

## NONLINEAR SOLID MECHANICS

This book covers solid mechanics for nonlinear elastic and elastoplastic materials, describing the behavior of ductile materials subjected to extreme mechanical loading and their eventual failure. The book highlights constitutive features to describe the behavior of frictional materials such as geological media. On the basis of this theory, including large strain and inelastic behaviors, bifurcation and instability are developed with a special focus on the modeling of the emergence of local instabilities such as shear band formation and flutter of a continuum. The former is regarded as a precursor of fracture, whereas the latter is typical of granular materials. The treatment is complemented with qualitative experiments, illustrations from everyday life and simple examples taken from structural mechanics.

Davide Bigoni is a professor in the faculty of engineering at the University of Trento, where he has been head of the Department of Mechanical and Structural Engineering. He was honored as a Euromech Fellow of the European Mechanics Society. He is co-editor of the *Journal of Mechanics of Materials and Structures* (an international journal founded by C. R. Steele) and is associate editor of *Mechanics Research Communications*.

# Nonlinear Solid Mechanics

BIFURCATION THEORY AND  
MATERIAL INSTABILITY

Davide Bigoni  
University of Trento



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## Preface

The purpose of this book is to present a research summary on solid mechanics at large strain, including the treatment of bifurcation and instability phenomena. The framework is crucial to the understanding of failure mechanisms in ductile materials, as connected to material instabilities, such as, shear banding.

I have employed Chapters 2 through 5 as a textbook for a graduate course on non-linear elasticity that I have offered at the University of Trento since 1999, whereas Chapters 8, 10, 11 and 13 have been the basis for a course held at CISM (no. 414, ‘Material Instabilities in Elastic and Inelastic Solids’, H. Petryk, ed.). Chapters 6, 7, 9, 12, 14 and 15 have been added to present the elasticity and the yield criteria in detail, including a treatment on elastic bifurcation and instability, wave propagation and multiple shear banding. This material has been taught during seminars for graduate students at various universities. Chapter 16 is devoted to the perturbative approach to material instability, developed by me in a series of articles in cooperation with D. Capuani, M. Brun, F. Dal Corso, M. Gei, A. Piccolroaz and J. R. Willis. Finally, I have to admit that the Introduction of the book is overlong; in fact, I have used it for a 20-hour graduate course on stability and bifurcation. The hope is to attract attention to the main topics presented in the book.

During preparation of this book, I have enjoyed help from a number of friends, who have read and commented on parts of the manuscript: L. Argani, M. Bacca, K. Bertoldi, M. Brun, F. Dal Corso, A. Gajo, M. Gei, G. Mishuris, D. Misseroni, A. B. Movchan, N. V. Movchan, G. Noselli, H. Petryk, A. Piccolroaz, G. Puglisi, A. Reali, S. Roccabianca and D. Veber.

The photos presented in this book have been taken by me (using a Nikon FG-20 traditional camera or a Panasonic DMC-FZ5 digital camera) or by students at the University of Trento (using a Nikon D100 or a Nikon D200 digital camera). Most of the experiments presented have been performed at the University of Trento in the Laboratory for Physical Modeling of Structures and Photoelasticity.

## Foreword

This book clearly exhibits some remarkable and unusual features. The central theme addresses one of the primary research challenges at present in solid and structural mechanics. In fact, research on nonlinearities owing to large deformations and inelastic behaviours of materials now has to be tackled for many systematic applications in mechanical and civil engineering because the evaluation of safety margins has become computationally possible, with obvious advantages when compared with “admissible stress” criteria, popular in past structural engineering practice.

The content of this book reflects the intensive and successful research work carried out by the author and his co-workers both at the University of Trento and at other institutions. The detailed introduction includes several clear illustrative descriptions of experiments and, hence, solid links with practical motivation and application for the book’s content. It seems that in his writing, Davide Bigoni has been mindful of Cicero’s admonition not always implemented in books on mechanics: *‘Non enim paranda nobis solum, sed fruenda sapientia est’* (‘The knowledge should not only be acquired; it should be utilized as well’). Isaac Newton expanded on Cicero’s advice when he wrote, *‘Exempla docent non minus quam praecepta’* (‘Examples are not less instructive than theories’). In fact, the subsequent chapters include many examples to clarify notions of applied mathematics and theoretical continuum mechanics.

The mathematics and physics covered in this volume are not easily found in the existing engineering-oriented literature in the consistent manner presented herein. At present, attention should be paid more than in the past to the warning addressed to engineers (*‘ingeniarii’*) by Leonardo da Vinci, namely, *‘Quelli che si innamorano di pratica senza scienza son come l’orchier . . . senza timone e bussola’* (‘Those who like practice without science are like a steersman without rudder and without steering compass’). More explicitly, Leonardo underlined the important role of mathematics: *‘Nessuna umana investigazione si può dimandare vera scienza se non passa per le matematiche dimostrazioni’* (‘No human research can be true science if it does not go through mathematical demonstrations’). Probably the author paid attention to this master’s wisdom in compiling this volume.

As a conclusion, I express the opinion that this book provides a remarkable and timely contribution both to scientific education at the doctoral level and to the updating of scientific approaches and analytical tools in several areas of mechanical, civil and materials technologies.

*Giulio Maier*