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LOSS OF STRONG ELLIPTICITY IN NON-ASSOCIATIVE ELASTOPLASTICITY

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Abstract

Loss of strong ellipticity is considered for elastoplastic solids in the presence of non-associative flow laws. Reference is made to the two comparison solids introduced by Raniecki and Bruhns. The loss of strong ellipticity is expressed in terms of a critical value of the hardening modulus. In the context of the infinitesimal theory, the loss of strong ellipticity is shown to occur simultaneously in the two comparison solids. Finally, an explicit form for the critical hardening modulus is given and applications are performed for the Drucker-Prager and Schleicher yield functions.

Notation

A notation is used which is standard in continuum mechanics [cf., for example, Gurtin (1981)]:

$\mathbf{Aa}$	vector that second-order tensor $\mathbf{A}$ assigns to vector $\mathbf{a}$
$\mathbb{A}[\mathbf{A}]$	second-order tensor that fourth-order tensor $\mathbb{A}$ assigns to second-order tensor $\mathbf{A}$
$\mathbf{AB}$	second-order tensor defined as $(\mathbf{AB})\mathbf{a} = \mathbf{A}(\mathbf{Ba})$ for every vector $\mathbf{a}$
$\mathbf{A}^T$	transpose of second-order tensor $\mathbf{A}$
$\mathbf{A}^{-1}$	inverse of second-order tensor $\mathbf{A}$
$\mathrm{tr}\,\mathbf{A}$	trace of second-order tensor $\mathbf{A}$
$\mathbb{O}$	fourth-order zero tensor
$\cdot$	scalar product
$\otimes$	tensorial product

1. Introduction

Local stability conditions for elastoplastic solids were formulated and discussed by Mróz (1963, 1966), Villaggio (1968), Hueckel and Maier (1977) and Maier and Hueckel (1979) in terms of second-order work positiveness, by Hill (1962) and Mandel (1966) in terms of the propagation of acceleration waves, by Hill and Hutchinson (1975), Rudnicki and Rice (1975), Rice (1976), Vardoulakis (1976) and Rice and Rudnicki (1980) in terms of strain localization, and, finally, by Benallal *et al.* (1989) and Schaeffer and Shearer (1990) in terms of the stability of the equations governing the elastoplastic rate problem. Recently, local stability criteria have been used in the numerical analysis of the nucleation and growth of localized deformations (FISH and BELYTSCHKO, 1990; NACAR *et al.*, 1989; ORTIZ *et al.*, 1987).

Sufficient conditions for the stability and uniqueness of the elastoplastic incremental response, allowing for large displacement gradients, were stated by HILL (1958, 1959, 1961). Hill's formulation was later extended to non-associative flow laws by MAIER (1970), RANIECKI (1979) and RANIECKI and BRUHNS (1981). However, the problem of stability and uniqueness in the non-associative case is far from being fully understood. Non-associative flow rules were introduced by MANDEL (1966) and MRÓZ (1963, 1966), and later widely used to model the behavior of pressure-sensitive materials, such as, for example, void-containing ductile solids, polymers, ceramics, concrete and rocks (LEE, 1988; NEEDLEMAN, 1979; NEMAT-NASSER, 1983; RICE, 1976; RUDNICKI and RICE, 1975).

This paper continues with a brief review of the general framework of stability and uniqueness for elastoplastic solids under dead loading. In this section, an attempt is made to establish a hierarchy between the various stability criteria and to emphasize all the physical motivations of the criteria. This paper then focuses on the local condition of strong ellipticity (the SE condition in the following). The SE condition is connected to the loss of infinitesimal stability through Hadamard's inequality (TRUESDELL and NOLL, 1965, Section 68 *bis*).

In Section 3, some new results are presented which were obtained without any assumption on the direction of the plastic flow, on the shape of the (smooth) yield surface, and on the type of hardening/softening law. The loss of the SE condition is expressed in terms of a critical value of the hardening modulus for both comparison solids (RANIECKI and BRUHNS, 1981). However, starting from Section 4, the co-rotational terms are neglected in the constitutive equations and the coaxiality of the tensors of the plastic flow mode and the yield surface gradient is assumed. In this context, a coincidence is shown between the critical hardening moduli of the comparison solids. Moreover, an explicit expression for the critical hardening modulus is given for the loss of the SE condition. Applications are performed for the yield surfaces proposed by DRUCKER and PRAGER (1952) and SCHLEICHER (1926) ("modified von Mises"). The non-associative flow rule has been selected to be the type used by RUDNICKI and RICE (1975). Finally, the possibility is discussed of generalizing the results obtained to the finite displacement theory.

2. STABILITY AND UNIQUENESS FOR ELASTOPLASTIC SOLIDS

2.1. *Constitutive equations*

A piecewise linear relationship is assumed (NEMAT-NASSER, 1983; RANIECKI and BRUHNS, 1981) to relate the Jaumann derivative ($^{\Delta}$) of the Kirchhoff stress $\mathbf{K}$ to the rate of deformation tensor $\mathbf{D}$

$$\overset{\Delta}{\mathbf{K}} = \mathbb{C}[\mathbf{D}]. \tag{2.1}$$

The fourth-order constitutive operator $\mathbb{C}$ is given by the following form:

$$\mathbb{C}[\mathbf{D}] = \mathbb{E}[\mathbf{D}] - \frac{\langle \mathbf{D} \cdot \mathbb{E}[\mathbf{Q}] \rangle}{H + H_o} \mathbb{E}[\mathbf{P}], \tag{2.2}$$

where $\mathbb{E}$ is the elastic (positive definite) constitutive tensor, $\mathbf{Q}$ is the gradient of the (smooth) yield surface, the (symmetric) second-order tensor $\mathbf{P}$ gives the mode of the plastic flow, the symbol $\langle \cdot \rangle$ denotes the McAulay brackets, H is the hardening modulus, and the scalar H_o represents the absolute value of the hardening modulus at the snap-back threshold, i.e.

$$H_o = \mathbf{P} \cdot \mathbb{E}[\mathbf{Q}]. \tag{2.3}$$

Note that no hypotheses are made for the evolution law of the hardening modulus. When the hardening modulus is negative, strain softening occurs. When $\mathbf{P} = \mathbf{Q}$, the constitutive equation (2.1)–(2.2) reduces to the usual constitutive equation in the presence of the associative flow rule (HILL, 1958).

In relation to the constitutive equation (2.1), the comparison solid "in loading" (Hill's type) is introduced:

$$\mathbb{C}^h = \mathbb{E} - \frac{\mathbb{E}[\mathbf{P}] \otimes \mathbb{E}[\mathbf{Q}]}{H + H_o}. \tag{2.4}$$

Moreover, the family of comparison solids introduced by RANIECKI (1979) is specified by the constitutive fourth-order tensor

$$\mathbb{C}^r = \mathbb{E} - \frac{\mathbb{E}[\mathbf{R}] \otimes \mathbb{E}[\mathbf{R}]}{4\psi(H + H_o)}, \qquad \psi \in \mathbb{R}^+, \tag{2.5}$$

where

$$\mathbf{R} = \mathbf{P} + \psi \mathbf{Q}. \tag{2.6}$$

Note that ψ is a free parameter and, therefore, the choice of a particular comparison solid of the family (2.5) can be optimized, depending on the problem at hand. Bifurcation analyses using Raniecki's comparison solid were performed by BRUHNS (1982) and TVERGAARD (1982).

2.2. *Stability and uniqueness for elastoplastic solids*

For the rate problem of an elastoplastic solid under dead loading the infinitesimal stability condition (TRUESDELL and NOLL, 1965, Section 68 *bis*; OGDEN, 1984, Section 6.2.3) states that

$$\int_{\mathscr{B}} \mathbf{L} \cdot (\mathbb{C} + \mathbb{G})[\mathbf{L}] \geq 0, \tag{2.7}$$

where $\mathbf{L}$ is the gradient of the spatial velocity and

$$\mathbb{G}_{ijhk} = \tfrac{1}{2}(K_{jk}\delta_{ih} - K_{ik}\delta_{jh} - K_{ih}\delta_{jk} - K_{jh}\delta_{ik}), \tag{2.8}$$

δ_{ij} being the Kronecker symbol. The tensor $(\mathbb{C} + \mathbb{G})[\mathbf{L}]$ represents the material derivative of the first Piola–Kirchhoff stress tensor.

Necessary conditions for (2.7) to hold have been derived by RYZHAK (1987):

$$\mathbf{g} \otimes \mathbf{n} \cdot (\mathbb{C}^h + \mathbb{G})\,[\mathbf{g} \otimes \mathbf{n}] \geqslant 0, \tag{2.9}$$

$$\mathbf{g} \otimes \mathbf{n} \cdot (\mathbb{E} + \mathbb{G})\,[\mathbf{g} \otimes \mathbf{n}] \geqslant 0, \tag{2.10}$$

for every pair of vectors $\mathbf{g}$ and $\mathbf{n}$ not equal to zero. The theorem derived by Ryzhak represents an extension to the non-associative elastoplasticity of the Hadamard theorem of finite elasticity, proved in a general form by CATTANEO (1946).

The following sufficient condition for the uniqueness of the rate problem for non-associative flow laws has been derived by RANIECKI (1979) and RANIECKI and BRUHNS (1981) through the introduction of the family of comparison solids (2.5):

$$\int_{\mathscr{B}} \mathbf{L} \cdot (\mathbb{C}^r + \mathbb{G})\,[\mathbf{L}] > 0, \tag{2.11}$$

for every velocity field satisfying homogeneous conditions on the boundary where displacements are prescribed. Based on the Hadamard theorem, a local necessary condition for (2.11) to hold can be written as

$$\mathbf{g} \otimes \mathbf{n} \cdot (\mathbb{C}^r + \mathbb{G})\,[\mathbf{g} \otimes \mathbf{n}] \geqslant 0, \qquad \forall \mathbf{g},\, \mathbf{n} \neq \mathbf{0}. \tag{2.12}$$

2.3. *Local stability criteria*

From (2.11), the following local sufficient condition for the uniqueness of the incremental elastoplastic response is immediately obtained:

$$\mathbf{L} \cdot (\mathbb{C}^r + \mathbb{G})\,[\mathbf{L}] > 0, \qquad \forall \mathbf{L} \in \text{Lin-}\{\mathbf{0}\}. \tag{2.13}$$

In the case of finite elasticity, condition (2.13) is due to HILL (1957). Moreover, when $\mathbf{L}$ is restricted to Sym-$\{\mathbf{0}\}$, condition (2.13) is analogous to the GCN^+ condition of finite elasticity (TRUESDELL and NOLL, 1965, Section 52). In the case of the infinitesimal theory ($\mathbb{G} = \mathbb{O}$), (2.13) is known as the condition of second-order work positiveness (MRÓZ, 1963). In this context, RANIECKI (1979) proved the coincidence of the loss of second-order work positiveness for both Hill's comparison solid and the optimum comparison solid of the Raniecki family.

Condition (2.13) excludes the possibility of localization of deformations into planar bands. Strain localization occurs when the constitutive equation (2.1) suffers a loss of ellipticity (HILL and HUTCHINSON, 1975; RUDNICKI and RICE, 1975; RICE, 1976). However, RICE and RUDNICKI (1980) proved that the first possibility of strain localization in the elastoplastic solid occurs when Hill's comparison solid suffers a loss of ellipticity, i.e. when

$$\exists \mathbf{n} \neq \mathbf{0}: \qquad \det \mathbf{A}_h(\mathbf{n}) = 0, \tag{2.14}$$

where the acoustic tensor $\mathbf{A}_h$ is defined as

$$\mathbf{A}_h(\mathbf{n})\mathbf{g} = (\mathbb{C}^h + \mathbb{G})\,[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}. \tag{2.15}$$

Condition (2.14) is the condition for the vanishing of the speed of an acceleration wave (HILL, 1962). When all acceleration waves are able to propagate with real speed, the material is considered stable in the MANDEL (1966) sense. Mandel's stability only proves to be a sufficient condition for excluding strain localization, due to the possibility of complex eigenvalues of the acoustic tensor (RICE, 1976).

Note that, due to the symmetry of $\mathbb{C}^r$, the condition of strain localization coincides, for Raniecki's comparison solid, with the loss of the SE condition.

Inequality (2.13) is obviously stronger than the SE condition for the comparison solid "in loading":

$$\mathbf{g} \otimes \mathbf{n} \cdot (\mathbb{C}^h + \mathbb{G}) [\mathbf{g} \otimes \mathbf{n}] > 0, \qquad \forall \mathbf{g}, \mathbf{n} \neq \mathbf{0}, \tag{2.16}$$

as well as for the family of comparison solids (2.5).

The SE condition plays a central role in the above-mentioned local criteria and, therefore, the present paper is devoted to a systematic study of this condition.

3. Strong Ellipticity for the Comparison Solids

The local criteria presented in the previous section place restrictions on the constitutive laws. This becomes particularly evident, for instance, in the infinitesimal theory, where it is well known that, when normality holds, the positiveness of the hardening modulus is sufficient for ensuring stability in all the mentioned senses (MELAN, 1938; HILL, 1950, pp. 53–60). The condition of loss of second-order work positiveness is expressed in terms of a critical value of the hardening modulus (MAIER and HUECKEL, 1979). For hardening modulus values exceeding the critical one, the second-order work is positive for every strain rate. Analogously, in the finite theory, the condition of strain localization (loss of ellipticity) is expressed in terms of a threshold for the hardening modulus (RICE, 1976). Clearly, the critical hardening modulus for localization is lower than that for the loss of second-order work positiveness.

In the following, the critical hardening modulus for the loss of the SE condition is derived for both comparison solids (2.4) and (2.5). These critical hardening modulus values are the validity thresholds for (2.9) and (2.12). Therefore, the interest in the SE conditions lies in the fact that they set a local bound sufficient for the validity of integral conditions (2.7) and (2.11). For this reason, the SE condition appears to be complementary to the condition of second-order work positiveness.

3.1. *Comparison solid "in loading" (Hill's type)*

For the comparison solid "in loading" (2.4), the critical hardening modulus, as a function of the unit vector $\mathbf{n}$, is given by

$$H^h(\mathbf{n}) = -H_o + \tfrac{1}{2}\{\mathbb{E}[\mathbf{P}]\mathbf{n} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{Q}]\mathbf{n} + [(\mathbb{E}[\mathbf{P}]\mathbf{n} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n})\,(\mathbb{E}[\mathbf{Q}]\mathbf{n} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{Q}]\mathbf{n})]^{1/2}\}, \tag{3.1}$$

where $\mathbf{A}_E(\mathbf{n})$ is the elastic acoustic tensor:

$$\mathbf{A}_E(\mathbf{n})\mathbf{g} = (\mathbb{E} + \mathbb{G}) [\mathbf{g} \otimes \mathbf{n}]\mathbf{n}. \tag{3.2}$$

Equation (3.1) requires that $\mathbf{A}_E(\mathbf{n})$ be non-singular. On the other hand, the components of the fourth-order tensor $\mathbb{G}$ ("geometrical terms") are of the order of magnitude of a stress component and, therefore, are generally small compared to the elastic moduli for metals and pressure-sensitive materials. The assumption that $\mathbf{A}_E^{-1}(\mathbf{n})$ exists is, therefore, common (RICE and RUDNICKI, 1980).

The vector $\mathbf{g}$ at the loss of the SE condition is

$$\mathbf{g}^h = \tfrac{1}{2}\{\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}+\eta\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{Q}]\mathbf{n}\}, \tag{3.3}$$

where

$$\eta = \left(\frac{\mathbb{E}[\mathbf{P}]\mathbf{n}\cdot\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}}{\mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{Q}]\mathbf{n}}\right)^{1/2}. \tag{3.4}$$

Therefore, the critical hardening modulus is given by the maximum value attained by (3.1) over all possible directions of $\mathbf{n}$, i.e.

$$H^h_{SE} = \max_{\mathbf{n}} H^h(\mathbf{n}), \qquad \text{subject to } \mathbf{n}\cdot\mathbf{n} = 1. \tag{3.5}$$

It is to be noted that condition (3.1) reduces to the condition of strain localization in a planar band having normal unit vector $\mathbf{n}$ (RICE, 1976) in the case of the associative flow law.

In order to prove (3.1), a procedure similar to that proposed in MAIER and HUECKEL (1979) is used. Hence, the constrained minimum of the functional

$$S(\mathbf{g}) = \mathbf{g}\cdot\mathbf{A}_E(\mathbf{n})\mathbf{g}-\mathbf{g}\cdot\mathbb{E}[\mathbf{P}]\mathbf{n}, \tag{3.6}$$

subject to the condition

$$\mathbf{g}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n} = H+H_o, \tag{3.7}$$

is set to equal zero (in Appendix 1 the equivalence is shown between the above problem and the loss of the SE condition). The constrained minimum of (3.6) is performed by minimizing the unconstrained functional

$$L(\mathbf{g},\eta) = S(\mathbf{g})-\eta(\mathbf{g}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n}-H-H_o), \tag{3.8}$$

where η is a Lagrangean multiplier. Then the extremum conditions of (3.8) and the vanishing of the minimum of the functional (3.6) are:

$$2\mathbf{A}_E(\mathbf{n})\mathbf{g}-\mathbb{E}[\mathbf{P}]\mathbf{n}-\eta\mathbb{E}[\mathbf{Q}]\mathbf{n} = \mathbf{0}, \tag{3.9}$$

$$\mathbf{g}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n}-H-H_o = 0, \tag{3.10}$$

$$\mathbf{g}\cdot\mathbf{A}_E(\mathbf{n})\mathbf{g}-\mathbf{g}\cdot\mathbb{E}[\mathbf{P}]\mathbf{n} = 0. \tag{3.11}$$

Vector $\mathbf{g}^h$ is obtained from (3.9) in the form of (3.3). A substitution of (3.3) into (3.11) yields (3.4). Finally, by substituting (3.3) and (3.4) into (3.10), the condition (3.1) is obtained.

3.2. *Family of comparison solids* (*Raniecki's type*)

As a function of the unit vector $\mathbf{n}$, the critical hardening modulus, for a given solid of the family (2.5), is given by

$$H^r(\mathbf{n},\psi) = -H_o + \frac{1}{2}\left\{\mathbb{E}[\mathbf{P}]\mathbf{n}\cdot\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{Q}]\mathbf{n}\right.$$

$$\left. + \frac{\psi}{2}\mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{Q}]\mathbf{n} + \frac{1}{2\psi}\mathbb{E}[\mathbf{P}]\mathbf{n}\cdot\mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}\right\}. \quad (3.12)$$

The vector $\mathbf{g}$ at the loss of the SE condition is

$$\mathbf{g}^r = \mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{R}]\mathbf{n}. \quad (3.13)$$

Finally, the critical hardening modulus is obtained by maximizing (3.12) as a function of the unit vector $\mathbf{n}$. The best choice in the family of comparison solids (2.5) for the loss of the SE condition is finally achieved by calculating the infimum of the hardening modulus as a function of ψ:

$$H_{SE}^r = \inf_{\psi}\max_{\mathbf{n}} H^r(\mathbf{n},\psi), \qquad \psi\in\mathbb{R}^+,\ \mathbf{n}\cdot\mathbf{n} = 1. \quad (3.14)$$

Equations (3.12) and (3.13) have been obtained by a procedure quite analogous to that employed to obtain (3.1) and (3.3). The Lagrangean multiplier is now equal to 1.

It is worth noting that the values of the critical hardening moduli (3.1) and (3.12) are not explicit, in the sense that a minimization with respect to the unit vector $\mathbf{n}$ is required. In the following section this minimization will be performed for the solid (2.5), so as to obtain an explicit form for the critical hardening modulus.

4. Critical Hardening Modulus for the Raniecki-type Comparison Solid

Problem (3.14) is solved, in the context of the infinitesimal theory, in order to obtain an explicit expression for the critical hardening modulus corresponding to the loss of the SE condition of the solids (2.5). In the case of the infinitesimal theory, all relations of Sections 2 and 3 still hold for $\mathbb{G} = \mathbb{O}$. Furthermore, $\mathbf{K}$ coincides with the Cauchy stress tensor $\mathbf{T}$.

The elastic behavior is assumed to be isotropic and is, therefore, characterized by the two Lamé constants λ and μ:

$$\mathbb{E} = \lambda(\mathbf{I}\otimes\mathbf{I}) + 2\mu\mathbb{S}, \quad (4.1)$$

where $\mathbf{I}$ is the identity tensor and $\mathbb{S}$ is the symmetric operator over Lin:

$$\mathbb{S}[\mathbf{X}] = \tfrac{1}{2}(\mathbf{X}+\mathbf{X}^T), \qquad \forall\mathbf{X}\in\mathrm{Lin}. \quad (4.2)$$

The tensors $\mathbf{P}$ and $\mathbf{Q}$ are assumed to be co-axial, i.e. $\mathbf{PQ} = \mathbf{QP}$. Due to objectivity requirements, this condition is always satisfied in the case of isotropic hardening. Furthermore, $\mathbf{P}$ and $\mathbf{Q}$ are co-axial if they are related through an isotropic tensorial function (Truesdell and Noll, 1965, Section 1.2), as in the particular cases studied by Needleman (1979) and Rudnicki and Rice (1975).

In order to maximize the hardening modulus (3.12) as a function of $\mathbf{n}$ (for a given ψ), condition (3.12) is rewritten in the equivalent form

$$H^* + H_o^* = \mathbb{E}[\mathbf{R}]\mathbf{n} \cdot \mathbf{A}_E^{-1}(\mathbf{n})\mathbb{E}[\mathbf{R}]\mathbf{n}, \tag{4.3}$$

having assumed that

$$H^* = 4\psi H^r - \mathbf{P} \cdot \mathbb{E}[\mathbf{P}] + 2\psi \mathbf{P} \cdot \mathbb{E}[\mathbf{Q}] - \psi^2 \mathbf{Q} \cdot \mathbb{E}[\mathbf{Q}], \tag{4.4}$$

$$H_o^* = \mathbf{R} \cdot \mathbb{E}[\mathbf{R}]. \tag{4.5}$$

It can now be noted that the hardening modulus H^* provided by (4.3) corresponds to the strain localization of an elastoplastic solid with an associative flow rule, characterized by the plastic strain rate mode **R**. For the associative flow rule, the maximization with respect to **n** of the critical hardening modulus, corresponding to the strain localization, can be performed as a particular case of that solved by BIGONI and HUECKEL (1990, 1991) for non-associative flow rules. This result is now used to obtain the maximum of H^* and, consequently, through (4.4), of H^r. The maximum of H^* can be stated as follows:

$$H^*(\psi) = \max_{k=1,2,3} H^*(\psi, k), \tag{4.6}$$

where

$$H^*(\psi, k) = -2G(1+\nu)R_k^2 - \frac{2G}{1-\nu}\left[\frac{\langle \beta_l \rangle}{\beta_l}(R_l + \nu R_k) + \frac{\langle \beta_m \rangle}{\beta_m}(R_m + \nu R_k)\right]^2. \tag{4.7}$$

In expression (4.7) G and ν indicate the tangential elasticity modulus and the Poisson's ratio, respectively, the indices denote principal components, (k, l, m) is a permutation of $(1, 2, 3)$, and, finally,

$$\beta_l = \frac{R_l + \nu R_k}{R_m - R_l}, \qquad \beta_m = \frac{R_m + \nu R_k}{R_l - R_m}. \tag{4.8}$$

Equations (4.8), obviously, are valid if $R_m \neq R_l$. In the case $R_m = R_l$, (4.7) and all the other following formulae still hold as limits for $\beta_l \to +\infty$ and $\beta_m \to -\infty$, or, equivalently, $\beta_l \to -\infty$ and $\beta_m \to +\infty$. The components of unit vector **n** corresponding to $H^*(\psi, k)$ are

$$\begin{aligned} n_k &= 0, \\ n_l^2 &= \langle 1 - \langle -\beta_m \rangle \rangle, \\ n_m^2 &= \langle 1 - \langle -\beta_l \rangle \rangle. \end{aligned} \tag{4.9}$$

Rewriting problem (4.6) in terms of H^r and making ψ explicit yields

$$H^r(\psi) = \max_{k=1,2,3} H^r(\psi, k), \tag{4.10}$$

where

$$H^r(\psi, k) = \frac{G}{2}\left(A\psi + \frac{B}{\psi} - 2C\right), \tag{4.11}$$

$$A(\psi,k) = \mathbf{Q}\cdot\mathbf{Q} + \frac{v}{1-2v}\operatorname{tr}^2\mathbf{Q} - (1+v)Q_k^2$$

$$- \frac{1}{1-v}\left[\frac{\langle\beta_l\rangle}{\beta_l}(Q_l+vQ_k) + \frac{\langle\beta_m\rangle}{\beta_m}(Q_m+vQ_k)\right]^2, \quad (4.12)$$

$$B(\psi,k) = \mathbf{P}\cdot\mathbf{P} + \frac{v}{1-2v}\operatorname{tr}^2\mathbf{P} - (1+v)P_k^2$$

$$- \frac{1}{1-v}\left[\frac{\langle\beta_l\rangle}{\beta_l}(P_l+vP_k) + \frac{\langle\beta_m\rangle}{\beta_m}(P_m+vP_k)\right]^2, \quad (4.13)$$

$$C(\psi,k) = \mathbf{P}\cdot\mathbf{Q} + \frac{v}{1-2v}\operatorname{tr}\mathbf{P}\operatorname{tr}\mathbf{Q} + (1+v)P_kQ_k$$

$$+ \frac{1}{1-v}\left[\frac{\langle\beta_l\rangle}{\beta_l}(P_l+vP_k)(Q_l+vQ_k) + \frac{\langle\beta_m\rangle}{\beta_m}(P_m+vP_k)(Q_m+vQ_k)\right]. \quad (4.14)$$

Note that $\langle\beta_m\rangle/\beta_m$ and $\langle\beta_l\rangle/\beta_l$, as functions of ψ, can assume only values of 0 and 1. Hence, functions A, B and C in (4.10) are independent of ψ inside the following subsets of $\mathbb{R}^+$:

$$I_{[0,1]} = \{\psi|\langle\beta_l\rangle = \langle\beta_m\rangle = 0\}, \quad (4.15)$$

$$I^l_{(-\infty,0]} = \{\psi|\langle\beta_l\rangle = \beta_l, \langle\beta_m\rangle = 0\}, \quad (4.16)$$

$$I^m_{(-\infty,0]} = \{\psi|\langle\beta_l\rangle = 0, \langle\beta_m\rangle = \beta_m\}. \quad (4.17)$$

It can be noted that the following implications hold:

$$\psi\in I_{[0,1]} \Rightarrow -\beta_l, -\beta_m \in [0,1], \quad (4.18)$$

$$\psi\in I^l_{(-\infty,0]} \Rightarrow -\beta_l \in (-\infty,0], \quad (4.19)$$

$$\psi\in I^m_{(-\infty,0]} \Rightarrow -\beta_m \in (-\infty,0]. \quad (4.20)$$

When the subsets (4.15)–(4.17) are determined, the maximization problem (4.10) can be solved. It then becomes possible to minimize the hardening modulus as a function of ψ so as to obtain H^r_{SE} and the corresponding values of ψ_{SE} and $\mathbf{n}^r_{SE}$.

5. Coincidence of Loss of Strong Ellipticity for Hill and Raniecki Comparison Solids

For a given unit vector **n**, the SE condition is lost in the best chosen Raniecki comparison solid (2.5) at the same critical level of the hardening modulus as the SE condition is lost in the Hill comparison solid (2.4) in the same direction **n**.

In fact, in the case of the family of Raniecki comparison solids, the minimum of (3.12) as a function of ψ is obtained for $\psi = \eta$, where η is given by (3.4). Substitution of this value of ψ into (3.12) yields (3.1), given previously for the Hill comparison solid.

Therefore, the critical hardening modulus for the loss of the SE condition in the comparison solid "in loading" may be expressed as

$$H^h_{SE} = \max_{\mathbf{n}} \min_{\psi} H^r(\mathbf{n}, \psi), \qquad \psi \in \mathbb{R}^+,\ \mathbf{n}\cdot\mathbf{n} = 1. \tag{5.1}$$

On the other hand, the analogous critical hardening modulus for the family of comparison solids (2.5) is expressed in the dual form (3.14). It will now be shown that the two dual problems (3.14) and (5.1) have the same solution. Therefore, the explicit solution (see Section 4) for the critical hardening modulus for the loss of the SE condition in the best chosen solid of family (2.5) also furnishes the critical hardening modulus for the solid (2.4).

5.1. *Proof that* (3.14) *and* (5.1) *have the same solution*

Let L be the functional defined as

$$L : \mathscr{C} \times \mathbb{R} \to \mathbb{R},\ (\mathbf{m},\varphi) \mapsto L(\mathbf{m}, \varphi) = H^r(\mathbf{n}, e^{\varphi}), \tag{5.2}$$

where e^{φ} is the exponential function, H^r is defined by (3.12) and

$$\mathscr{C} = \{\mathbf{m} \in \mathbb{R}^3 | \exists \mathbf{n} \in \mathbb{R}^3 : \mathbf{n}\cdot\mathbf{n} = 1,\ m_i = n_i^2\ (i = 1,2,3)\}. \tag{5.3}$$

It can be verified that:

(1) $\mathscr{C}$ is closed, bounded, convex and non-empty;
(2) $\forall \varphi \in \mathbb{R}$, $\mathbf{m} \mapsto L(\mathbf{m}, \varphi)$ is concave and continuous (see Appendix 2);
(3) $\forall \mathbf{m} \in \mathscr{C}$, $\varphi \mapsto L(\mathbf{m}, \varphi)$ is convex and continuous;
(4) $\exists \mathbf{m}_o \in \mathscr{C}$ such that

$$\lim_{|\varphi| \to \infty} L(\mathbf{m}_o, \varphi) = +\infty. \tag{5.4}$$

Therefore, an application of the KY FAN-SION theorem [see EKELAND and TEMAN (1976, pp. 171–174)] shows that the functional L possesses at least a saddle point $(\bar{\mathbf{m}}, \bar{\varphi}) \in \mathscr{C} \times \mathbb{R}$ and

$$L(\bar{\mathbf{m}}, \bar{\varphi}) = \min_{\varphi \in \mathbb{R}} \max_{\mathbf{m} \in \mathscr{C}} L(\mathbf{m}, \varphi) = \max_{\mathbf{m} \in \mathscr{C}} \min_{\varphi \in \mathbb{R}} L(\mathbf{m}, \varphi). \tag{5.5}$$

From the definition (5.2) of L, it can finally be deduced that

$$H^h_{SE} = H^r_{SE}. \tag{5.6}$$

Result (5.6) is analogous to that obtained by RANIECKI (1979) concerning the coincidence of the loss of second-order work positiveness for the solid (2.4) and in the best chosen solid (2.5). Here, however, the proof technique used by RANIECKI (1979) does not seem to be applicable in the case of the loss of the SE condition.

As a consequence of the existence of a saddle point of the functional $L(\mathbf{m}, \varphi)$, problem (3.14) can be written as follows:

$$H^r_{SE} = \min_{\psi} H^r(\psi), \tag{5.7}$$

where $H^r(\psi)$ is given by (4.10). In order to solve (5.7), we observe that (5.6) implies that

$$\eta(\mathbf{n}^r_{SE}) = \psi_{SE}, \tag{5.8}$$

where η is given by (3.4). In fact, let us suppose that $\eta(\mathbf{n}^r_{SE}) \neq \psi_{SE}$. Then the Raniecki comparison solid defined by $\psi = \eta(\mathbf{n}^r_{SE})$ loses the SE condition in the direction $\mathbf{n}^h_{SE} = \mathbf{n}^r_{SE}$ at the level H^r_{SE}. On the other hand, the best chosen Raniecki comparison solid, defined by ψ_{SE}, loses the SE condition in the direction $\mathbf{n}^h_{SE}$ at the level $\bar{H}$ satisfying $H^h_{SE} < \bar{H} \leqslant H^r_{SE}$. Thus, from (5.6) we obtain (5.8).

From (5.8), it is easy to conclude that ψ_{SE} is expressible in the form

$$\psi_{SE} = \sqrt{B_{SE}/A_{SE}}. \tag{5.9}$$

Therefore, the solution of problem (5.7) can be written as

$$H^r_{SE} = \min_{\psi,k} \{G(\sqrt{B/A} - C) | \psi \in \mathbb{R}^+,\ k = 1,2,3\}, \tag{5.10}$$

where the coefficients A, B and C are defined by (4.12)–(4.14). The minimization (5.10) is to be performed over all possible values of A, B and C, with the condition that $\sqrt{B/A}$ belongs to the subset of $\mathbb{R}^+$ in which A and B are defined, and with the condition that

$$H^r(\sqrt{B/A}, k) \geqslant H^r(\sqrt{B/A}, j), \qquad j = 1,2,3, \tag{5.11}$$

where H^r is the function (4.11). It may be noted that, at most, nine different values of $G(\sqrt{B/A} - C)$ exist. Therefore the minimization problem (5.10) can be easily solved.

6. Applications

The results obtained in the preceding sections hold for all non-associative elasto-plastic models with a smooth yield surface with coaxial tensors of plastic flow mode and yield surface gradient.

The critical hardening modulus for the loss of the SE condition is now evaluated for the model proposed by Rudnicki and Rice (1975) to represent the behavior of pressure-sensitive materials and for the model of Schleicher (1926) ("modified von Mises") suitable in the modelling of polymers and metals exhibiting the strength-differential (SD) effect (Drucker, 1973; Spitzig *et al.*, 1976; Raghava and Caddel, 1973; Lee, 1988). The model of Rudnicki and Rice (1975) is founded on the constitutive assumptions described hereafter.

The Drucker–Prager (1952) yield criterion is assumed:

$$f(\mathbf{T}) = \frac{\alpha}{3}\operatorname{tr}\mathbf{T} + \sqrt{J_2} - c, \tag{6.1}$$

where $\mathbf{T}$ is the Cauchy stress, J_2 is the second invariant of the deviatoric stress $\mathbf{S}$, α is a material parameter, and c may be a function of state variables. The yield surface gradient associated with (6.1) is

$$\mathbf{Q} = \frac{\alpha}{3}\mathbf{I} + \frac{1}{2\sqrt{J_2}}\mathbf{S}. \tag{6.2}$$

The plastic flow rule is assumed as follows:

$$\mathbf{P} = \mathbf{Q} - \frac{\alpha-\beta}{3}\mathbf{I}, \tag{6.3}$$

where β is a material constitutive parameter. It is worth noting that condition (6.3) represents the simplest form of an isotropic tensorial function relating the tensors **P** and **Q**.

The values of the critical hardening moduli for the loss of second-order work positive definiteness, for the loss of the SE condition, and for strain localization, are reported in Table 1 in the case of axially-symmetric compression under constant lateral pressure. The angle between the unit vector **n** and the direction of compression is reported in brackets. The cases $\nu = 0$ and $\nu = 0.3$ are investigated for values of α and β ranging from 0 to 0.9. Analogously, in Table 2, the case of axially-symmetric tension under constant lateral pressure is reported. The values of the critical hardening modulus for the loss of the SE condition have been obtained by using the explicit solution (5.10). Second-order work positive definiteness is lost when the hardening modulus becomes equal to the critical value H_w derived by MAIER and HUECKEL (1979):

$$H_w = \tfrac{1}{2}\{[(\mathbf{P}\cdot\mathbb{E}[\mathbf{P}])\,(\mathbf{Q}\cdot\mathbb{E}[\mathbf{Q}])]^{1/2} - \mathbf{P}\cdot\mathbb{E}[\mathbf{Q}]\}. \tag{6.4}$$

Finally, the critical values of the hardening modulus for the loss of ellipticity have been calculated by using the explicit solution obtained by BIGONI and HUECKEL (1990).

It can be noted that the values of the critical hardening modulus for localization reported in Table 1, for $\nu = 0.3$, $\alpha = 0.6$ and $\alpha = 0.9$, do not coincide with the same values given in RUDNICKI and RICE (1975). This non-coincidence is due to the fact that the results in RUDNICKI and RICE (1975) were obtained by neglecting terms like those multiplying $2G/(1-\nu)$ in (4.7) [see the explicit solution for localization given by BIGONI and HUECKEL (1990)].

It is seen from Tables 1 and 2 that the second-order work is lost well before the SE condition. Moreover, localization of deformation coincides with the loss of the SE condition in the case of associative flow rule ($\alpha = \beta$) only.

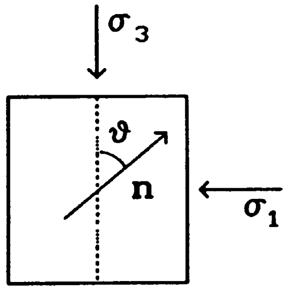
The yield surface of SCHLEICHER (1926) represents a modification of the Huber–Hencky–von Mises yield criterion to include a different behavior in tension and compression. This yield criterion is able to model the behavior of polymers (RAGHAVA and CADDEL, 1973) as well as metals showing the SD effect (DRUCKER, 1973). In these materials, however, adopting the Schleicher criterion, the plastic volumetric deformation is generally overestimated using the associative flow rule (SPITZIG *et al.*, 1976; WHITNEY and ANDREWS, 1967). Therefore, a non-associative flow rule was proposed corresponding to zero plastic volumetric strain rate (LEE, 1988). The model is founded on the SCHLEICHER (1926) yield function:

$$3J_2 + (c-t)\,\mathrm{tr}\,\mathbf{T} - ct = 0, \tag{6.5}$$

where c and t are the absolute values of the compressive and tensile yield stresses,

TABLE 1. *Critical values of* H/G *in axially-symmetric compression* (ϑ *in degrees is given in parentheses*)

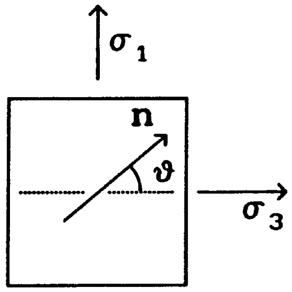
$|\sigma_1| = |\sigma_2| < |\sigma_3|$



α	β	Second-order work	Strong ellipticity	Ellipticity
		$\nu = 0$		
0.3	0.00	0.015	−0.215 (38.8)	−0.219 (38.7)
0.3	0.15	0.004	−0.261 (40.4)	−0.262 (40.4)
0.3	0.30	0.000	−0.302 (42.1)	−0.302 (42.1)
0.6	0.00	0.057	−0.243 (42.2)	−0.262 (42.1)
0.6	0.15	0.031	−0.309 (43.8)	−0.318 (43.7)
0.6	0.30	0.013	−0.370 (45.4)	−0.375 (45.4)
0.6	0.45	0.003	−0.426 (47.0)	−0.427 (47.0)
0.6	0.60	0.000	−0.478 (48.7)	−0.478 (48.7)
0.9	0.00	0.120	−0.250 (45.3)	−0.295 (45.4)
0.9	0.15	0.080	−0.337 (46.9)	−0.367 (47.0)
0.9	0.30	0.049	−0.418 (48.5)	−0.438 (48.7)
0.9	0.45	0.026	−0.495 (50.2)	−0.505 (50.4)
0.9	0.60	0.011	−0.566 (52.0)	−0.570 (52.1)
0.9	0.75	0.003	−0.632 (53.8)	−0.633 (53.8)
0.9	0.90	0.000	−0.693 (55.5)	−0.693 (55.5)
		$\nu = 0.3$		
0.3	0.00	0.047	−0.250 (45.5)	−0.280 (45.5)
0.3	0.15	0.011	−0.332 (47.6)	−0.339 (47.6)
0.3	0.30	0.000	−0.393 (49.8)	−0.393 (49.8)
0.6	0.00	0.167	−0.207 (49.0)	−0.318 (49.8)
0.6	0.15	0.086	−0.346 (51.3)	−0.403 (52.0)
0.6	0.30	0.034	−0.460 (53.8)	−0.482 (54.2)
0.6	0.45	0.007	−0.550 (56.4)	−0.554 (56.5)
0.6	0.60	0.000	−0.621 (58.9)	−0.621 (58.9)
0.9	0.00	0.330	−0.118 (51.4)	−0.333 (54.2)
0.9	0.15	0.204	−0.316 (54.0)	−0.443 (56.5)
0.9	0.30	0.115	−0.480 (56.9)	−0.547 (58.9)
0.9	0.45	0.057	−0.615 (60.1)	−0.644 (61.4)
0.9	0.60	0.022	−0.726 (63.4)	−0.736 (64.1)
0.9	0.75	0.005	−0.820 (66.7)	−0.821 (66.9)
0.9	0.90	0.000	−0.901 (70.1)	−0.901 (70.1)

TABLE 2. *Critical values of* H/G *in axially-symmetric extension* (ϑ *in degrees is given in parentheses*)

$|\sigma_1| > |\sigma_2| = |\sigma_3|$



α	β	Second-order work	Strong ellipticity	Ellipticity
		$\nu = 0$		
0.3	0.00	0.015	−0.101 (58.2)	−0.104 (58.3)
0.3	0.15	0.004	−0.088 (60.2)	−0.089 (60.2)
0.3	0.30	0.000	−0.071 (62.2)	−0.071 (62.2)
0.6	0.00	0.057	−0.021 (61.4)	−0.031 (62.2)
0.6	0.15	0.031	−0.026 (63.8)	−0.031 (64.2)
0.6	0.30	0.013	−0.027 (66.2)	−0.028 (66.4)
0.6	0.45	0.003	−0.023 (68.7)	−0.023 (68.8)
0.6	0.60	0.000	−0.016 (71.3)	−0.016 (71.3)
0.9	0.00	0.120	0.070 (64.4)	0.052 (66.4)
0.9	0.15	0.080	0.047 (67.2)	0.037 (68.8)
0.9	0.30	0.049	0.029 (70.2)	0.024 (71.3)
0.9	0.45	0.026	0.016 (73.5)	0.014 (74.3)
0.9	0.60	0.011	0.007 (77.3)	0.007 (77.8)
0.9	0.75	0.003	0.002 (82.6)	0.002 (82.8)
0.9	0.90	0.000	−0.0005 (90.0)	−0.0005 (90.0)
		$\nu = 0.3$		
0.3	0.00	0.047	−0.106 (52.9)	−0.130 (53.2)
0.3	0.15	0.011	−0.109 (55.4)	−0.114 (55.5)
0.3	0.30	0.000	−0.093 (57.8)	−0.093 (57.8)
0.6	0.00	0.167	0.054 (56.0)	−0.018 (57.8)
0.6	0.15	0.086	0.004 (59.1)	−0.028 (60.3)
0.6	0.30	0.034	−0.020 (62.2)	−0.031 (62.9)
0.6	0.45	0.008	−0.027 (65.4)	−0.029 (65.6)
0.6	0.60	0.000	−0.020 (68.6)	−0.020 (68.6)
0.9	0.00	0.330	0.239 (58.3)	0.117 (62.9)
0.9	0.15	0.204	0.145 (61.9)	0.082 (65.6)
0.9	0.30	0.115	0.082 (65.9)	0.054 (68.6)
0.9	0.45	0.057	0.041 (70.2)	0.031 (72.0)
0.9	0.60	0.022	0.017 (75.1)	0.015 (76.1)
0.9	0.75	0.005	0.004 (81.4)	0.004 (81.8)
0.9	0.90	0.000	−0.001 (90.0)	−0.001 (90.0)

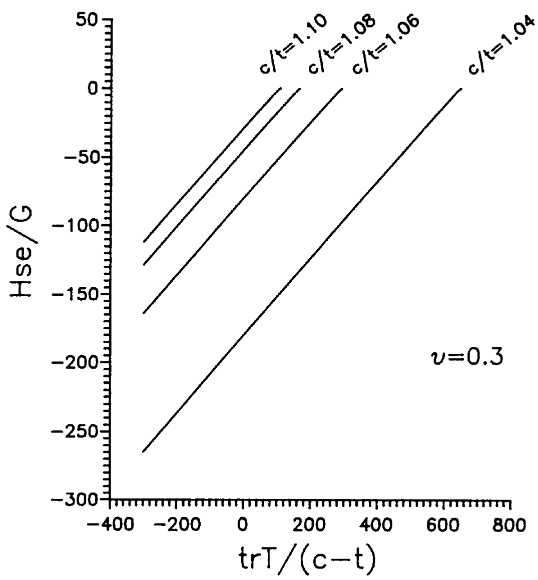


FIG. 1. Schleicher (modified von Mises) yield criterion with zero plastic volumetric deformation. Critical values of H/G for the loss of the SE condition vs $\mathrm{tr}\,\mathbf{T}/(c-t)$.

respectively. The yield surface gradient, normalized with respect to the parameter $3(c-t)$, is

$$\mathbf{Q} = \frac{1}{3}\mathbf{I} + \frac{1}{c-t}\mathbf{S} \tag{6.6}$$

and the flow rule is assumed in the form (LEE, 1988)

$$\mathbf{P} = \frac{1}{c-t}\mathbf{S}. \tag{6.7}$$

The critical hardening modulus for the loss of the SE condition has been evaluated in the case of axially-symmetric extension at various lateral pressures (the principal components of tensor $\mathbf{S}$ satisfy $S_1 > 0$ and $S_2 = S_3 \leqslant 0$). In Fig. 1, for the values of c/t equal to 1.04, 1.06, 1.08 and 1.10, the critical values of the hardening modulus (normalized with respect to the elastic shear modulus) are reported as functions of the mean normal stress [normalized with respect to the parameter $(c-t)/3$]. Figure 2 concerns the analogous case of c/t equal to 1.20, 1.25, 1.30 and 1.35. By inspection, it can be noted that the curves are almost rectilinear and the critical hardening modulus becomes strongly negative when all the principal stress components are negative.

7. REMARKS ON THE FINITE-STRAIN INCREMENTAL THEORY

Due to Raniecki's comparison theorem, the following inequality holds at finite strain ($\mathbb{G} \neq \mathbb{O}$):

$$\mathbf{g}\cdot\mathbf{A}^h(\mathbf{n})\mathbf{g} \geqslant \mathbf{g}\cdot\mathbf{A}^r(\mathbf{n})\mathbf{g}, \qquad \forall\mathbf{g} \neq \mathbf{0}. \tag{7.1}$$

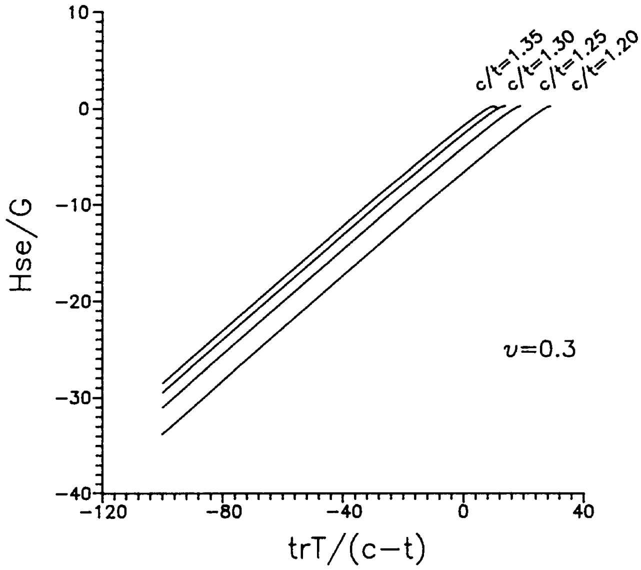


FIG. 2. Schleicher (modified von Mises) yield criterion with zero plastic volumetric deformation. Critical values of H/G for the loss of the SE condition vs $\mathrm{tr}\,\mathbf{T}/(c-t)$.

Hence, the loss of the SE condition in Raniecki's solid always precedes the loss of the SE condition in Hill's solid. Moreover, in the case of non-associative flow rules, certain directions of the velocity of deformation can result in a stiffer response to plastic loading than that corresponding to elastic unloading (MRÓZ, 1966). Therefore, in the finite theory, a loss of the SE condition corresponding to elastic unloading can, in principle, occur between the loss of the SE condition in Raniecki's and Hill's comparison solids.

As stressed in Section 2, the loss of the SE condition can be used as an exclusion principle for the integral conditions of uniqueness (2.11) and infinitesimal stability (2.7). Let us assume that the tensor $\mathbf{A}_G(\mathbf{n})$,

$$\mathbf{A}_G(\mathbf{n})\mathbf{g} = \mathbb{G}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}, \tag{7.2}$$

is negative definite. The result of Section 5, that tensors $\mathbf{A}^h$ and $\mathbf{A}^r$ lose positive definiteness at the same level in the case $\mathbb{G} = \mathbb{O}$, allows us to define, in the finite-strain context, a necessary condition for the integral conditions (2.11) and (2.7). However, this necessary condition seems to be too unrestrictive in the context of finite strain when the importance of the "geometrical terms" is relevant. On the other hand, the results obtained in Sections 4 and 5 can be easily extended to the finite-strain theory only when tensor $\mathbb{G}$ is isotropic and, therefore, tensor $\mathbb{E}+\mathbb{G}$ admits a representation of type (4.1). This is the case when the Kirchhoff stress tensor $\mathbf{K}$ is spherical.

8. CONCLUSIONS

In the context of non-associative elastoplasticity, the condition of the loss of strong ellipticity has been expressed in terms of a critical value of the hardening modulus for

Hill and Raniecki comparison solids. Under the small-strain assumption, and assuming the coaxiality of the plastic flow mode and the yield surface gradient, the following results have been obtained:

(1) loss of strong ellipticity has been shown to occur simultaneously in the two comparison solids;
(2) an explicit form is given that allows for a direct calculation of the critical hardening modulus, corresponding to the loss of strong ellipticity.

The fact that the loss of strong ellipticity does not coincide with the localization of deformation, as verified using the elastoplastic model by RUDNICKI and RICE (1975), suggests that the localization can be regarded as an extreme form of the loss of uniqueness. This result fits consistently in the framework given by RICE (1976).

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APPENDIX 1

For a given vector $\mathbf{n}$, the strong ellipticity condition (2.16) can be regarded as the condition of positiveness of the function of $\mathbf{g}$:

$$F(\mathbf{g}) = \mathbf{g}\cdot\mathbf{A}_E(\mathbf{n})\mathbf{g} - \frac{\mathbf{g}\cdot\mathbb{E}[\mathbf{Q}]\mathbf{n}}{H+H_o}\,\mathbf{g}\cdot\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{A1.1}$$

The critical hardening modulus H^h is defined as the value of the hardening modulus corresponding to the semi-definite positiveness of $F(\mathbf{g})$. Therefore, the critical hardening modulus is determined through the solution of the problem

$$\text{Find}\quad (H^h, \mathbf{g}^h):\qquad \min_{\mathbf{g}} F(\mathbf{g}) = 0. \tag{A1.2}$$

This problem is equivalent to the following problem:

$$\text{Find}\quad (\bar{H}, \bar{\mathbf{g}}):\qquad \min_{\mathbf{g}} F(\mathbf{g}) = 0 \quad \text{subject to (3.7).} \tag{A1.3}$$

In fact, if $(H^h, \mathbf{g}^h)$ is a solution to (A1.2), then $(H^h, \alpha\mathbf{g}^h)$ is a solution as well, for every $\alpha \in \mathbb{R}$. Therefore $(H^h, \bar{\alpha}\mathbf{g}^h)$ with $\bar{\alpha} = (H+H_0)/(\mathbf{g}^h \cdot \mathbb{E}[\mathbf{Q}]\mathbf{n})$ is the (unique) solution to (A1.3). The converse is trivial.

Problem (A1.3) admits the same solution as problem (3.6)–(3.7). This may be easily verified by performing the derivative of the Lagrangean functions associated to problems (A1.3) and (3.6)–(3.7). The obtained solutions differ for the values of the Lagrangean multipliers only.

Appendix 2. Concavity of $\mathbf{m} \mapsto L_\varphi(\mathbf{m}) = L(\mathbf{m}, \varphi)$

Functional (5.2), as a function of $\mathbf{m}$ only, can be written as follows:

$$L_\varphi(\mathbf{m}) = \frac{G}{2\psi}\left(2R_i^2 m_i - \frac{1}{1-\nu}(R_i m_i)^2 + \frac{2\nu}{1-\nu}(R_i m_i)\,\mathrm{tr}\,\mathbf{R} + \frac{\nu^2}{(1-\nu)(1-2\nu)}\mathrm{tr}^2\,\mathbf{R}\right) - 2G\left(P_i Q_i + \frac{\nu}{1-2\nu}\mathrm{tr}\,\mathbf{P}\,\mathrm{tr}\,\mathbf{Q}\right), \quad \text{(A2.1)}$$

where indices (summation convention) denote components in the principal reference system and the dependence on φ is implicit in $\mathbf{R}$.

Concavity requires that

$$L_\varphi(\xi\mathbf{m}' + (1-\xi)\mathbf{m}'') - \xi L_\varphi(\mathbf{m}') - (1-\xi)L_\varphi(\mathbf{m}'') \geqslant 0, \qquad \forall \xi \in [0,1], \quad \forall m', m'' \in \mathscr{C}. \quad \text{(A2.2)}$$

Using (A2.1), inequality (A2.2) becomes

$$\frac{1}{1-\nu}\xi(1-\xi)\,(R_i m_i' - R_i m_i'')^2 \geqslant 0, \qquad \forall \xi \in [0,1], \quad \text{(A2.3)}$$

which is always satisfied.