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On uniqueness for frictional contact rate problems

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Abstract

A linear elastic solid having part of the boundary in unilateral frictional contact with a stiffer constraint is considered. Bifurcations of the quasistatic velocity problem are analyzed, making use of methods developed for elastoplasticity. An exclusion principle for bifurcation is proposed which is similar, in essence, to the well-known exclusion principle given by Hill (1958). Sufficient conditions for uniqueness are given for a broad class of contact constitutive equations. The uniqueness criteria are based on the introduction of 'linear comparison interfaces' defined both where the contact rate constitutive equations are piece-wise incrementally linear and where these are thoroughly nonlinear. Structural examples are proposed which give evidence to the applicability of the exclusion criteria. © 1999 Elsevier Science Ltd. All rights reserved

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Notation

Vectors and second-order tensors are denoted by bold letters. The natural inner product of two vectors \mathbf{a} and \mathbf{b} is denoted by $\mathbf{a} \cdot \mathbf{b} = a_i b_i$. The tensor product $\mathbf{a} \otimes \mathbf{b}$ of two vectors is the tensor that assigns to each vector \mathbf{u} the vector $(\mathbf{b} \cdot \mathbf{u})\mathbf{a}$. In components $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. The symbols ∇ and div indicate the gradient and the divergence of a vector or tensor field respectively. Also, $L^2(\Omega)$ and $H^1(\Omega)$ denote the well-known Hilbert spaces of real functions defined on Ω , equipped with the associated norms, respectively:

$$\|\mathbf{u}\|_{L^2(\Omega)} = \left\{ \int_{\Omega} \mathbf{u} \cdot \mathbf{u} \right\}^{1/2}, \quad \|\mathbf{u}\|_{H^1(\Omega)} = \left\{ \int_{\Omega} (\mathbf{u} \cdot \mathbf{u} + \nabla \mathbf{u} \cdot \nabla \mathbf{u}) \right\}^{1/2}.$$

1. Introduction

The problem of deformation of a solid body in contact with a stiffer frictional constraint has an evident interest in many mechanical and civil engineering problems and represents an interesting challenge in continuum mechanics. From the physical point of view, this is due to the occurrence of many instabilities, including slip-stick motion and flutter (Rice and Ruina, 1983; Gu et al., 1984; Oden and Martins, 1985; Simões and Martins, 1997). Moreover, bifurcation threshold stresses are strongly reduced by the presence of interfacial effects (Bigoni et al., 1997). It follows that a proper description of constitutive laws at interfaces becomes an essential ingredient in buckling analyses. From the mathematical point of view difficulties are related to the fact that an interfacial constitutive operator representing dry-frictional contact is typically non-symmetric (Michalowski and Mróz, 1978; Curnier, 1984; Jarzebowski and Mróz, 1994; Mróz and Jarzebowski, 1994; Mróz and Stupkiewicz, 1994). These and other peculiarities complicate the numerical analysis of problems involving contact, which is the subject of a number of contributions (see e.g. Kikuchi and Oden, 1988; Raous et al., 1988; Laursen and Simo, 1993; Laursen and Oancea, 1997 and references cited therein).

The problem of uniqueness of solutions of boundary value problems during quasistatic, unilateral, frictional contact has been analyzed from a number of perspectives. In particular, Cocu (1984) has given uniqueness and existence results for the Signorini problem with holonomic Coulomb friction. Necessary and sufficient conditions for bifurcation for finite-dimensional contact incremental problems were stated by Curnier and Alart (1988). Examples of non-uniqueness of the rate solution have been presented by Klarbring (1990b) and reconsidered by Martins et al. (1994) including the possibility of discontinuities in time. Sufficient conditions for uniqueness in the rate problem were given by Klarbring (1987), Klarbring et al. (1988), Klarbring (1990a), Chateau and Nguyen (1991), and Strömberg et al. (1996).

In the present article, uniqueness of the incremental response during quasistatic deformation of a linear, elastic body in unilateral, frictional contact with a stiffer constraint is considered afresh, borrowing concepts from elastoplasticity theory. In particular, the constitutive equations of a frictional interface and of non-associative elastoplasticity (Hill, 1967; Mandel, 1966; Mróz, 1963, 1966) have a similar structure, a fact which reflects an intimate connection between underlying physical micro-mechanisms of deformation and slip. In particular, it is well-known (Klarbring, 1990a) that the zone of contact in the friction problem may be divided into four parts, corresponding to the occurrence of separation, grazing, stick, and stick or slip. In the grazing zone, we refer to a general constitutive equation embracing a broad class of specific constitutive laws employed in the literature. In particular, we assume a thoroughly non-linear incremental constitutive equation with tangential and normal compliance. Results presented remain valid in the specific case where this compliance is assumed to vanish for zero contact pressure (Oden and Martins, 1985; Klarbring, 1990a; Buczkowski and Kleiber, 1977). Tangential and normal compliance are also assumed in the stick and slip zones. In the region where the friction condition is verified, stick or slip may occur depending on the incremental fields. In this zone the

contact laws are incrementally piece-wise linear and formally similar to the equations of incremental non-associative elastoplasticity with smooth yield and plastic potential functions. Therefore, it is natural to employ the general framework for bifurcation given by Hill (1958) and extended to non-associative plasticity by Raniecki (1979) and Raniecki and Bruhns (1981). We present an integral exclusion condition for bifurcation in essence similar to the Hill (1958) exclusion functional, which was also given—in a similar form—by Chateau and Nguyen (1991). Basically, our exclusion functional consists of the sum of two terms: a volumetric term and a surface term, corresponding to the contact area. The first term is always positive, when the body is elastic (with positive definite strain energy), whereas the surface term may be negative. The surface term may be bounded in the stick/slip zone by introducing a family of linear comparison interfaces formally analogous to the comparison solids introduced by Raniecki (1979) and Raniecki and Bruhns (1981). These comparison solids were formulated for incrementally piece-wise linear constitutive laws and therefore do not cover the situation corresponding to the grazing zone, where a thoroughly non-linear incremental constitutive law is assumed. Thus we introduce a new linear comparison interface specifically valid in the grazing zone. Nothing analogous is known for thoroughly non-linear incremental theories of plasticity, where all available results are restricted to self-adjoint constitutive operators (Petryk, 1989).

The introduction of the comparison interface makes possible the formulation of a quadratic exclusion functional for bifurcation. From this functional, global and local criteria of uniqueness are derived. In cases where the grazing zone is absent, a sufficient criterion for uniqueness may be easily obtained by imposing the positive-definiteness of the linear comparison interface operator. This is valid for certain interfacial constitutive laws often employed in the literature (Michalowski and Mróz, 1978; Cheng and Kikuchi, 1985). A more precise bound to the exclusion functional is obtained through a comparison between the volume and the surface terms. The comparison is possible making use of functional analysis arguments based on a Korn-type inequality and the trace theorem. As shown by an example in which a square elastic domain with frictional boundary on a side is considered, this condition may be of difficult practical use. This is a consequence of the well-known fact that the numerical value of the constants appearing in the Korn-type inequality and in the trace theorem are only known for very special cases. In an additional example, our approach is applied to a simple 2-D.O.F. elastic structure with a frictional constraint. For this case and when elastic compliance of the constraint is zero, the bifurcation condition (in terms of a critical value of friction) is known from Curnier and Alart (1988) and Klarbring (1990b). The example reveals that our exclusion condition (in the limit of zero constraint compliance) is in this case over-conservative. Moreover, it should be noted that our condition of uniqueness may not be critical for bifurcation, in the sense that failure of our condition does not imply bifurcation, a circumstance in full agreement with the non-associative elastoplasticity counterpart (Raniecki, 1979; Raniecki and Bruhns, 1981).

The approach presented in this paper may be considered complementary to that proposed by Klarbring (1990a). One of the advantages of our formulation is that an extension to large strains seems to be possible. In particular, the generalization of the

quadratic exclusion functional to elastoplastic solids subject to large deformations is straightforward. Moreover, an analysis may be performed of higher-order bifurcations, similar to that developed by Bigoni (1996) for non-associative elastoplasticity.

2. Frictional contact rate problem

A linear elastic body occupying a bounded region Ω of the Euclidean point space is considered (Fig. 1), with Lipschitz boundary $\partial\Omega = S_u \cup S_t \cup S_c$. In the body, the usual incremental field equations hold:

$$\operatorname{div} \dot{\boldsymbol{\sigma}} + \dot{\mathbf{f}} = \mathbf{0}, \quad \dot{\boldsymbol{\sigma}} = \mathbb{E} \boldsymbol{\varepsilon}(\dot{\mathbf{u}}), \quad \boldsymbol{\varepsilon}(\dot{\mathbf{u}}) = \frac{1}{2}(\nabla \dot{\mathbf{u}} + \nabla \dot{\mathbf{u}}^T), \quad \text{in } \Omega, \quad (1)$$

where $\dot{\mathbf{f}}$ is the increment of body force, $\dot{\boldsymbol{\sigma}}$ the stress rate, $\dot{\mathbf{u}}$ the velocity and \mathbb{E} the elastic constitutive fourth-order tensor of the body. On S_u and S_t velocities $\bar{\mathbf{v}}$ and traction rates $\dot{\mathbf{t}}$ are prescribed, respectively, i.e.

$$\dot{\boldsymbol{\sigma}} \mathbf{n} = \dot{\mathbf{t}}, \quad \text{on } S_t, \quad \dot{\mathbf{u}} = \bar{\mathbf{v}}, \quad \text{on } S_u, \quad (2)$$

where \mathbf{n} is the outer unit normal vector (see Fig. 1). The problem under consideration is evolutionary, in the sense that the extension of the contact zone fully depends on the history of loading up to the current time. However, this zone, as well as its four parts that will be defined in the following, is completely independent of the incremental fields (Klarbring, 1990a).

We denote by S_c the part of the boundary in possible frictional, unilateral contact with a stiff constraint (Fig. 1). Initially, when the body is completely unstressed, there may not be complete contact on S_c . In order to model this situation in an infinitesimal theory, the presence of a gap g between the contact boundary and the constraint is introduced, measured along the outward normal direction to S_c (as in Klarbring et al., 1988). In a generic situation of the loading process, the zone S_c may be divided into four different parts S_1, S_2, S_3 and S_4 , where separation, grazing, stick and stick or slip, may respectively occur, depending on the current values of tractions and displacements. Therefore, when the body is completely unstressed, S_c consists of S_1 and S_2 only, but during increasing loading the four regions S_1, \dots, S_4 form and evolve. These may be defined as (Klarbring, 1990a):

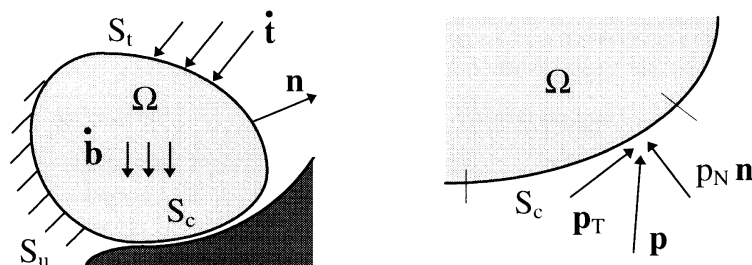


Fig. 1. Sketch of the frictional contact rate problem.

$$\begin{aligned}
 S_1 &= \{ \mathbf{x} \in S_c : u_N - w < g, p_N = 0, \mathbf{p}_T = \mathbf{0} \} \quad (\text{separation}), \\
 S_2 &= \{ \mathbf{x} \in S_c : u_N - w = g, p_N = 0, \mathbf{p}_T = \mathbf{0} \} \quad (\text{grazing}), \\
 S_3 &= \{ \mathbf{x} \in S_c : u_N - w > g, p_N > 0, |\mathbf{p}_T| - \mu p_N < 0 \} \quad (\text{stick}), \\
 S_4 &= \{ \mathbf{x} \in S_c : u_N - w > g, p_N > 0, |\mathbf{p}_T| - \mu p_N = 0 \} \quad (\text{stick or slip}),
 \end{aligned} \tag{3}$$

where μ is the friction coefficient, w is the irreversible component of displacement normal to the boundary, $\mathbf{p} = -\boldsymbol{\sigma}\mathbf{n}$, and p_N , \mathbf{p}_T and u_N , \mathbf{u}_T are the normal and tangential components of tractions (with reversed sign) and displacement, namely

$$p_N = \mathbf{p} \cdot \mathbf{n}, \quad \mathbf{p}_T = \mathbf{p} - p_N \mathbf{n}, \quad u_N = \mathbf{u} \cdot \mathbf{n}, \quad \mathbf{u}_T = \mathbf{u} - u_N \mathbf{n}. \tag{4}$$

It should be noted from conditions (3) that irreversible displacement in the contact zone may consist of both a tangential and a normal component. The normal component can model situations corresponding to interfacial dilatancy or contractivity as related to the presence of asperities or wear. Note also that the assumed contact law allows for positive normal displacements in the contact zone. This may be related to a compliance of the frictional constraint, which may be constant or dependent on the current state. Moreover, it should also be noted that in the definition of S_3 and S_4 a Coulomb law of friction has been assumed, relating the modulus of the tangential component of traction to the contact pressure.

2.1. Interfacial constitutive rate equations

The rate boundary conditions on S_c may be expressed through the interfacial constitutive laws relating traction rate and velocity. The constitutive laws are different in the various zones in which S_c is divided.

In the separation region S_1 the traction rate vector vanishes:

$$\dot{\mathbf{p}} = \mathbf{0}. \tag{5}$$

In the stick region S_3 , a linear incremental relation is assumed:

$$\dot{\mathbf{p}} = \mathbf{C}\dot{\mathbf{u}}, \tag{6}$$

where the positive definite tensor \mathbf{C} indicates the elastic interface stiffness, may be constant or dependent on the current state and may take the special simple form, isotropic in the plane tangent to the contact

$$\mathbf{C} = k_T \mathbf{I} + (k_N - k_T) \mathbf{n} \otimes \mathbf{n}, \tag{7}$$

where k_N and k_T are the normal and tangential stiffness parameter of the interface, respectively.

In the region S_4 , where the friction condition is satisfied, a linear relation between traction rate and the reversible part of the velocity is assumed, namely $\dot{\mathbf{p}} = \mathbf{C}\dot{\mathbf{u}}^r$, where $\dot{\mathbf{u}} = \dot{\mathbf{u}}^r + \dot{\mathbf{u}}^s$ and $\dot{\mathbf{u}}^s$ is the irreversible part of the velocity due to slip. The reversible part of the velocity can be attributed to the elastic deformations of the asperities of the surfaces in contact, and it is usually called adherence. The irreversible part of velocity has normal component $\dot{\mathbf{u}}^s \cdot \mathbf{n} = \dot{w}$, where w has been defined in (3). The slip term $\dot{\mathbf{u}}^s$

occurs only if the consistency condition $\mathbf{a} \cdot \dot{\mathbf{p}} = 0$ is verified, where \mathbf{a} is the gradient of the friction condition. In this case, the slip is given by $\dot{\mathbf{u}}^s = \Lambda \mathbf{b}$, being $\Lambda > 0$ a scalar multiplier and \mathbf{b} a vector defining the direction of the irreversible part of velocity. From the friction and slip conditions (Fig. 2), vectors \mathbf{a} and \mathbf{b} can be defined as:

$$\mathbf{a} = \frac{\mathbf{p}_T}{|\mathbf{p}_T|} - \mu \mathbf{n}, \quad \mathbf{b} = \frac{\mathbf{p}_T}{|\mathbf{p}_T|} - \beta(\mathbf{p}) \mathbf{n}, \quad (8)$$

where parameter $\beta(\mathbf{p})$ may describe dilatancy effects due to asperities or wear of the contact and is assumed to vanish for zero normal pressure, i.e. $\beta(\mathbf{0}) = 0$. Finally, by imposing the consistency condition, the following value of Λ can be derived:

$$\Lambda = \frac{\mathbf{a} \cdot \mathbf{C} \dot{\mathbf{u}}}{\mathbf{a} \cdot \mathbf{C} \mathbf{b}},$$

and thus, the piecewise linear relation between traction rate vector and velocity, holding in S_4 , results in:

$$\dot{\mathbf{p}} = \mathbf{C} \dot{\mathbf{u}}, \quad \text{if } \mathbf{a} \cdot \mathbf{C} \dot{\mathbf{u}} \leq 0 \quad (\text{stick}), \quad (9_1)$$

$$\dot{\mathbf{p}} = \left(\mathbf{C} - \frac{\mathbf{C} \mathbf{b} \otimes \mathbf{C} \mathbf{a}}{\mathbf{C} \mathbf{a} \cdot \mathbf{b}} \right) \dot{\mathbf{u}}, \quad \text{if } \mathbf{a} \cdot \mathbf{C} \dot{\mathbf{u}} > 0 \quad (\text{slip}), \quad (9_2)$$

so that stick or slip may occur depending on the incremental fields. Locking behavior of contact interface is assumed to be excluded, thus $\mathbf{C} \mathbf{a} \cdot \mathbf{b} > 0$ is assumed in (9)₂.

At this point, the formal analogy is evident between the interfacial constitutive laws holding on S_3 and S_4 and the rate of equations of non-associative, elastic–perfectly plastic solids. In particular, a similarity may be noted with the Drucker–Prager model, where the yield surface is a cone, and in its corner the incremental constitutive equations become thoroughly non-linear. Analogously, vectors \mathbf{a} and \mathbf{b} , eqn (8), and thus constitutive equations (9) are not defined when $\mathbf{p} = \mathbf{0}$. Zero contact pressure occurs in the grazing zone S_2 , where separation or stick/slip may occur depending on the sign of normal velocity \dot{u}_N , namely

$$\dot{\mathbf{p}} = \mathbf{0}, \quad \text{if } \dot{u}_N \leq 0 \quad (\text{separation}). \quad (10_1)$$

$$\dot{p}_N = k_N \dot{u}_N, \quad \dot{\mathbf{p}}_T = k_T \dot{\mathbf{u}}_T, \quad \text{if } \dot{u}_N > 0 \quad \text{and} \quad k_T |\dot{\mathbf{u}}_T| \leq \mu k_N \dot{u}_N \quad (\text{stick}), \quad (10_2)$$

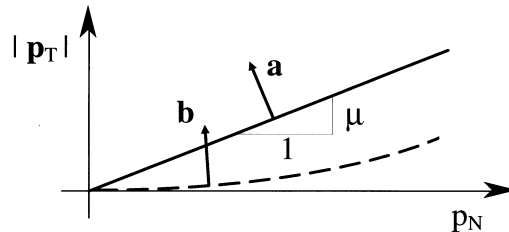


Fig. 2. Friction condition and sliding rule.

$$\dot{p}_N = k_N \dot{u}_N, \quad \dot{\mathbf{p}}_T = \mu k_N \dot{u}_N \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|}, \quad \text{if } \dot{u}_N > 0 \quad \text{and} \quad k_T |\dot{\mathbf{u}}_T| > \mu k_N \dot{u}_N \quad (\text{slip}). \tag{10_3}$$

Observe that $\dot{w} = 0$ as a consequence of $\beta(\mathbf{0}) = 0$. It may be important to note that the incremental constitutive equation (10) is thoroughly nonlinear and corresponds to the vertex behavior of model (9) for constant stiffness parameters of the interface, eqn (7).

It should also be noted that an approximation to Signorini’s law of contact with Coulomb friction can be recovered in our model as a limit behavior when the elastic stiffness tends to infinity, as the elastic–perfectly plastic model tends to the limit of the rigid–perfectly plastic behavior, when the elastic stiffness tends to infinity.

2.2. A possible generalization of the interfacial constitutive equations

It is interesting to note that, extending the analogy between friction and plasticity along the lines drawn in (Michalowski and Mróz, 1978; Curnier, 1984; Cheng and Kikuchi, 1985), the constitutive equations (9), which hold on S_4 , can be generalized to include some hardening as follows:

$$\dot{\mathbf{p}} = \mathbf{L} \dot{\mathbf{u}}, \tag{11}$$

where \mathbf{L} is the frictional interface constitutive tensor, namely

$$\mathbf{L} = \mathbf{C} \quad \text{if } \mathbf{a} \cdot \mathbf{C} \dot{\mathbf{u}} \leq 0 \quad (\text{stick}), \tag{12_1}$$

$$\mathbf{L} = \mathbf{C} - \frac{\mathbf{C} \mathbf{b} \otimes \mathbf{C} \mathbf{a}}{h + \mathbf{C} \mathbf{a} \cdot \mathbf{b}} \quad \text{if } \mathbf{a} \cdot \mathbf{C} \dot{\mathbf{u}} > 0 \quad (\text{slip}), \tag{12_2}$$

and h is the interface hardening coefficient (describing softening when negative, and assumed null for vanishing contact pressure, i.e. $h = 0$ for $\mathbf{p} = \mathbf{0}$). In the following, the quantity $h + \mathbf{C} \mathbf{a} \cdot \mathbf{b}$ is assumed positive. Even if the normality rule is generally considered inadequate to describe frictional behavior, Bowden and Tabor (1964) have proposed a model of interfacial behavior based on a normality rule. In the present context, this case simply corresponds to $\mathbf{a} = \mathbf{b}$.

3. Uniqueness criteria

Uniqueness criteria are obtained in this section following the method introduced by Hill (1958). Let $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ be two different solutions (under the same external loading rates $\dot{\mathbf{f}}$ and $\dot{\mathbf{t}}$). Let us introduce the admissible velocity function manifold

$$\mathcal{V} = \{ \mathbf{v} \in H^1(\Omega) : \mathbf{v} = \bar{\mathbf{v}} \text{ on } S_u \},$$

which is a subset of the Hilbert space $H^1(\Omega)$, and the tangent space of the manifold \mathcal{V} :

$$\mathcal{H} = \{\mathbf{v} \in H^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } S_u\}.$$

On application of the divergence theorem it follows that

$$\int_{\Omega} \boldsymbol{\varepsilon}(\Delta \mathbf{v}) \cdot \mathbb{E} \boldsymbol{\varepsilon}(\Delta \mathbf{v}) + \int_{S_c} \Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} = 0, \quad \forall \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathcal{V}, \quad (13)$$

where $\Delta \mathbf{v} = \mathbf{v}^{(1)} - \mathbf{v}^{(2)}$ and $\Delta \dot{\mathbf{p}} = \dot{\mathbf{p}}^{(1)} - \dot{\mathbf{p}}^{(2)}$ is the incrementally non-linear function of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ defined by (5)–(9) or by (5)–(8) and (11). From relation (13), an exclusion condition for bifurcation in Hill's sense can be given in the form

$$J_{\Omega}(\Delta \mathbf{v}) + J_c(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) > 0, \quad \forall \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathcal{V}, \quad (14)$$

where

$$J_{\Omega}(\Delta \mathbf{v}) = \int_{\Omega} \boldsymbol{\varepsilon}(\Delta \mathbf{v}) \cdot \mathbb{E} \boldsymbol{\varepsilon}(\Delta \mathbf{v}), \quad J_c(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) = \int_{S_c} \Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v}. \quad (15)$$

The quadratic functional (15)₁ is positive definite in the present context, being proportional to an elastic stored energy. Note that, in the absence of friction, condition (14) reduces to the well-known Kirchhoff uniqueness argument of elasticity. The non-linear functional (15)₂ may assume negative values and therefore uniqueness may be lost. A uniqueness condition similar to (14) was proposed by Chateau and Nguyen (1991).

3.1. Raniecki type linear comparison interface

Following Hill (1958), Raniecki (1979) and Raniecki and Bruhns (1981), we introduce a family of linear comparison interfaces, thus bonding the non-linear function (15)₂ in the zone S_4 from below with a quadratic functional of velocity difference. This family of comparison interfaces provides a lower bound to bifurcation as is defined through the constitutive tensor

$$\mathbf{L}^R = \mathbf{C} - \frac{\mathbf{C}(\mathbf{b} + \psi \mathbf{a}) \otimes \mathbf{C}(\mathbf{b} + \psi \mathbf{a})}{4\psi(h + \mathbf{C}\mathbf{a} \cdot \mathbf{b})}, \quad (16)$$

where the arbitrary parameter $\psi > 0$ defines the family of comparison interfaces. The following comparison theorem holds true:

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} \geq \Delta \mathbf{v} \cdot \mathbf{L}^R \Delta \mathbf{v}, \quad \forall \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathcal{V}. \quad (17)$$

Proof: The proof is similar to that of Raniecki (1979) for non-associative plasticity and is only summarized here. In the two cases $\mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(i)} < 0$ or $\mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(i)} > 0$, with $i = 1, 2$, $\Delta \dot{\mathbf{p}} = \mathbf{L}\Delta \mathbf{v}$ and it can be shown that

$$\Delta \mathbf{v} \cdot (\mathbf{L} - \mathbf{L}^R) \Delta \mathbf{v} = \begin{cases} \frac{[\mathbf{C}(\mathbf{b} + \psi \mathbf{a}) \cdot \Delta \mathbf{v}]^2}{4\psi(h + \mathbf{C}\mathbf{a} \cdot \mathbf{b})} \geq 0 & \text{if } \mathbf{C}\mathbf{v}^{(i)} \cdot \mathbf{a} < 0 \\ \frac{[\mathbf{C}(\mathbf{b} - \psi \mathbf{a}) \cdot \Delta \mathbf{v}]^2}{4\psi(h + \mathbf{C}\mathbf{a} \cdot \mathbf{b})} \geq 0 & \text{if } \mathbf{C}\mathbf{v}^{(i)} \cdot \mathbf{a} > 0 \end{cases}$$

so that (17) is verified.

The condition $\mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(i)} < 0$ and $\mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(j)} > 0$, with $i, j = 1, 2 (i \neq j)$, only needs to be examined. In this case, algebraic manipulations yield

$$\begin{aligned} &4\psi(h + \mathbf{C}\mathbf{a} \cdot \mathbf{b})(\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} - \Delta \mathbf{v} \cdot \mathbf{L}^R \Delta \mathbf{v}) \\ &= [2\psi \mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(i)} + (j - i)(\mathbf{b} - \psi \mathbf{a}) \cdot \mathbf{C}\Delta \mathbf{v}]^2 - 4\psi^2(\mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(i)})(\mathbf{a} \cdot \mathbf{C}\mathbf{v}^{(j)}) \geq 0. \end{aligned}$$

3.2. A new linear comparison interface for the grazing zone

In the grazing zone S_2 , the incremental constitutive equations are thoroughly non-linear. In this zone we define a linear comparison interface bounding the non-linear functional (15)₂ from below with a quadratic functional of velocity difference. As for the Raniecki comparison solid, also the new comparison interface provides a lower bound to bifurcation.

We prove the following comparison theorem at every point of S_2 , under the constitutive assumption (10):

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} \geq -k_N \frac{\sqrt{1 + \mu^2} - 1}{2} |\Delta \mathbf{v}|^2, \quad \forall \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathcal{V}. \tag{18}$$

Proof: The proof follows from the preliminary lemma:

$$\mu \Delta v_N \Delta |\mathbf{v}_T| + (\Delta v_N)^2 \geq -\frac{\sqrt{1 + \mu^2} - 1}{2} |\Delta \mathbf{v}|^2, \quad \forall \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathcal{V}, \tag{19}$$

which can be obtained from

$$\begin{aligned} \mu \Delta v_N \Delta |\mathbf{v}_T| + (\Delta v_N)^2 &= [\Delta v_N, \Delta |\mathbf{v}_T|] \begin{bmatrix} 1 & \mu/2 \\ \mu/2 & 0 \end{bmatrix} \begin{bmatrix} \Delta v_N \\ \Delta |\mathbf{v}_T| \end{bmatrix} \\ &\geq -\frac{\sqrt{1 + \mu^2} - 1}{2} [(\Delta v_N)^2 + (\Delta |\mathbf{v}_T|)^2] \end{aligned}$$

and $|\Delta \mathbf{v}|^2 \geq [(\Delta v_N)^2 + (\Delta |\mathbf{v}_T|)^2]$.

The proof is now divided into six cases, in which two solutions belonging to different constitutive situations are considered.

- (1) $\mathbf{v}^{(1)}$ corresponds to separation ($v_N^{(1)} \leq 0, \dot{\mathbf{p}}^{(1)} = \mathbf{0}$) and $\mathbf{v}^{(2)}$ to slip (10)₃:

$$\begin{aligned}\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} &= k_N v_N^{(2)} \left[\mu \left(|\mathbf{v}_T^{(2)}| - \frac{\mathbf{v}_T^{(2)} \cdot \mathbf{v}_T^{(1)}}{|\mathbf{v}_T^{(2)}|} \right) - \Delta v_N \right] \\ &\geq \min \{0, k_N [\mu \Delta v_N \Delta |\mathbf{v}_T| + (\Delta v_N)^2]\},\end{aligned}$$

where $-\Delta v_N \geq v_N^{(2)} > 0$ has been used.

(2) Both $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ correspond to slip (10)₃:

$$\begin{aligned}\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} &= \mu k_N \left[v_N^{(2)} |\mathbf{v}_T^{(2)}| + v_N^{(1)} |\mathbf{v}_T^{(1)}| - \mathbf{v}_T^{(1)} \cdot \mathbf{v}_T^{(2)} \left(\frac{v_N^{(1)}}{|\mathbf{v}_T^{(1)}|} + \frac{v_N^{(2)}}{|\mathbf{v}_T^{(2)}|} \right) \right] + k_N (\Delta v_N)^2 \\ &\geq k_N [\mu \Delta v_N \Delta |\mathbf{v}_T| + (\Delta v_N)^2],\end{aligned}$$

where $\mathbf{v}_T^{(1)} \cdot \mathbf{v}_T^{(2)} \leq |\mathbf{v}_T^{(1)}| |\mathbf{v}_T^{(2)}|$ has been used.

(3) $\mathbf{v}^{(1)}$ corresponds to separation ($v_N^{(1)} \leq 0$, $\dot{\mathbf{p}}^{(1)} = \mathbf{0}$) and $\mathbf{v}^{(2)}$ to stick (10)₂:

$$\begin{aligned}\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} &= k_T |\mathbf{v}_T^{(2)}|^2 - k_T \mathbf{v}_T^{(1)} \cdot \mathbf{v}_T^{(2)} + (k_N v_N^{(2)})^2 - k_N v_N^{(1)} v_N^{(2)} \\ &\geq -k_T |\mathbf{v}_T^{(2)}| \left(\Delta |\mathbf{v}_T| + \frac{1}{\mu} \Delta v_N \right) \\ &\geq \min \{0, k_N [\mu \Delta v_N \Delta |\mathbf{v}_T| + (\Delta v_N)^2]\},\end{aligned}$$

where $-\Delta v_N \geq v_N^{(2)} > 0$ and $k_T |\mathbf{v}_T^{(2)}| \leq \mu k_N v_N^{(2)}$ have been used.

(4) $\mathbf{v}^{(1)}$ corresponds to stick (10)₂ and $\mathbf{v}^{(2)}$ to slip (10)₃:

$$\begin{aligned}\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} &= \mu k_N v_N^{(2)} \left(|\mathbf{v}_T^{(2)}| - \frac{\mathbf{v}_T^{(2)} \cdot \mathbf{v}_T^{(1)}}{|\mathbf{v}_T^{(2)}|} \right) - k_T \mathbf{v}_T^{(1)} \cdot (\mathbf{v}_T^{(2)} - \mathbf{v}_T^{(1)}) + k_N (\Delta v_N)^2 \\ &\geq -(\mu k_N v_N^{(2)} - k_T |\mathbf{v}_T^{(1)}|) \Delta |\mathbf{v}_T| + k_N (\Delta v_N)^2.\end{aligned}$$

Now, $\Delta |\mathbf{v}_T| > 0 \Rightarrow \mu k_N v_N^{(2)} - k_T |\mathbf{v}_T^{(1)}| < 0 \Rightarrow \Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} \geq 0$, whereas if $\Delta |\mathbf{v}_T| \leq 0$:

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} \geq k_N \Delta v_N (\mu \Delta |\mathbf{v}_T| + \Delta v_N).$$

(5) Both $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ correspond to stick (10)₂:

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} = k_T |\Delta \mathbf{v}_T|^2 + k_N (\Delta v_N)^2 \geq 0.$$

(6) Both $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ correspond to separation ($v_N^{(1)} \leq 0$ and $v_N^{(2)} \leq 0$, $\dot{\mathbf{p}}^{(1)} = \dot{\mathbf{p}}^{(2)} = \mathbf{0}$):

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} = 0.$$

In the special case in which k_N is a function of the current state, null for $\mathbf{p} = \mathbf{0}$, the comparison solid (18) gives $\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} \geq 0$. The case in which k_T vanishes for $\mathbf{p} = \mathbf{0}$ and k_N is not negative may be also interesting and is analyzed in the Appendix.

3.3. Exclusion condition for bifurcation with quadratic functionals

The functional J_c in the exclusion condition (14) is the sum of the four contributions relative to the zones into which S_c has been divided, namely: $J_c = J_1 + J_2 + J_3 + J_4$, where

$$J_k(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) = \int_{S_k} \Delta \mathbf{p} \cdot \Delta \mathbf{v} \quad (k = 1, 2, 3, 4). \tag{20}$$

It is necessary to separately analyze the four contributions J_k . Firstly, it may be easily checked that $J_1 = 0$, since $\Delta \mathbf{p} = \mathbf{0}$ on S_1 as follows from (5).

We are now in a position to formulate the sufficient condition for uniqueness in terms of two quadratic functionals: The velocity problem of an elastic body with part of the boundary in frictional, unilateral contact with a stiffer constraint, defined by relations (5)–(11), is unique if

$$J_\Omega(\mathbf{v}) + J_c(\mathbf{v}) > 0, \quad \forall \mathbf{v} \in \mathcal{H}, \tag{21}$$

where

$$J_c(\mathbf{v}) = J_2(\mathbf{v}) + J_3(\mathbf{v}) + J_4(\mathbf{v}),$$

$$J_2(\mathbf{v}) = \int_{S_2} k_N \frac{1 - \sqrt{1 + \mu^2}}{2} |\mathbf{v}|^2, \quad J_3(\mathbf{v}) = \int_{S_3} \mathbf{v} \cdot \mathbf{C} \mathbf{v}, \quad J_4(\mathbf{v}) = \int_{S_4} \mathbf{v} \cdot \mathbf{L}^R \mathbf{v}. \tag{22}$$

Therefore, the contributions to J_c from S_2 , S_3 and S_4 can be bounded by three quadratic functionals.

3.4. Lower bounds for J_2 , J_3 and J_4

Functionals J_2 , J_3 and J_4 defined in (22) can be estimated. In particular,

$$J_2(\mathbf{v}) \geq \inf_{S_2} \left\{ k_N \frac{1 - \sqrt{1 + \mu^2}}{2} \right\} \|\mathbf{v}\|_{L^2(S_2)}^2 \geq 0, \quad \forall \mathbf{v} \in \mathcal{H}, \tag{23}$$

and a lower bound to J_3 is consequent to the positive definiteness of tensor \mathbf{C} :

$$J_3(\mathbf{v}) \geq \inf_{S_3} \{ \gamma \} \|\mathbf{v}\|_{L^2(S_3)}^2 \geq 0, \quad \forall \mathbf{v} \in \mathcal{H}, \tag{24}$$

where γ is the minimum (positive) eigenvalue of \mathbf{C} . A lower bound for J_4 can be found on the basis of the following proposition.

For every vector \mathbf{v} , the following inequality holds true:

$$\mathbf{v} \cdot \mathbf{L}^R \mathbf{v} \geq -\rho^R \Gamma |\mathbf{v}|^2, \tag{25}$$

where

$$\rho^R = \frac{1}{2} \left[\frac{\sqrt{(\mathbf{Ca} \cdot \mathbf{a})(\mathbf{Cb} \cdot \mathbf{b})} - h}{h + \mathbf{Ca} \cdot \mathbf{b}} - 1 \right] \quad (26)$$

and Γ is the maximum eigenvalue of \mathbf{C} in the case $\rho^R > 0$.¹ Note that ρ^R is not less than zero for $h \leq 0$, but it may be less than zero for $h > 0$.

Proof: The Cauchy–Schwarz inequality in the metric induced by \mathbf{C} (which is symmetric and positive definite), i.e. $(\mathbf{a} \cdot \mathbf{Cb})^2 \leq (\mathbf{a} \cdot \mathbf{Ca})(\mathbf{b} \cdot \mathbf{Cb})$ yields for every vector \mathbf{v} :

$$\begin{aligned} \mathbf{v} \cdot \mathbf{L}^R \mathbf{v} &\geq \left[1 - \frac{(\mathbf{b} + \psi \mathbf{a}) \cdot \mathbf{C}(\mathbf{b} + \psi \mathbf{a})}{4\psi(h + \mathbf{Ca} \cdot \mathbf{b})} \right] (\mathbf{v} \cdot \mathbf{Cv}) \\ &= \frac{1}{2} \left[1 - \frac{\mathbf{b} \cdot \mathbf{Cb} + \psi^2 \mathbf{a} \cdot \mathbf{Ca} - 2\psi h}{2\psi(h + \mathbf{Ca} \cdot \mathbf{b})} \right] (\mathbf{v} \cdot \mathbf{Cv}). \end{aligned} \quad (27)$$

The last term of the right hand side of (27) attains a maximum for

$$\psi = \sqrt{\frac{\mathbf{Cb} \cdot \mathbf{b}}{\mathbf{Ca} \cdot \mathbf{a}}},$$

and therefore, for every vector \mathbf{v} :

$$\mathbf{v} \cdot \mathbf{L}^R \mathbf{v} \geq -\frac{1}{2} \left[\frac{\sqrt{(\mathbf{Ca} \cdot \mathbf{a})(\mathbf{Cb} \cdot \mathbf{b})} - h}{h + \mathbf{Ca} \cdot \mathbf{b}} - 1 \right] (\mathbf{v} \cdot \mathbf{Cv}).$$

It should be noted that the bound (25) is optimal because equality holds for $\mathbf{v} = \mathbf{b} + \psi \mathbf{a}$.

From the above proposition, the following lower bound to $J_4(\mathbf{v})$ can be given:

$$J_4(\mathbf{v}) \geq -\sup_{S_4} \{ \Gamma \rho^R \} \|\mathbf{v}\|_{L^2(S_4)}^2, \quad \forall \mathbf{v} \in \mathcal{H}. \quad (28)$$

Just neglecting the positive contribution of J_3 , the following lower bound for J_c trivially follows:

$$J_c(\mathbf{v}) \geq -\max \left\{ \sup_{S_2} \left\{ k_N \frac{\sqrt{1 + \mu^2} - 1}{2} \right\}, \sup_{S_4} \{ \Gamma \rho^R \} \right\} \|\mathbf{v}\|_{L^2(S_2 \cup S_4)}^2, \quad \forall \mathbf{v} \in \mathcal{H}. \quad (29)$$

3.5. Relations between the two linear comparison interfaces

Let us consider for \mathbf{C} representation (7) and $\beta = h = 0$. The following inequality holds true:

¹ In the case $\rho^R \leq 0$, Γ should correspond to the minimum eigenvalue. This case is less interesting, because it is $J_4 \geq 0$ (see also the following Section 3.6).

$$k_N \frac{\sqrt{1+\mu^2}-1}{2} \leq \max\{k_N, k_T\} \rho^R, \quad (30)$$

where, in this case (26) reduces to

$$\rho^R = \frac{1}{2} \left(\sqrt{1 + \mu^2 \frac{k_N}{k_T}} - 1 \right).$$

Equality occurs in (30) for $k_N = k_T$ or $\mu = 0$. A consequence of inequality (30) is that the Raniecki comparison interface may still be used to provide a bound also in the grazing zone S_2 . However, this bound is not optimal.

3.6. A local condition for uniqueness

Invoking experimental evidence, it is often assumed in the literature that the compliance of the contact constraint, k_N and k_T , is a function of the contact pressure (Oden and Martins, 1985; Klarbring, 1990a; Buczkowski and Kleiber, 1997). In particular, it is often assumed that both the normal and tangential compliances vanish at zero contact pressure. In this case, $\dot{\mathbf{p}} = \mathbf{0}$, the grazing and separation zones can be included in the same zone, for which $J_1 = J_2 = 0$. This situation also applies in the case where the tangential compliance is null for zero contact pressure, but the normal compliance is not (see the Appendix). In other cases, as for instance in Stankowski et al. (1993), the separation and consequently the vertex behavior in the grazing zone are assumed not to exist. In all these cases (where J_2 is a priori null) as J_Ω is always positive, $\rho^R < 0$ is a local sufficient condition for uniqueness. This condition can be written in terms of a critical value of the interfacial hardening modulus. Therefore, the solution of the contact problem is unique when

$$h > h_{cr} = \frac{1}{2} \{ \sqrt{(\mathbf{Ca} \cdot \mathbf{a})(\mathbf{Cb} \cdot \mathbf{b})} - \mathbf{Ca} \cdot \mathbf{b} \} \geq 0. \quad (31)$$

Condition (31) is analogous to the condition of positiveness of second order work in plasticity (Maier and Hueckel, 1979). Extending this analogy, it may be interesting to note that the interfacial operator \mathbf{L} is positive definite for $h > h_{cr}$. This follows immediately from the comparison theorem, noting that for every vector \mathbf{v} ,

$$\mathbf{v} \cdot \mathbf{L}\mathbf{v} \geq \mathbf{v} \cdot \mathbf{L}^R \mathbf{v}.$$

But \mathbf{L}^R is positive definite for $h > h_{cr}$, and $\mathbf{v} \cdot \mathbf{L}\mathbf{v} = 0$ for $h = h_{cr}$ and

$$\mathbf{v} = \mathbf{b} \sqrt{\mathbf{Ca} \cdot \mathbf{a}} + \mathbf{a} \sqrt{\mathbf{Cb} \cdot \mathbf{b}}.$$

Note that the coincidence between loss of positive definiteness of the two operators \mathbf{L}^R and \mathbf{L} implies that bifurcation is excluded when \mathbf{L} is positive definite. This result is analogous to the situation of elastoplasticity (Raniecki, 1979).

3.7. Lower bounds for J_Ω

Estimates of the functional J_Ω are presented in the following by using classical arguments of functional analysis. When $\text{meas}\{S_u\} > 0$, it may be proved that there is a positive constant K , which depends on the geometry of the body and the extent of S_u , such that

$$\|\mathbf{e}(\mathbf{u})\|_{L^2(\Omega)} \geq K\|\mathbf{u}\|_{H^1(\Omega)}, \quad \forall \mathbf{u} \in \mathcal{H}, \quad (32)$$

where $\mathbf{e}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$. Note that inequality (32) is an immediate consequence of the Korn and Poincaré inequalities when $\partial\Omega \equiv S_u$. The generalization to the case $\text{meas}\{S_u\} > 0$ was given in (Fichera, 1972; see also Brenner and Scott, 1994, Section 9.2).

If $\alpha > 0$ denotes the minimum eigenvalue of \mathbb{E} , by using standard coercivity arguments we can write

$$J_\Omega(\mathbf{v}) \geq \alpha\|\mathbf{e}(\mathbf{v})\|_{L^2(\Omega)}^2 \geq \alpha K^2\|\mathbf{v}\|_{H^1(\Omega)}^2, \quad \forall \mathbf{v} \in \mathcal{H}. \quad (33)$$

An application of the trace theorem:

$$c_0\|\gamma_0(\mathbf{u})\|_{L^2(\partial\Omega)} \leq \|\mathbf{u}\|_{H^1(\Omega)}, \quad \forall \mathbf{u} \in H^1(\Omega), \quad (34)$$

where γ_0 is the trace operator, c_0 is a positive constant (depending on the geometry of the body, for instance for a unit disk $c_0 = 8^{-1/4}$; see Brenner and Scott, 1994, Section 1.6), yields

$$J_\Omega(\mathbf{v}) \geq \alpha(c_0K)^2\|\gamma_0(\mathbf{v})\|_{L^2(\partial\Omega)}^2 \geq \alpha(c_0K)^2\|\mathbf{v}\|_{L^2(S_2 \cup S_4)}^2, \quad \forall \mathbf{v} \in \mathcal{H}. \quad (35)$$

It should be noted that the last inequality of (35) clearly underestimates J_Ω , since the bound is obtained considering only the contribution of zones S_2 and S_4 .

3.8. Lower bounds to the sufficient conditions for uniqueness

Finally, from the lower bound (29) to J_c and (35) to J_Ω , it follows:

$$J_\Omega(\mathbf{v}) + J_c(\mathbf{v}) \geq \left[\alpha(c_0K)^2 - \max \left\{ \sup_{S_2} \left\{ k_N \frac{\sqrt{1+\mu^2}-1}{2} \right\}, \sup_{S_4} \{ \Gamma \rho^R \} \right\} \right] \|\mathbf{v}\|_{L^2(S_2 \cup S_4)}^2, \quad (36)$$

$$\forall \mathbf{v} \in \mathcal{H},$$

and thus, an exclusion condition for bifurcation can be given in the form

$$\max \left\{ \sup_{S_2} \left\{ k_N \frac{\sqrt{1+\mu^2}-1}{2} \right\}, \sup_{S_4} \{ \Gamma \rho^R \} \right\} < \alpha(c_0K)^2, \quad (37)$$

where Γ and ρ^R depend on the constitutive laws of the interface. Moreover, the positive constants α , c_0 and K depend on the elastic constitutive tensor, the geometry and boundary conditions of the solid Ω . A condition analogous to (37) was obtained by Klarbring et al. (1988) for a particular constitutive assumption concerning the

normal compliance. It may be important to remark that condition (37) depends on the extent of the grazing zone S_2 and of the slip/stick zone S_4 . However, in the interesting case in which $h = \beta = 0$, μ is constant and \mathbf{C} admits representation (7) with constant values of k_N and k_T , condition (37) can be made independent of these zones, just observing from (30) that

$$\max \left\{ \sup_{S_2} \left\{ k_N \frac{\sqrt{1 + \mu^2} - 1}{2} \right\}, \sup_{S_4} \{ \Gamma \rho^R \} \right\} \leq \max \{ k_N, k_T \} \frac{1}{2} \left(\sqrt{1 + \mu^2 \frac{k_N}{k_T}} - 1 \right), \tag{38}$$

and therefore the exclusion condition becomes

$$\max \{ k_N, k_T \} \frac{1}{2} \left(\sqrt{1 + \mu^2 \frac{k_N}{k_T}} - 1 \right) < \alpha (c_0 K)^2. \tag{39}$$

It should be noted that condition (39) is independent of the loading program and therefore uniqueness can be a priori established.

4. Examples

We present in this section applications of the exclusion criterion (39) to a two-dimensional elastic system and to a 2 D.O.F. elastic structure. The conditions of bifurcation of the elastic structure were already known from Curnier and Alart (1988) and Klarbring (1990b), in the limit case of Signorini’s contact with Coulomb friction. All the following examples are referred to a simple constitutive interface model without hardening and wear, i.e. with $\beta = h = 0$, which is explained in the following.

4.1. Specialization of exclusion condition to a simple constitutive model

The elastic constitutive tensor of the body is assumed isotropic, namely

$$\mathbb{E} = 2G\mathbb{I} + \lambda\mathbf{I} \otimes \mathbf{I}, \tag{40}$$

where G and λ are the Lamé constants and the interface stiffness tensor \mathbf{C} is assumed in the form (7) with constant k_N and k_T . Therefore, the eigenvalue α is equal to $2G$. Furthermore, wear and hardening at the interface are neglected (i.e. $\beta = h = 0$), and thus

$$\mathbf{a} = \frac{\mathbf{p}_T}{|\mathbf{p}_T|} - \mu \mathbf{n}, \quad \mathbf{b} = \frac{\mathbf{p}_T}{|\mathbf{p}_T|}, \tag{41}$$

where μ is assumed constant. In this case, Γ coincides with $\max \{ k_N, k_T \}$, and the sufficient exclusion condition for bifurcation (39) becomes

$$\mu^2 < \frac{8Gc_0^2 K^2 k_T}{k_N \max\{k_N, k_T\}} \left(1 + \frac{2Gc_0^2 K^2}{\max\{k_N, k_T\}} \right). \quad (42)$$

In the limit case when $k_N \rightarrow \infty$, we obtain an approximation of the impenetrability condition, corresponding to the Signorini problem with Coulomb friction. In this case, condition (42) gives as a limit value $\mu = 0$, a circumstance also occurring in an analogous condition given by Klarbring et al. (1988). This trivial result may be related to the fact that the sufficient condition for uniqueness turns out to be in this limit case over-sufficient.

In the other limit case when $k_T \rightarrow \infty$, a finite limit is obtained from (42):

$$\mu^2 < \frac{8Gc_0^2 K^2}{k_N}. \quad (43)$$

4.2. Elastic square domain with friction on one side

A linear-elastic, isotropic square domain (having size dimension a) is considered, as shown in Fig. 3. On the left side of the domain displacements are prescribed to be zero, i.e. $u_i(0, x_2) = 0$ ($i = 1, 2$).

On the right side a frictional constraint is present, corresponding to the constitutive equations (7) and (41) and on the upper and lower sides a generic, but symmetric (about the x_1 -axis) system of forces is prescribed, i.e.

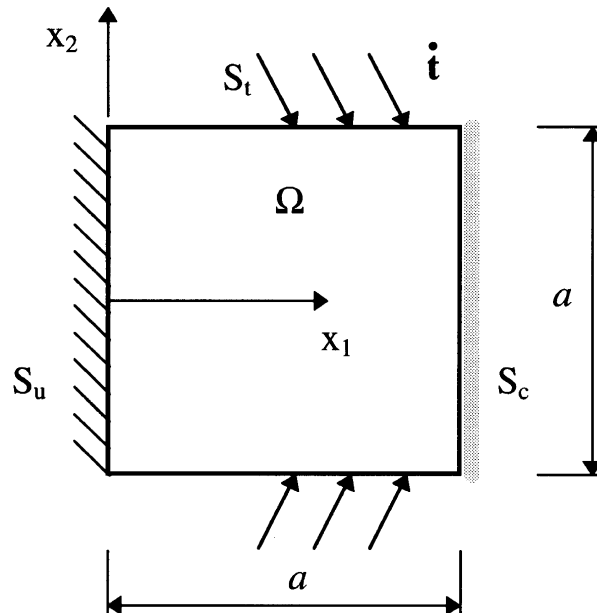


Fig. 3. Two-dimensional elastic domain with one side in frictional contact.

$$t_2(x_1, a/2) = -t_2(x_1, -a/2), \quad t_1(x_1, a/2) = t_1(x_1, -a/2). \tag{44}$$

Bifurcations with null mean spin (including symmetric bifurcations about the x_1 -axis) can be excluded as follows. First, we want to bound the norm in $L^2(\Omega)$ of the velocity gradient field in the body, with the norm in $L^2(S_c)$ of the velocity in the zone of contact. To this purpose, following an argument similar to Villaggio (1977, Section 9.3) and Brenner and Scott (1994, Section 1.6), we write, for $v_i \in C^1(\bar{\Omega})$,

$$v_i(x_1, x_2) - v_i(0, x_2) = \int_0^{x_1} \frac{\partial v_i(t, x_2)}{\partial t} dt \quad (i = 1, 2), \tag{45}$$

which, taking into account the condition $v_i(0, x_2) = 0$, squaring each member and applying the Cauchy–Schwarz inequality, becomes

$$v_i^2(x_1, x_2) \leq x_1 \int_0^{x_1} \left(\frac{\partial v_i}{\partial x_1} \right)^2 dx_1 \leq a \int_0^a \left(\frac{\partial v_i}{\partial x_1} \right)^2 dx_1 \quad (i = 1, 2). \tag{46}$$

Evaluating (46) for $x_1 = a$ and summing the components, we obtain

$$v_1^2(a, x_2) + v_2^2(a, x_2) \leq a \int_0^a |\nabla \mathbf{v}|^2 dx_1. \tag{47}$$

Finally, integration for x_2 between $-a/2$ and $a/2$ of both sides of (47) yields

$$\|\mathbf{v}\|_{L^2(S_c)}^2 \leq a \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2. \tag{48}$$

Now, we restrict the attention to bifurcation satisfying null mean spin, i.e.

$$\int_{\Omega} (v_{i,j} - v_{j,i}) d\Omega = 0,$$

which includes symmetric bifurcations. For this case Korn’s constant has been bounded by Horgan and Payne (1983) (see also Horgan, 1995) between 4 and $8 + 4\sqrt{2}$ and its precise value has been conjectured to be seven. Assuming this conjecture value, we obtain

$$\|\mathbf{v}\|_{L^2(S_2 \cup S_4)}^2 \leq \|\mathbf{v}\|_{L^2(S_c)}^2 \leq a \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2 < 7a \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)}^2, \tag{49}$$

and we can conclude that in this case that the constant $(c_0 K)^2$ in (42) takes the value $1/(7a)$. Note that in this way, condition (42) depends on the dimensional parameter a .

4.3. A structural example

Let us consider the elastic frame in Fig. 4, having a frictional constraint on one edge. The elastic incremental relations between forces and displacements at the point in frictional unilateral contact, are given by

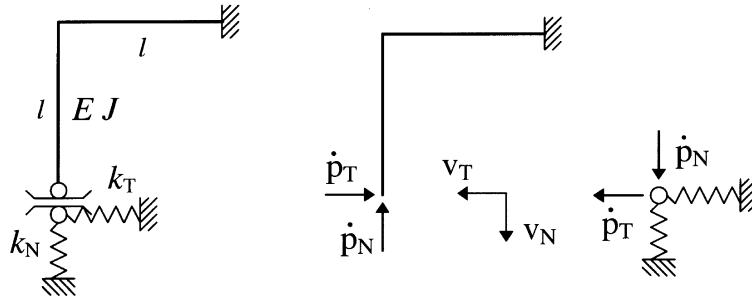


Fig. 4. Elastic frame with frictional constraint.

$$-\dot{\mathbf{p}} = \mathbf{K}\mathbf{v}, \quad (50)$$

where

$$\mathbf{p} = \{p_N, p_T\}, \quad \mathbf{v} = \{v_N, v_T\}, \quad \mathbf{K} = \kappa \begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}, \quad (51)$$

and $\kappa = 6EJ/(7l^3)$, being EJ the flexural rigidity of the frame elements.

Moreover, for the frictional support with elastic compliance, we have

$$\mathbf{a} = \left\{ -\mu, \frac{p_T}{|p_T|} \right\}, \quad \mathbf{b} = \left\{ 0, \frac{p_T}{|p_T|} \right\}, \quad \mathbf{C} = \begin{bmatrix} k_N & 0 \\ 0 & k_T \end{bmatrix}, \quad (52)$$

and thus a substitution of (52) into (26), by considering $h = 0$, gives

$$\rho^R = \frac{1}{2} \left(\sqrt{1 + \frac{k_N}{k_T} \mu^2} - 1 \right). \quad (53)$$

In this case, the two functionals in the sufficient condition for uniqueness (21) may be directly estimated to be

$$J_\Omega(\mathbf{v}) = \int_\Omega \frac{\dot{M}^2}{EJ} dx = \mathbf{v} \cdot \mathbf{K}\mathbf{v}, \quad J_c(\mathbf{v}) \geq -\rho^R(\mathbf{v} \cdot \mathbf{C}\mathbf{v}), \quad (54)$$

where M is the bending moment. Therefore, bifurcation is excluded when

$$\mathbf{v} \cdot (\mathbf{K} - \rho^R \mathbf{C})\mathbf{v} > 0, \quad \forall \mathbf{v} \neq \mathbf{0}. \quad (55)$$

The 2×2 symmetric matrix $(\mathbf{K} - \rho^R \mathbf{C})$ turns out to be positive definite if both its determinant and trace are positive, namely if $\rho^R < \rho_c$, where

$$\rho_c = \frac{\kappa}{k_T k_N} \{4k_T + k_N - \sqrt{16k_T^2 + k_N^2 + k_T k_N}\}. \quad (56)$$

Therefore, by using results (53) and (56) uniqueness is ensured when $\mu < \mu_R$, where

$$\mu_R = \left[4\rho_c(1 + \rho_c) \frac{k_T}{k_N} \right]^{1/2}. \quad (57)$$

It should be noted that in the limit case of Signorini's contact with Coulomb friction, obtained as $k_N \rightarrow \infty$, the critical value of the frictional coefficient (57) reduces to $\mu_R = 0$. On the other hand, it is known from Klarbring (1990b) that bifurcation may occur for $\mu \geq \mu_H = 2/3$. Therefore, the criterion of exclusion of bifurcation turns out to be in this limit case over-sufficient.

5. Conclusions

Conditions for bifurcation in velocities of linear elastic solids in frictional, quasi-static, unilateral contact with a stiffer constraint on a part of the boundary have been examined. A normal and tangential compliance has been assumed in the zone of contact. This is a largely used assumption for contact problems. Global exclusion conditions for bifurcation in velocity have been proposed, which are similar to conditions formulated for elastoplasticity (Hill, 1958; Raniecki, 1979; Raniecki and Bruhns, 1981). Global and local exclusion conditions for bifurcation are derived making use of results from functional analysis. It may be important to mention that in the specific case where the contact pressure is zero, the assumed incremental constitutive laws for the contact are thoroughly non-linear. For this behavior, an incrementally linear composition contact law has been formulated.

The main advantage of the proposed approach to bifurcation, an alternative to that of Klarbring et al. (1988) and Klarbring (1990a), is the possibility of a generalization to include large strain and elastoplastic behavior of the solid in contact. Its main shortcoming is related to the fact that the conditions for uniqueness may well be often over-sufficient, a fact already known in the context of elastoplasticity. Due to its connections with elastoplasticity theory, the proposed method of analysis furnishes a new key to explore the behavior of non-associative elastoplastic solids, taking advantage of the analogy with friction.

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Appendix: A special interfacial constitutive law

We consider here the case in which the elastic compliance of the interface \mathbf{C} reduces, for vanishing contact pressure $\mathbf{p} = \mathbf{0}$, to the law

$$\mathbf{C}_0 = k_0 \mathbf{n} \otimes \mathbf{n}, \quad (\text{A1})$$

where k_0 is a (non-negative) normal stiffness coefficient. As a consequence of (A1), tangential stiffness is assumed not to occur for vanishing normal pressure. For this choice of contact law in the grazing zone, the comparison interface (18) still works, but is not optimal. It may in fact be proved that

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} \geq 0, \quad \forall \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \in \mathcal{V}, \quad (\text{A2})$$

when (A1) holds.

Proof: The proof is divided into three cases.

(i) $v_N^{(i)} \leq 0$, for $i = 1, 2$. Then, from (10)₁, $\dot{\mathbf{p}}^{(i)} = \mathbf{0}$, and thus

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} = 0.$$

(ii) $v_N^{(i)} > 0$ for $i = 1, 2$. Then, from (10)₂ and (8), $\dot{\mathbf{p}}^{(i)} = k_0 v_N^{(i)} \mathbf{n}$, and thus

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} = k_0 (\Delta v_N)^2 \geq 0.$$

(iii) $v_N^{(i)} \leq 0$, and $v_N^{(j)} > 0$, for $i, j = 1, 2$ ($i \neq j$). Then, from (10) and (8), $\dot{\mathbf{p}}^{(i)} = \mathbf{0}$, and $\dot{\mathbf{p}}^{(j)} = k_0 v_N^{(j)} \mathbf{n}$, and thus

$$\Delta \dot{\mathbf{p}} \cdot \Delta \mathbf{v} = k_0 [(v_N^{(j)})^2 - v_N^{(i)} v_N^{(j)}] \geq 0.$$

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