may yield a strong increase or decrease in the critical hardening modulus for localization with respect to the isotropic value.
- The elastic directional properties dictate the band inclination.
- Depending on the relative position of anisotropy parameters with respect to the

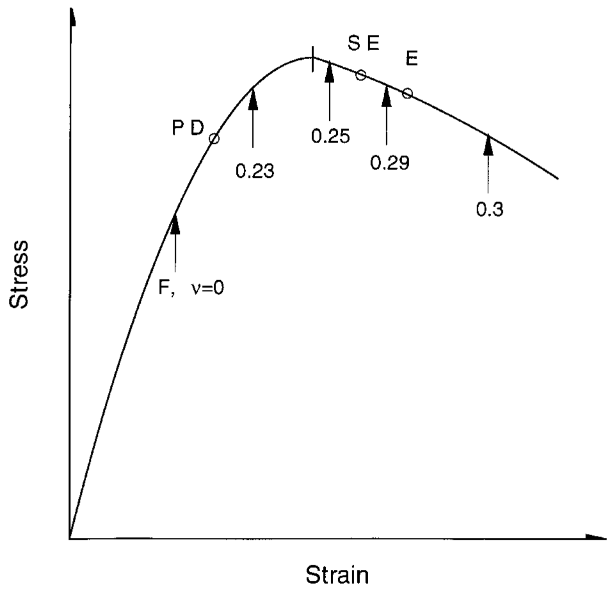

Fig. 4. Sketch of instability thresholds in uniaxial traction of an isotropic elastic–plastic Drucker–Prager material with $\psi = 15^\circ$, $\chi = 0^\circ$, and different values of Poisson's ratio ν. Thresholds are denoted by acronyms: (PD) for positive definiteness, (SE) for strong ellipticity, (E) for strain localization of the constitutive operator, and (F) for coincidence of three eigenvalues of the acoustic tensor (numbers with arrows denote different values of Poisson's ratio).

isotropic case, two shear bands may form with normals belonging to the plane defined by the traction direction and the anisotropy axis, or may form symmetrically with respect to this plane.

The latter observation may have implications in view of experiments. The treatment of general non-associativity, which implies noncoaxiality between plastic characteristics, requires isotropic solutions that need to be developed. In view of the algebraic difficulties to obtain analytical solutions to strain localization for general elastic anisotropy, this correspondence principle may be an incentive to exploit the capabilities of the elastic tensor (5) to describe both the inherent and induced anisotropies of structural and damaged materials and of geological deposits. Whether or not this correspondence principle can be extended to other classes of elastic anisotropy, defined by a fabric tensor or through other mathematical devices, remains an open question.

While certainly necessary for an accurate description of the behavior of many engineering materials, elastic anisotropy has been shown to facilitate flutter instability. In particular, it has been shown that flutter can occur for the specific models presented in Sections 3 and 4, and this even for very small amounts of elastic anisotropy. Obviously, this does not mean that flutter occurs in all situations, rather it depends on the very details of the elastic anisotropy law and on the current value of state variables and elastic parameters. Another characteristic of flutter instability is that its occurrence seems to be practically unrelated to the occurrence of other familiar local instabilities as evidenced in Fig. 4.

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Appendix A: Fabric anisotropy and structural anisotropy

A.1. *Orthotropy*

Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be an orthonormal triad defining the directions of orthotropy. Then the free energy is assumed to be an orthotropic function of $\mathbf{E}$, or equivalently an isotropic function of $\{\mathbf{E}, \mathbf{M}_1 \equiv \mathbf{b}_1 \otimes \mathbf{b}_1, \mathbf{M}_2 \equiv \mathbf{b}_2 \otimes \mathbf{b}_2\}$. Due to the identity $\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \mathbf{I}$, tensor $\mathbf{M}_3 = \mathbf{b}_3 \otimes \mathbf{b}_3$ need not to be included. Representation theorems (Wang, 1970) show that, for linear elasticity, the free energy involves nine coefficients a_i, $i \in [1, 9]$. Tensor $\mathbf{M}_3$ can be finally reintroduced in order to enhance the equal influence of the three axes, yielding

$$\begin{aligned}\Psi^{\text{ortho}} &= \frac{a_1}{2}(\text{tr}\,\mathbf{M}_1\mathbf{E})^2 + \frac{a_2}{2}(\text{tr}\,\mathbf{M}_2\mathbf{E})^2 + \frac{a_3}{2}(\text{tr}\,\mathbf{M}_3\mathbf{E})^2 \\ &+ a_4\,\text{tr}\,\mathbf{M}_1\mathbf{E}\,\text{tr}\,\mathbf{M}_2\mathbf{E} + a_5\,\text{tr}\,\mathbf{M}_1\mathbf{E}\,\text{tr}\,\mathbf{M}_3\mathbf{E} + a_6\,\text{tr}\,\mathbf{M}_2\mathbf{E}\,\text{tr}\,\mathbf{M}_3\mathbf{E} \\ &+ a_7\,\text{tr}\,\mathbf{M}_1\mathbf{E}^2 + a_8\,\text{tr}\,\mathbf{M}_2\mathbf{E}^2 + a_9\,\text{tr}\,\mathbf{M}_3\mathbf{E}^2. \end{aligned} \tag{A.1}$$

The resulting fourth-order elastic tensor $\mathscr{E}^{\text{ortho}} = \partial^2\Psi^{\text{ortho}}/\partial\mathbf{E}\,\partial\mathbf{E}$, endowed with minor and major symmetries, takes the form:

$$\begin{aligned}\mathscr{E}^{\text{ortho}} &= a_1\mathbf{M}_1 \otimes \mathbf{M}_1 + a_2\mathbf{M}_2 \otimes \mathbf{M}_2 + a_3\mathbf{M}_3 \otimes \mathbf{M}_3 \\ &+ a_4(\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1) + a_5(\mathbf{M}_1 \otimes \mathbf{M}_3 \\ &+ \mathbf{M}_3 \otimes \mathbf{M}_1) + a_6(\mathbf{M}_2 \otimes \mathbf{M}_3 + \mathbf{M}_3 \otimes \mathbf{M}_2) \quad + a_7(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{M}_1 \\ &+ \mathbf{M}_1\,\overline{\underline{\otimes}}\,\mathbf{I}) + a_8(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{M}_2 + \mathbf{M}_2\,\overline{\underline{\otimes}}\,\mathbf{I}) + a_9(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{M}_3 + \mathbf{M}_3\,\overline{\underline{\otimes}}\,\mathbf{I}). \end{aligned} \tag{A.2}$$

For the fabric tensor $\mathbf{B}$ given by eqn (16), the elastic tensor eqn (5) can be recast in the format (A.2) if coefficients a_i, $i \in [1, 9]$, are identified with

$$a_7 = \mu(b_1b_2 + b_1b_3 - b_2b_3), \quad a_8 = \mu(b_1b_2 - b_1b_3 + b_2b_3),$$

$$a_9 = \mu(-b_1 b_2 + b_1 b_3 + b_2 b_3), \quad a_1 = (\lambda + 2\mu) b_1^2 - 2a_7,$$
$$a_2 = (\lambda + 2\mu) b_2^2 - 2a_8, \quad a_3 = (\lambda + 2\mu) b_3^2 - 2a_9, \tag{A.3}$$
$$a_4 = \lambda b_1 b_2, \quad a_5 = \lambda b_1 b_3, \quad a_6 = \lambda b_2 b_3.$$

The inverse relations expressing the parameters of the model (5) in terms of the coefficients a_i, $i \in [1, 9]$, are

$$\frac{\lambda}{\mu} = \frac{2a_4}{a_7 + a_8}, \quad \lambda + 2\mu = \frac{1}{3}\sum_{k=1}^{3} a_k + 2a_{k+6}, \quad b_k^2 = \frac{a_k + 2a_{k+6}}{\lambda + 2\mu}, \quad k \in [1, 3]. \tag{A.4}$$

A.2. Transverse isotropy

Let **b** be the axis of material symmetry. The free energy is assumed to be a transversely isotropic function of **E**, or equivalently an isotropic function of $\{\mathbf{E}, \mathbf{M} = \mathbf{b} \otimes \mathbf{b}\}$. Representation theorems (Wang, 1970) show that, for linear elasticity, the free energy involves five coefficients a_i, $i \in [1, 5]$,

$$\Psi^{\mathrm{trans}} = \frac{a_1}{2}(\mathrm{tr}\,\mathbf{E})^2 + \frac{a_2}{2}\mathrm{tr}\,\mathbf{E}^2 + a_3\,\mathrm{tr}\,\mathbf{E}\,\mathrm{tr}\,\mathbf{ME} + \frac{a_4}{2}(\mathrm{tr}\,\mathbf{ME})^2 + a_5\,\mathrm{tr}\,\mathbf{ME}^2. \tag{A.5}$$

The resulting fourth-order elastic tensor $\mathscr{E}^{\mathrm{trans}} = \partial^2\Psi^{\mathrm{trans}}/\partial\mathbf{E}\,\partial\mathbf{E}$, endowed with minor and major symmetries, takes the form:

$$\mathscr{E}^{\mathrm{trans}} = a_1 \mathbf{I} \otimes \mathbf{I} + a_2 \mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} + a_3(\mathbf{I} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{I}) + a_4 \mathbf{M} \otimes \mathbf{M}$$
$$+ a_5(\mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{M} + \mathbf{M} \,\overline{\underline{\otimes}}\, \mathbf{I}). \tag{A.6}$$

For the fabric tensor **B** endowed with an eigenplane of normal **b**, eqn (37), the elastic tensor (5) can be recast in the format (A.6) with the following coefficients a_i, $i \in [1, 5]$:

$$a_1 = \lambda b_2^2, \quad a_2 = 2\mu b_2^2, \quad a_3 = \lambda b_2(b_1 - b_2), \tag{A.7}$$
$$a_4 = (\lambda + 2\mu)(b_1 - b_2)^2, \quad a_5 = 2\mu b_2(b_1 - b_2).$$

Appendix B: Solution of strain localization for associative plasticity

The analytic solution of Bigoni and Hueckel (1990, 1991) is briefly reported, in the special case of associative flow rule, namely $\mathbf{P} = \mathbf{Q}$ in the transformed space. The solution is expressed in the principal reference system of **Q**, where its components are denoted with an index subscript, e.g. Q_i. The solution is given for the critical hardening modulus, the squared components of the transformed direction of propagation **n**, and the transformed mode of propagation **g**. While there is a single critical hardening modulus, there are several, at least two, associated **n** and **g**. The critical hardening modulus for strain localization is the maximum among the three values:

$$h^{\text{crit}} = \max\{h_1, h_2, h_3\}. \tag{B.1}$$

For (i, j, k) circular permutations of $(1, 2, 3)$, one has to perform the following calculations:

(1) set

$$x_k = \frac{\tilde{Q}_j + v\tilde{Q}_k}{Q_i - Q_j}; \tag{B.2}$$

(2) obtain h_k:

$$\frac{h_k}{\mu} = -2(1+v)\tilde{Q}_k^2 - \frac{2}{1-v}(\tilde{Q}_j + v\tilde{Q}_k)^2 \mathscr{H}(x_k - 1) - \frac{2}{1-v}(\tilde{Q}_i + v\tilde{Q}_k)^2 \mathscr{H}(-x_k); \tag{B.3}$$

(3) obtain the squares of the components of the unit vector $\tilde{\mathbf{n}}$:

$$\tilde{n}_i^2 = \min\{\langle x_k \rangle, 1\}, \quad \tilde{n}_j^2 = 1 - \tilde{n}_i^2, \quad \tilde{n}_k^2 = 0; \tag{B.4}$$

(4) obtain the components of the eigenvector $\tilde{\mathbf{g}}$:

$$\tilde{g}_i = (Q_i - Q_j)\tilde{n}_i, \quad \tilde{g}_j = (Q_j - Q_i)\tilde{n}_j, \quad \tilde{g}_k = 0. \tag{B.5}$$

In the above expressions, the symbol $\langle \cdot \rangle$ denotes the Macaulay brackets, and $\mathscr{H}$ the Heaviside step function.

If $Q_i = Q_j$, a cone of solutions for the shear band normals exists and the eigenvectors i and j of $\mathbf{Q}$ are arbitrary. Then x_k can be taken indifferently equal to $+\infty$ or $-\infty$ and either $g_i = 1$, $g_j = 0$ or $g_i = 0$, $g_j = 1$ with, in both cases, $g_k = 0$.

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