# Non-holonomic constraints inducing flutter instability in structures under conservative loadings

**Alessandro Cazzolli, Francesco Dal Corso, Davide Bigoni***

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**- SUPPLEMENTARY MATERIAL -**

***DICAM, University of Trento, via Mesiano 77, I-38123 Trento, Italy.***

****E-mail: bigoni@ing.unitn.it***

## 1 Explicit expression for the coefficients $\rho_i$ of the characteristic polynomial for the non-holonomic double pendulum

From Eqs.(58), the explicit expressions for the coefficients $\rho_i$ $(i = 0, ..., 4)$ can be obtained as

$$\rho_0 = \frac{1}{4}\left(\tilde{I}_{r,L}\,\tilde{M}_L + \frac{1}{16}\tilde{d}^4 + \frac{1}{2}\tilde{d}^2\,\tilde{I}_{r,L} + \frac{1}{4}\tilde{d}^2\,\tilde{M}_L - \frac{1}{4}\tilde{d}\,\tilde{M}_L + \frac{1}{2}\tilde{I}_{r,L} + \frac{1}{8}\tilde{M}_L\right), \tag{SM 1}$$

$$\begin{aligned}\rho_1 =& \frac{1}{64}\left(\frac{5}{3}\tilde{c}_e\,\tilde{d}^2 - \tilde{c}_e\,\tilde{d} + \frac{1}{3}\tilde{c}_e\right) + \frac{1}{16}\left(\frac{1}{3}\tilde{c}_e\tilde{M}_L + \tilde{c}_{t,L}\,\tilde{d}^2 - \tilde{c}_{t,L}\,\tilde{d} + \frac{1}{2}\tilde{c}_{t,L}\right) + \frac{1}{8}\Big(\tilde{c}_{r,L}\,\tilde{d}^2 + \tilde{c}_{r,L} + 2\tilde{c}_{r,L}\,\tilde{M}_L + \\ &+ 2\tilde{c}_{t,L}\,\tilde{I}_{r,L} + 3\,\tilde{c}_i\,\tilde{d}^2 + 2\tilde{c}_i\,\tilde{d} + \tilde{c}_i + 10\,\tilde{c}_i\tilde{M}_L\Big) + 2\,\tilde{c}_i\,\tilde{I}_{r,L} + \frac{1}{6}\tilde{c}_e\,\tilde{I}_{r,L},\end{aligned} \tag{SM 2}$$

$$\begin{aligned}\rho_2 =& \frac{7}{2304}\tilde{c}_e^2 + \frac{1}{16}\left(\frac{1}{3}\tilde{c}_e\,\tilde{c}_{t,L} - \tilde{d}^2\tilde{p}^{\text{QS}} - \tilde{d}\,\tilde{p}^{\text{QS}}\right) + \frac{1}{8}\left((2+\tilde{k}_1)\,\tilde{d}^2 + 3\,\tilde{c}_e\,\tilde{c}_i + 1\right) + \frac{1}{4}\Big(\tilde{c}_{r,L}\,\tilde{c}_{t,L} + 5\,\tilde{c}_{t,L}\,\tilde{c}_i \\ &+ \tilde{d} - \tilde{p}^{\text{QS}}\tilde{M}_L + (4+\tilde{k}_1)\tilde{M}_L\Big) + \frac{1}{6}\tilde{c}_e\,\tilde{c}_{r,L} - \frac{1}{2}\tilde{I}_{r,L}\,\tilde{p}^{\text{QS}} + 2\,\tilde{c}_{r,L}\,\tilde{c}_i + \tilde{c}_i^2 + (1+\tilde{k}_1)\,\tilde{I}_{r,L},\end{aligned} \tag{SM 3}$$

$$\rho_3 = -\frac{5}{96}\tilde{c}_e\,\tilde{p}^{\text{QS}} + \frac{8+\tilde{k}_1}{24}\tilde{c}_e + \frac{1}{4}\left((4+\tilde{k}_1)\,\tilde{c}_{t,L} - \tilde{c}_{t,L}\,\tilde{p}^{\text{QS}}\right) - \frac{1}{2}\tilde{c}_{r,L}\,\tilde{p}^{\text{QS}} + (1+\tilde{k}_1)\,(\tilde{c}_i + \tilde{c}_{r,L})\,, \tag{SM 4}$$

$$\rho_4 = \tilde{k}_1. \tag{SM 5}$$

## 2 Critical flutter load with a single source of viscosity

### 2.1 Presence of internal damping $\tilde{c}_i$

The critical load for flutter in the presence of only the internal damping $\tilde{c}_i = r$ is given by

$$\begin{aligned}&\mathcal{P}_d\,(\tilde{c}_i,\,\boldsymbol{\xi}) \\ &= \mathcal{P}_d\left(r,\,\frac{\pi}{2},\,\frac{\pi}{2},\,0,\,\boldsymbol{\xi}\right) \\ &= 2\Bigg(\frac{\tilde{d}^4(2(\tilde{k}_1-1)\tilde{k}_1+5) + 2\tilde{d}^3\left(\tilde{k}_1^2+5\right) + \tilde{d}^2\left(16\tilde{I}_{r,L}(\tilde{k}_1(2\tilde{k}_1-1)+3) + \tilde{k}_1^2 + 4(\tilde{k}_1(3\tilde{k}_1-2)+10)\tilde{M}_L + 9\right)}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)} \\ &\qquad + \frac{4\tilde{d}\left(4\tilde{I}_{r,L}\left(\tilde{k}_1^2+3\right) + 2\left(\tilde{k}_1^2+\tilde{k}_1+5\right)\tilde{M}_L + 1\right) + 4\left[4\tilde{I}_{r,L}\left(8\tilde{I}_{r,L}\left(\tilde{k}_1^2+1\right) - \tilde{k}_1 + 1\right)\right]}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)} \\ &\qquad + \frac{\left[\tilde{M}_L(8\tilde{I}_{r,L}(\tilde{k}_1(3\tilde{k}_1-1)+6) - \tilde{k}_1) + 5\left(\tilde{k}_1^2+4\right)\tilde{M}_L^2\right] + 16\tilde{M}_L + 1}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)}\Bigg) \\ &+ \tilde{c}_i^2\,\frac{16\,(\tilde{k}_1+1)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)}\end{aligned} \tag{SM 6}$$

For $\tilde{c}_i \to 0$ (or equivalently $r \to 0$) the limit critical load is given by

$$\mathcal{P}_d^*\left(\frac{\pi}{2},\,\frac{\pi}{2},\,0,\,\boldsymbol{\xi}\right) = 2\Bigg(\frac{\tilde{d}^4(2(\tilde{k}_1-1)\tilde{k}_1+5)+2\tilde{d}^3\left(\tilde{k}_1^2+5\right)+\tilde{d}^2\left(16\tilde{I}_{r,L}(\tilde{k}_1(2\tilde{k}_1-1)+3)+\tilde{k}_1^2+4(\tilde{k}_1(3\tilde{k}_1-2)+10)\tilde{M}_L+9\right)}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)}$$
$$+\frac{4\tilde{d}\left(4\tilde{I}_{r,L}\left(\tilde{k}_1^2+3\right)+2\left(\tilde{k}_1^2+\tilde{k}_1+5\right)\tilde{M}_L+1\right)+4\left[4\tilde{I}_{r,L}\left(8\tilde{I}_{r,L}\left(\tilde{k}_1^2+1\right)-\tilde{k}_1+1\right)\right]}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)}$$
$$+\frac{\left[\tilde{M}_L(8\tilde{I}_{r,L}(\tilde{k}_1(3\tilde{k}_1-1)+6)-\tilde{k}_1)+5\left(\tilde{k}_1^2+4\right)\tilde{M}_L^2\right]+16\tilde{M}_L+1}{(\tilde{k}_1+1)\left(\tilde{d}^2+\tilde{d}+8\tilde{I}_{r,L}+4\tilde{M}_L\right)(\tilde{d}(3\tilde{d}+2)+16\tilde{I}_{r,L}+10\tilde{M}_L+1)}\Bigg) \qquad \text{(SM 7)}$$

### 2.2 Presence of external damping $\tilde{c}_e$

The critical load for flutter in the presence of only the external damping $\tilde{c}_e = r$ is given by

$$\mathcal{P}_d\left(\tilde{c}_e,\,\boldsymbol{\xi}\right) = \mathcal{P}_d\left(r,\,\frac{\pi}{2},\,\frac{\pi}{2},\,\frac{\pi}{2},\,\boldsymbol{\xi}\right)$$
$$=\Bigg(175\,\tilde{c}_e^2\tilde{d}^2-105\,\tilde{c}_e^2\tilde{d}+1120\,\tilde{c}_e^2\tilde{I}_{r,L}+140\,\tilde{c}_e^2\tilde{M}_L+35\,\tilde{c}_e^2$$
$$-(\tilde{d}(5\tilde{d}-3)+32\tilde{I}_{r,L}+4\tilde{M}_L+1)\Big[1225\,\tilde{c}_e^4+20160\,\tilde{c}_e^2\left(3\tilde{d}^2(\tilde{k}_1-2)-2\tilde{d}(\tilde{k}_1+3)+8\tilde{I}_{r,L}(3\tilde{k}_1-11)+2(\tilde{k}_1-12)\tilde{M}_L+5\right)$$
$$+82944\Big(\tilde{d}^4(\tilde{k}_1(9\tilde{k}_1+64)+36)+\tilde{d}^3(4\tilde{k}_1(17-3\tilde{k}_1)+72)+2\tilde{d}^2(8\tilde{I}_{r,L}(\tilde{k}_1(9\tilde{k}_1+79)+66)+$$
$$+\tilde{k}_1(6\tilde{k}_1\tilde{M}_L+2\tilde{k}_1+196\tilde{M}_L-13)+144\tilde{M}_L-12)+4\tilde{d}(-24\tilde{I}_{r,L}((\tilde{k}_1-4)\tilde{k}_1-11)+$$
$$+\tilde{k}_1(-2\tilde{k}_1\tilde{M}_L+38\tilde{M}_L+5)+72\tilde{M}_L-15)+64\tilde{I}_{r,L}^2(\tilde{k}_1(9\tilde{k}_1+94)+121)+16\tilde{I}_{r,L}(\tilde{k}_1(6\tilde{k}_1\tilde{M}_L+206\tilde{M}_L-15)+264\tilde{M}_L-55)+$$
$$+4\tilde{M}_L((\tilde{k}_1(\tilde{k}_1+136)+144)\tilde{M}_L-5(\tilde{k}_1+12))+25\Big)\Big]^{\frac{1}{2}}+7200\tilde{d}^4\tilde{k}_1+14400\tilde{d}^4-3168\tilde{d}^3\tilde{k}_1+14976\tilde{d}^3+122112\tilde{d}^2\tilde{I}_{r,L}\tilde{k}_1$$
$$+297216\tilde{d}^2\tilde{I}_{r,L}+22464\tilde{d}^2\tilde{k}_1\tilde{M}_L+288\tilde{d}^2\tilde{k}_1+87552\tilde{d}^2\tilde{M}_L-7776\tilde{d}^2-29952\tilde{d}\tilde{I}_{r,L}\tilde{k}_1+94464\tilde{d}\tilde{I}_{r,L}-1728\tilde{d}\tilde{k}_1\tilde{M}_L$$
$$+576\tilde{d}\tilde{k}_1+32256\tilde{d}\tilde{M}_L+3168\tilde{d}+516096\tilde{I}_{r,L}^2\tilde{k}_1+1548288\tilde{I}_{r,L}^2+184320\tilde{I}_{r,L}\tilde{k}_1\tilde{M}_L-6912\tilde{I}_{r,L}\tilde{k}_1+783360\tilde{I}_{r,L}\tilde{M}_L$$
$$-89856\tilde{I}_{r,L}+20736\tilde{k}_1\tilde{M}_L^2-576\tilde{k}_1\tilde{M}_L+119808\tilde{M}_L^2-10368\tilde{M}_L$$
$$+1440\Bigg)\Big/\left(720\left(5\tilde{d}^4+4\tilde{d}^3+4\tilde{d}^2(26\tilde{I}_{r,L}+7\tilde{M}_L-1)+2\tilde{d}(8\tilde{I}_{r,L}+2\tilde{M}_L+1)+8\left(30\tilde{I}_{r,L}\tilde{M}_L+\tilde{I}_{r,L}(64\tilde{I}_{r,L}-3)+4\tilde{M}_L^2\right)-2\tilde{M}_L\right)\right) \qquad \text{(SM 8)}$$

For $\tilde{c}_e \to 0$ (or equivalently $r \to 0$), the limit critical load is given by

$$\mathcal{P}_d^*\left(\frac{\pi}{2},\,\frac{\pi}{2},\,\frac{\pi}{2},\,\boldsymbol{\xi}\right) = 2\Big[25\tilde{d}^4(\tilde{k}_1+2)+\tilde{d}^3(52-11\tilde{k}_1)+\tilde{d}^2(8\tilde{I}_{r,L}(53\tilde{k}_1+129)+78\tilde{k}_1\tilde{M}_L+\tilde{k}_1+304\tilde{M}_L-27)$$
$$+\tilde{d}(8\tilde{I}_{r,L}(41-13\tilde{k}_1)-6\tilde{k}_1\tilde{M}_L+2\tilde{k}_1+112\tilde{M}_L+11)+1792\tilde{I}_{r,L}^2(\tilde{k}_1+3)+8\tilde{I}_{r,L}(80\tilde{k}_1\tilde{M}_L-3\tilde{k}_1+340\tilde{M}_L-39)$$
$$+8(9\tilde{k}_1+52)\tilde{M}_L^2-2(\tilde{k}_1+18)\tilde{M}_L+5-\left(5\tilde{d}^2-3\tilde{d}+32\tilde{I}_{r,L}+4\tilde{M}_L+1\right)\Big\{\tilde{d}^4\left(9\tilde{k}_1^2+64\tilde{k}_1+36\right)$$
$$+\tilde{d}^3\left(-12\tilde{k}_1^2+68\tilde{k}_1+72\right)+2\tilde{d}^2\left(8\tilde{I}_{r,L}\left(9\tilde{k}_1^2+79\tilde{k}_1+66\right)+\tilde{k}_1^2(6\tilde{M}_L+2)+\tilde{k}_1(196\tilde{M}_L-13)+144\tilde{M}_L-12\right)$$
$$-4\tilde{d}\left(24\tilde{I}_{r,L}\left(\tilde{k}_1^2-4\tilde{k}_1-11\right)+2\tilde{k}_1^2\tilde{M}_L-\tilde{k}_1(38\tilde{M}_L+5)-72\tilde{M}_L+15\right)+64\tilde{I}_{r,L}^2\left(9\tilde{k}_1^2+94\tilde{k}_1+121\right)$$
$$+16\tilde{I}_{r,L}\left(6\tilde{k}_1^2\tilde{M}_L+\tilde{k}_1(206\tilde{M}_L-15)+264\tilde{M}_L-55\right)+4\tilde{k}_1^2\tilde{M}_L^2+544\tilde{k}_1\tilde{M}_L^2-20\tilde{k}_1\tilde{M}_L+576\tilde{M}_L^2-240\tilde{M}_L$$
$$+25\Big\}^{\frac{1}{2}}\Big]\Big/\left[5\left(5\tilde{d}^4+4\tilde{d}^3+4\tilde{d}^2(26\tilde{I}_{r,L}+7\tilde{M}_L-1)+2\tilde{d}(8\tilde{I}_{r,L}+2\tilde{M}_L+1)+512\tilde{I}_{r,L}^2+24\tilde{I}_{r,L}(10\tilde{M}_L-1)+2\tilde{M}_L(16\tilde{M}_L-1)\right)\right] \qquad \text{(SM 9)}$$

## 3 Critical flutter load with two sources of viscosity

### 3.1 Presence of internal and external damping

For the sake of simplicity, the critical load is particularized for the case $\hat{\boldsymbol{\xi}} = [1/2,\, 15,\, 15,\, 50]$. By setting $\tilde{c}_e = r\sin\phi_3$, $\tilde{c}_i = r\cos\phi_3$ and $r = \sqrt{\tilde{c}_e^2 + \tilde{c}_i^2}$ the critical load is given by

$$
\begin{aligned}
\mathcal{P}_d\left(r,\, \frac{\pi}{2},\, \frac{\pi}{2},\, \phi_3,\, \hat{\boldsymbol{\xi}}\right) = \\
\Big[25235r^2\sin^2\phi_3 + 3554600r^2\sin\phi_3\cos\phi_3 + 96\left(240r^2\cos^2\phi_3(6284\cot\phi_3 + 2717)\right. \\
\left. + 288\cot\phi_3(57927483\cot\phi_3 + 10504342) + 120487001\right) \\
- (12568\cot\phi_3 + 721)\left\{1225r^4\sin^4\phi_3 + 1152\left(4225r^4\sin^2(2\phi_3) + 35552208708\right)\right. \\
+ 132710400r^4\cos^4\phi_3 + 99532800r^4\sin\phi_3\cos^3\phi_3 + 359009280r^2\sin^2\phi_3 \\
- 9953280r^2\cos^2\phi_3(294984\cot\phi_3 + 98747) + 302400r^2\sin\phi_3\cos\phi_3\left(r^2\sin^2\phi_3 + 117036\right) \\
\left. + 1439244288\cot\phi_3(11283138\cot\phi_3 - 711833)\right\}^{\frac{1}{2}}\Big] / [60(54519984\cot\phi_3 + 2826493)]
\end{aligned}
\tag{SM 10}
$$

Moreover, the maximum value of the critical limit load (79) can be obtained by taking the limit of vanishing viscosities along the particular direction

$$
\bar{\phi}_3 = 2\tan^{-1}\left(\frac{\sqrt{15690571665841035300\sqrt{120482} + 262945884657140045244 10} - 52150850614 - 150434475\sqrt{120482}}{144389423358}\right)
$$

and corresponding in this case to the critical value for the ideal case without damping $\mathcal{P}_0(\hat{\boldsymbol{\xi}})$, namely

$$
\mathcal{P}_d^*\left(\frac{\pi}{2},\, \frac{\pi}{2},\, \bar{\phi}_3,\, \hat{\boldsymbol{\xi}}\right) = \frac{20}{723}\left(3102 - \sqrt{120482}\right) = \mathcal{P}_0\left(\hat{\boldsymbol{\xi}}\right) \tag{SM 11}
$$

### 3.2 Presence of translational and rotational damping for the non-holonomic constraint

For the sake of simplicity, the critical load is particularized to the case $\hat{\boldsymbol{\xi}} = [1/2,\, 15,\, 15,\, 50]$. By setting $\tilde{c}_{r,L} = r\sin\phi_1$, $\tilde{c}_{t,L} = r\cos\phi_1$ and $r = \sqrt{\tilde{c}_{r,L}^2 + \tilde{c}_{t,L}^2}$ the critical load is given by

$$
\begin{aligned}
\mathcal{P}_d\left(r,\, \phi_1,\, 0,\, 0,\, \hat{\boldsymbol{\xi}}\right) = \Big[-4\sin(2\phi_1)\left(r^2(732\sin(2\phi_1) - 259\cos(2\phi_1)) + 741r^2 + 6641121\right) + 440805\cos(2\phi_1) - 18295539 \\
+ 2(250\sin\phi_1 + 241\cos\phi_1)\left\{64r^4\sin^2\phi_1\cos^4\phi_1 + \cos^2\phi_1\left(256r^4\sin^4\phi_1 + 155027401\right)\right. \\
+ 2\sin(2\phi_1)\left(8r^2\sin(2\phi_1)\left(2r^2\sin(2\phi_1) + 2823\right) - 52890047\right) + 183984r^2\sin\phi_1\cos^3\phi_1 \\
\left. - 374592r^2\sin^3\phi_1\cos\phi_1 + 161137636\sin^2\phi_1\right\}^{\frac{1}{2}}\Big] / [-288272\sin(2\phi_1) + 6534\cos(2\phi_1) \\
- 234538]
\end{aligned}
\tag{SM 12}
$$

Moreover, the maximum value of the critical flutter load at vanishing viscosity, equation (81), can be obtained as

$$
\bar{\phi}_1 = 2\tan^{-1}\left(\frac{\sqrt{5012544564868560760800\sqrt{120482} + 1243347484183875671201 22197} - 315372600\sqrt{120482} - 7947019755154}{7820974511191}\right)
$$

which corresponds to the critical value for the ideal case without damping $\mathcal{P}_0(\hat{\boldsymbol{\xi}})$, namely

$$
\mathcal{P}_d^*\left(\bar{\phi}_1,\, 0,\, 0,\, \hat{\boldsymbol{\xi}}\right) = \frac{20}{723}\left(3102 - \sqrt{120482}\right) = \mathcal{P}_0\left(\hat{\boldsymbol{\xi}}\right) \tag{SM 13}
$$