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HERAUSGEGEBEN VON
L. COLLATZ · F. LÖSCH · C. TRUESDELL

6

PLANE ELASTIC SYSTEMS

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BY

L. M. MILNE-THOMSON

WITH 76 FIGURES

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Preface

In an epoch-making paper entitled "On an approximate solution for the bending of a beam of rectangular cross-section under any system of load with special reference to points of concentrated or discontinuous loading", received by the Royal Society on June 12, 1902, L. N. G. FILON introduced the notion of what was subsequently called by LOVE "generalized plane stress". In the same paper FILON also gave the fundamental equations which express the displacement (u, v) in terms of the complex variable. The three basic equations of the theory of KOLOSOV (1909) which was subsequently developed and improved by MUSKHELISHVILI (1915 and onwards) can be derived directly from Filon's equations. The derivation is indicated by FILONENKO-BORODICH. Although FILON proceeded at once to the real variable, historically he is the founder of the modern theory of the application of the complex variable to plane elastic problems. The method was developed independently by A. C. STEVENSON in a paper received by the Royal Society in 1940 but which was not published, for security reasons, until 1945.

Complex variable methods were also used by the author in lectures during the War to Officers of the Royal Corps of Naval Constructors at the Royal Naval College, Greenwich. Arising from this experience the need became evident for a thorough overhaul and systematization of the theory. The opportunity to undertake this at last presented itself when the author joined the Mathematics Research Center, United States Army. Here it was possible to undertake research leading to a series of technical reports which form the basis of the present volume. In the course of this research new methods and points of view were discovered and are here incorporated in the text.

This book is concerned with plane elastic systems in equilibrium or steady motion, within the framework of the linear theory. A companion volume on antiplane elastic systems is in preparation.

Although the notion of tensors is used occasionally in the opening chapter for the sake of conciseness and physical clarity, tensor theory forms no part of the subsequent development which is consistently by means of the complex variable.

The threefold aim of the book is (i) to state clearly what problem is under discussion, (ii) to approach the boundary value problem directly rather than by inverse or guessing methods, (iii) to provide the proper

mathematical tools. In particular the necessary parts of the theory of functions of a complex variable are given in Chapter III so that in this respect the book is reasonably self-contained. On the other hand existence and uniqueness of solutions are assumed, for there are many places, e. g. SOKOLNIKOFF, in which the reader can find general discussions of these matters.

Although the treatment takes a theoretical form, it is directly adapted to practical physical and engineering applications. The last chapter on the influence of anisotropy will, it is hoped, show that the principal effect of this phenomenon is one of arithmetical rather than of theoretical complication, and indeed in these days of efficient computing machines arithmetical complication can be largely discounted.

Each chapter is followed by a set of examples, there are 150 in all, which in many cases point to problems otherwise untreated in the text.

The references to literature are mainly directed to elucidating the text. No attempt whatever has been made to achieve even the semblance of completeness, or to settle questions of priority. For an extensive bibliography up to 1953, the reader can do no better than to consult TRUESDELL's monograph.

The sections of the book are numbered in decimal notation; the integer preceding the decimal point gives the number of the chapter. Thus 2.725 indicates that the section so numbered is in Chapter II and that it occurs later than 2.72 but earlier than 2.73. Reference to the diagrams is greatly facilitated by giving them the number of the section to which they belong.

It is a pleasure to express my lively thanks to my friend, Dr. R. E. LANGER, Director of the Mathematics Research Center, United States Army, for affording me the opportunity to carry out this research. To another friend, Professor W. M. SHEPHERD of Bristol University I owe a great debt of gratitude for his kindness in undertaking the arduous task of proof reading and for making many valuable suggestions. I also acknowledge with pleasure the really excellent work of Mrs KERN and Mrs MCMAHAN in preparing the typescript of the technical reports on which the book is based.

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1 January, 1960

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Chapter I

The role of Airy's stress function

1.01. The stress tensor

Consider a tetrahedron, centroid P, of continuous material, isolated in thought from the material which surrounds it. Let the lengths of the edges be infinitesimal of order l.

Let the vector areas of the four faces directed along the outward normals be $d\boldsymbol{\Sigma}_k$, $k = 1, 2, 3, 4$.

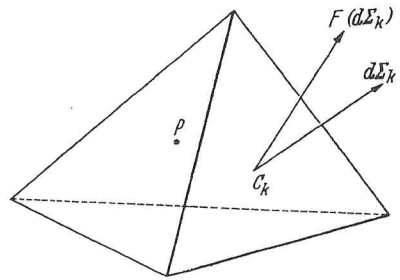

Fig. 1.01

Then by projection on an arbitrary plane we have

$$d\boldsymbol{\Sigma}_1 + d\boldsymbol{\Sigma}_2 + d\boldsymbol{\Sigma}_3 + d\boldsymbol{\Sigma}_4 = 0 . \quad (1)$$

Hypothesis. The force exerted on any face k by the material on that side of the face into which $d\boldsymbol{\Sigma}_k$ points is a vector $F(d\boldsymbol{\Sigma}_k)$ which is a function of the vector area $d\boldsymbol{\Sigma}_k$ of the face, and acts at the centroid C_k of the face.

From this hypothesis it follows by the law of action and reaction that

$$F(-d\boldsymbol{\Sigma}_k) = -F(d\boldsymbol{\Sigma}_k) , \quad (2)$$

so that F is an odd function.

We can neglect the inertial forces which are proportional to the volume (of order l^3) in comparison with the surface forces which are proportional to the area (of order l^2). Therefore the surface forces form a system in equilibrium, that is to say

$$F(d\boldsymbol{\Sigma}_1) + F(d\boldsymbol{\Sigma}_2) + F(d\boldsymbol{\Sigma}_3) + F(d\boldsymbol{\Sigma}_4) = 0 . \quad (3)$$

Therefore

$$\begin{aligned} F(d\boldsymbol{\Sigma}_1) + F(d\boldsymbol{\Sigma}_2) + F(d\boldsymbol{\Sigma}_3) &= -F(d\boldsymbol{\Sigma}_4) \\ &= F(-d\boldsymbol{\Sigma}_4) = F(d\boldsymbol{\Sigma}_1 + d\boldsymbol{\Sigma}_2 + d\boldsymbol{\Sigma}_3) , \end{aligned}$$

on using (2) and then (1).

Thus $F(d\boldsymbol{\Sigma})$ is a *linear function* of $d\boldsymbol{\Sigma}$ so that we can write, the dot denoting the scalar product,

$$F(d\boldsymbol{\Sigma}) = d\boldsymbol{\Sigma} \cdot \mathsf{S}\,,$$

where S is a tensor of the second rank, called the *stress tensor* at P.

The vector $d\boldsymbol{\Sigma} \cdot \mathsf{S}$ is the force exerted on the directed element of area $d\boldsymbol{\Sigma}$. Let $d\boldsymbol{\Sigma} = \boldsymbol{n}\, d\Sigma$, where $\boldsymbol{n}$ is the unit normal vector. The force per unit area $\boldsymbol{n} \cdot \mathsf{S}$ is called the *stress vector* for an infinitesimal area whose unit normal is $\boldsymbol{n}$.

Let $\boldsymbol{i}_h$ be a unit vector in the direction h. The component of the stress vector in this direction is denoted by $\widehat{nh}$, where

$$\widehat{nh} = (\boldsymbol{n} \cdot \mathsf{S}) \cdot \boldsymbol{i}_h\,,$$

so that in words; $\widehat{nh}$ is the component of stress in the h-direction across an element of area which is perpendicular to the n-direction.

Thus, for example, the components of stress referred to cartesian axes x, y, z across an element of area perpendicular to the x-axis will be $\widehat{xx}$, $\widehat{xy}$, $\widehat{xz}$ in the x-, y-, z-directions respectively.

1.02. The deformation tensor

Consider the particles of deformable material which lie on a line C_0 at time t_0. At time t the same particles will occupy a line C.

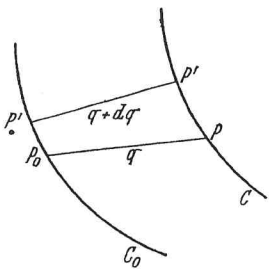

Fig. 1.02

Let P_0, P_0' be two neighbouring particles on C_0 at time t_0; P, P' the *same* particles on C at time t. We shall denote the position vectors of P_0, P by $\boldsymbol{P}_0, \boldsymbol{P}$ respectively and write

$$\overrightarrow{P_0 P_0'} = d\boldsymbol{P}_0, \quad \overrightarrow{PP'} = d\boldsymbol{P}\,, \tag{1}$$

where $d\boldsymbol{P}$ will be also supposed infinitesimal.

Consider the scalar quantity

$$d\boldsymbol{P} \cdot d\boldsymbol{P} - d\boldsymbol{P}_0 \cdot d\boldsymbol{P}_0\,. \tag{2}$$

If the material is rigid, the distance between the same two particles is invariant in time and (2) vanishes.

If the material is deformable, we take (2) as the measure of the deformation.

Introduce the displacement

$$\boldsymbol{q} = \boldsymbol{P} - \boldsymbol{P}_0\,. \tag{3}$$

Then $\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{q}$, $d\boldsymbol{P} = d\boldsymbol{P}_0 + d\boldsymbol{q}$ and therefore

$$\begin{aligned} d\boldsymbol{P} \cdot d\boldsymbol{P} - d\boldsymbol{P}_0 \cdot d\boldsymbol{P}_0 &= (d\boldsymbol{P}_0 + d\boldsymbol{q}) \cdot (d\boldsymbol{P}_0 + d\boldsymbol{q}) - d\boldsymbol{P}_0 \cdot d\boldsymbol{P}_0 \\ &= d\boldsymbol{P}_0 \cdot d\boldsymbol{q} + d\boldsymbol{q} \cdot d\boldsymbol{P}_0 + d\boldsymbol{q} \cdot d\boldsymbol{q}\,. \end{aligned} \tag{4}$$

Now in terms of Hamilton's vector differentiation operator ∇ *nabla* [which in cartesian coordinates has the form $\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$ where (x, y, z) are the coordinates of P_0] and dyadic multiplication, MILNE-THOMSON (5), we have

$$d\boldsymbol{q} = d\boldsymbol{P}_0 \cdot (\nabla\,;\boldsymbol{q}) = (\boldsymbol{q};\nabla) \cdot d\boldsymbol{P}_0\,, \tag{5}$$

where $\boldsymbol{q};\nabla$ is the *conjugate* of $\nabla;\boldsymbol{q}$.

Therefore from (4)

$$d\boldsymbol{P} \cdot d\boldsymbol{P} - d\boldsymbol{P}_0 \cdot d\boldsymbol{P}_0 = d\boldsymbol{P}_0 \cdot \{\nabla;\boldsymbol{q} + \boldsymbol{q};\nabla + (\nabla;\boldsymbol{q}) \cdot (\boldsymbol{q};\nabla)\} \cdot d\boldsymbol{P}_0\,. \tag{6}$$

The tensor

$$\mathsf{D} = \frac{1}{2}[\nabla;\boldsymbol{q} + \boldsymbol{q};\nabla + (\nabla;\boldsymbol{q}) \cdot (\boldsymbol{q};\nabla)] \tag{7}$$

is the *deformation tensor*.

When $\nabla;\boldsymbol{q}$ is infinitesimal we can neglect the second order term and

$$\mathsf{D} = \frac{1}{2}[\nabla;\boldsymbol{q} + \boldsymbol{q};\nabla]\,. \tag{8}$$

1.1. The equation of motion

For deformable material in equilibrium or in motion there exist at any instant two characteristic tensors of the second rank; the stress tensor S and the deformation tensor D.

Consider the material which occupies the entire region τ within a closed convex surface Σ. If $\boldsymbol{n}$ is the unit vector drawn normally outwards at the area element $d\Sigma$ of the surface, the force exerted upon the material in the region τ by the material outside Σ is

$$\int_\Sigma \boldsymbol{n} \cdot \mathsf{S}\, d\Sigma\,.$$

Let ϱ be the density, $\boldsymbol{g}$ the body-force per unit mass, $\boldsymbol{v}$ the velocity and $\boldsymbol{a}$ the acceleration at any point of the material. Then Newton's second law of motion gives

$$\int_\Sigma \boldsymbol{n} \cdot \mathsf{S}\, d\Sigma + \int_\tau \boldsymbol{g}\,\varrho\, d\tau = \frac{d}{dt}\int_\tau \boldsymbol{v}\,\varrho\, d\tau = \int_\tau \boldsymbol{a}\,\varrho\, d\tau\,, \tag{1}$$

since conservation of mass demands that $d(\varrho\, d\tau)/dt = 0$. To the first integral on the left of (1) we apply Gauss's theorem [MILNE-THOMSON (5)] and so obtain the equation of motion in the intrinsic form

$$\nabla \cdot \mathsf{S} = \varrho\,(\boldsymbol{a} - \boldsymbol{g})\,, \tag{2}$$

where ∇ is Hamilton's vector differentiation operator *nabla*, 1.02.

The vector

$$\boldsymbol{b} = \varrho\,(\boldsymbol{a} - \boldsymbol{g}) \tag{3}$$

will be called the *body-vector*. It includes mass acceleration and reversed body force.

We shall assume the stress tensor to be symmetric. In the absence of volume couples and certain types of discontinuity of stress, the symmetry follows by equating to zero the sum of the moments about the centroid of the stress forces which act upon the faces of an infinitesimal rectangular parallelepiped of the material.

The condition of symmetry is expressed by

$$\boldsymbol{x} \cdot \mathsf{S} = \mathsf{S} \cdot \boldsymbol{x} \tag{4}$$

where $\boldsymbol{x}$ is an arbitrary vector.

Take the orthogonal cartesian axes* Ox, Oy, OR with corresponding unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The stress tensor can then be written in the form

$$\begin{aligned} \mathsf{S} = {} & \widehat{xx}\,(\boldsymbol{i};\boldsymbol{i}) + \widehat{xy}\,(\boldsymbol{i};\boldsymbol{j}) + \widehat{xz}\,(\boldsymbol{i};\boldsymbol{k}) \\ & + \widehat{yx}\,(\boldsymbol{j};\boldsymbol{i}) + \widehat{yy}\,(\boldsymbol{j};\boldsymbol{j}) + \widehat{yz}\,(\boldsymbol{j};\boldsymbol{k}) \\ & + \widehat{zx}\,(\boldsymbol{k};\boldsymbol{i}) + \widehat{zy}\,(\boldsymbol{k};\boldsymbol{j}) + \widehat{zz}\,(\boldsymbol{k};\boldsymbol{k}) , \end{aligned} \tag{5}$$

where the semi-colon denotes dyadic multiplication [MILNE-THOMSON(5)]. The symmetry is expressed by the equalities

$$\widehat{xy} = \widehat{yx}, \quad \widehat{yz} = \widehat{zy}; \quad \widehat{zx} = \widehat{xz} \tag{6}$$

the first pair of which follow by writing $\boldsymbol{j}$ for $\boldsymbol{x}$ in (4).

The equation of motion (2) is then equivalent to the three scalar equations

$$\begin{aligned} \frac{\partial \widehat{xx}}{\partial x} + \frac{\partial \widehat{yx}}{\partial y} + \frac{\partial \widehat{zx}}{\partial R} &= b_1 , \\ \frac{\partial \widehat{xy}}{\partial x} + \frac{\partial \widehat{yy}}{\partial y} + \frac{\partial \widehat{zy}}{\partial R} &= b_2 , \\ \frac{\partial \widehat{xz}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} + \frac{\partial \widehat{zz}}{\partial R} &= b_3 , \end{aligned} \tag{7}$$

where b_1, b_2, b_3 are components of the body-vector $\boldsymbol{b}$.

We have preferred the above notation since in the plane systems with which we shall deal, the asymmetric part played by the applicate R makes the use of x, y convenient.

The general tensor notation in terms of cartesian coordinates x_1, x_2, x_3 with corresponding unit vectors $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, and the usual summation convention, in which a repeated index is given the values 1, 2, 3 in turn and the resulting terms are then summed, would replace (5) by

$$\mathsf{S} = \widehat{x_\alpha x_\beta}(\boldsymbol{i}_\alpha ; \boldsymbol{i}_\beta) . \tag{8}$$

* The reason for calling the third axis OR instead of Oz is that in the sequel we shall require z for the complex variable $x + iy$, and differentiations with respect to this variable will occur. On the other hand there need be no confusion in using $\widehat{xz}$, $\widehat{zz}$ etc. for stress components. In this notation x is the abscissa, y is the ordinate and R is the applicate.

The corresponding representation of the deformation tensor in terms of displacement of a particle from (x_1, x_2, x_3) to $(x_1 + q_1, x_2 + q_2, x_3 + q_3)$ is

$$\mathsf{D} = e_{\alpha\beta}\,(\boldsymbol{i}_\alpha ; \boldsymbol{i}_\beta) , \tag{9}$$

$$e_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial q_\alpha}{\partial x_\beta} + \frac{\partial q_\beta}{\partial x_\alpha}\right) + \frac{1}{2}\frac{\partial q_\gamma}{\partial x_\alpha}\frac{\partial q_\gamma}{\partial x_\beta} . \tag{10}$$

The deformation tensor D used here is always symmetric.

1.2. Plane systems

Definition 1. A *plane system* is one for which there exists a plane such that the stress tensor is the same at all material points of any normal to this plane as at the material point in which that normal meets the plane.

It follows from 1.1 (2) that the body-vector has the same property as the stress tensor of invariability along a normal.

The plane referred to in the above definition will, for convenience, be called the *plane of the system*. It is clear from the definition that this plane can be replaced by any parallel plane.

In the case of a plane system we can take, without loss of generality, the xy-plane as the plane of the system. It then follows from the definition that all the components of stress are independent of the coordinate R and so equations 1.1 (7) reduce to

$$\frac{\partial \widehat{xx}}{\partial x} + \frac{\partial \widehat{yx}}{\partial y} = b_1 , \tag{1}$$

$$\frac{\partial \widehat{xy}}{\partial x} + \frac{\partial \widehat{yy}}{\partial y} = b_2 , \tag{2}$$

$$\frac{\partial \widehat{xz}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} = b_3 . \tag{3}$$

Since the left-hand members are independent of R, it follows that b_1, b_2, b_3 are also independent of R in conformity with the property of the body-vector deduced from the definition.

The foregoing puts in evidence the general formulation for plane systems.

1.22. Fundamental stress combinations

We now proceed to discuss more particularly those plane systems for which

$$\widehat{yz} = \widehat{zx} = 0 \tag{1}$$

so that, in 1.2 (3), $b_3 = 0$ and the body-vector is parallel to the plane of the system.

We introduce the complex variable z and its conjugate $\bar{z}$,

$$z = x + iy , \quad \bar{z} = x - iy .$$

We then have the following equivalences of differentiation operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z}, \qquad \frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar z}, \tag{2}$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{3}$$

Substituting (2) in 1.2 (1) and (2) and forming the combination $b_1 - i b_2$ we find that

$$\frac{\partial}{\partial z}(\widehat{xx} + \widehat{yy}) - \frac{\partial}{\partial \bar z}(\widehat{yy} - \widehat{xx} + 2i\,\widehat{xy}) = b_1 - i\,b_2$$

or

$$\frac{\partial \Theta}{\partial z} - \frac{\partial \Phi}{\partial \bar z} = b_1 - i\,b_2, \tag{4}$$

where

$$\Theta = \widehat{xx} + \widehat{yy}, \qquad \Phi = \widehat{yy} - \widehat{xx} + 2i\,\widehat{xy}. \tag{5}$$

We shall call Θ and Φ the *fundamental stress combinations*. Their introduction into elasticity theory* is due to Kolosov.

Let Θ_0, Φ_0 be a particular solution of (4) so that

$$\frac{\partial \Theta_0}{\partial z} - \frac{\partial \Phi_0}{\partial \bar z} = b_1 - i b_2. \tag{6}$$

Then the appropriate solution of (4) is

$$\Theta = \Theta_0 + 4\frac{\partial^2 \chi}{\partial z\,\partial \bar z}, \qquad \Phi = \Phi_0 + 4\frac{\partial^2 \chi}{\partial z^2}, \tag{7}$$

where χ is an arbitrary real-valued function of x and y and therefore of z and $\bar z$ since

$$x = \frac{1}{2}(z + \bar z), \qquad y = -\frac{1}{2}i\,(z - \bar z). \tag{8}$$

This function χ is the stress function introduced in 1862 by the Astronomer Royal, G. B. Airy.

If Θ and Φ are given, so is the complex conjugate $\bar\Phi$ and the stress components can be determined from the equations

$$\left.\begin{aligned} \widehat{xx} &= \frac{1}{2}\Theta - \frac{1}{4}(\Phi + \bar\Phi), \\ \widehat{yy} &= \frac{1}{2}\Theta + \frac{1}{4}(\Phi + \bar\Phi), \\ \widehat{xy} &= -\frac{1}{4}i\,(\Phi - \bar\Phi). \end{aligned}\right\} \tag{9}$$

* Here they are introduced without any reference to elastic response of the material.

An equivalent formulation is

$$\left.\begin{aligned} \widehat{xx} &= \widehat{xx}_0 + \frac{\partial^2 \chi}{\partial y^2}, \\ \widehat{yy} &= \widehat{yy}_0 + \frac{\partial^2 \chi}{\partial x^2}, \\ \widehat{xy} &= \widehat{xy}_0 - \frac{\partial^2 \chi}{\partial x\,\partial y}, \end{aligned}\right\} \tag{10}$$

where $\widehat{xx}_0$, $\widehat{yy}_0$, $\widehat{xy}_0$ form a particular solution of equations 1.2 (1), (2).

1.24. Rotation of the axes of reference

Let us rotate the axes of reference x, y in their own plane, about the origin, through the angle α, so that they coincide with new axes x_1, y_1. We then have

$$z_1 = e^{-i\alpha} z, \qquad \bar z_1 = e^{i\alpha}\bar z, \qquad z_1 = x_1 + i y_1, \tag{1}$$

and Θ, Φ, $b_1 - i b_2$ become, say, Θ_1, Φ_1, $B_1 - i B_2$. Equation 1.22 (4) is replaced by

$$\frac{\partial \Theta_1}{\partial z_1} - \frac{\partial \Phi_1}{\partial \bar z_1} = B_1 - i B_2,$$

which, using (1), can be written

$$e^{i\alpha}\frac{\partial \Theta_1}{\partial z} - e^{-i\alpha}\frac{\partial \Phi_1}{\partial \bar z} = e^{i\alpha}(b_1 - i b_2).$$

Comparison of this with 1.22 (4) shows that

$$\Theta_1 = \Theta \tag{2}$$

$$\Phi_1 = \Phi\, e^{2i\alpha}. \tag{3}$$

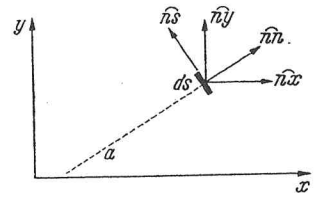

Fig. 1.24

These results could also be obtained by direct transformation of 1.22 (5). As a corollary, for the new axes, Θ_0 and Φ_0 become Θ_{01} and Φ_{01}, where

$$\Theta_{01} = \Theta_0, \qquad \Phi_{01} = \Phi_0\, e^{2i\alpha}. \tag{4}$$

From the definition of Θ and Φ we have

$$2\,(\widehat{y_1 y_1} + i\,\widehat{x_1 y_1}) = \Theta_1 + \Phi_1 = \Theta + \Phi\, e^{2i\alpha} \tag{5}$$

$$2\,(\widehat{x_1 x_1} - i\,\widehat{x_1 y_1}) = \Theta_1 - \Phi_1 = \Theta - \Phi\, e^{2i\alpha}. \tag{6}$$

Now consider a line element ds whose normal makes an angle α with Ox (fig. 1.24). The stress vector across this line has normal component

$\widehat{nn}$ and tangential component $\widehat{ns}$, or equivalently $\widehat{nx}$ parallel to the x-axis and $\widehat{ny}$ parallel to the y-axis. Therefore

$$\widehat{nx} + i\,\widehat{ny} = e^{i\alpha}(\widehat{nn} + i\,\widehat{ns}) \tag{7}$$

and so

$$2\,(\widehat{nx} - i\,\widehat{ny}) = 2\,e^{-i\alpha}(\widehat{nn} - i\,\widehat{ns}) = \Theta\,e^{-i\alpha} - \Phi\,e^{i\alpha}, \tag{8}$$

by using (6).

Now $dz = ds\,e^{i(\alpha+\pi/2)} = i\,ds\,e^{i\alpha}$, $d\bar{z} = -i\,ds\,e^{-i\alpha}$. Therefore, using 1.22 (7),

$$\begin{aligned} 2\,(\widehat{nx} - i\,\widehat{ny}) &= \left(4\,i\frac{\partial^2\chi}{\partial z\,\partial\bar{z}} + i\Theta_0\right)\frac{d\bar{z}}{ds} + \left(4\,i\frac{\partial^2\chi}{\partial z^2} + i\Phi_0\right)\frac{dz}{ds} \\ &= 4\,i\frac{d}{ds}\left(\frac{\partial\chi}{\partial z}\right) + i\Theta_0\frac{d\bar{z}}{ds} + i\Phi_0\frac{dz}{ds} \\ &= 4\,i\frac{d}{ds}\left(\frac{\partial\chi}{\partial z}\right) + \Theta_0\,e^{-i\alpha} - \Phi_0\,e^{i\alpha}. \end{aligned} \tag{9}$$

The last two terms are stresses due to the body-vector. Therefore on the model of (8) we can write

$$2\,(\widehat{nx}_B - i\,\widehat{ny}_B) = \Theta_0 e^{-i\alpha} - \Phi_0\,e^{i\alpha} \tag{10}$$

and then (9) yields

$$(\widehat{nx} - i\,\widehat{ny}) - (\widehat{nx}_B - i\,\widehat{ny}_B) = 2\,i\frac{d}{ds}\left(\frac{\partial\chi}{\partial z}\right). \tag{11}$$

This gives the difference between the complex conjugate stress vector and its correspondent due to the body-vector in terms of the stress function.

From (8) we obtain

$$2\,(\widehat{nn} - i\,\widehat{ns}) - 2\,(\widehat{nn}_B - i\,\widehat{ns}_B) = \frac{d}{d\bar{z}}\left(4\frac{\partial\chi}{\partial z}\right). \tag{12}$$

1.26. Force and moment

Let AB be an arc of a curve drawn in the material in the xy-plane, fig. 1.26 (i). On AB we choose a sense of description, say from A to B, in which the arc length s increases. This enables us to distinguish material L on the left from material R on the right as we move along AB in the chosen sense.

The resultant force exerted by the material on the right upon the material on the left of the arc AB can be reduced at the *base-point* O to components (X, Y) along the axes and a moment M about O. The part of this stress-resultant due to the body-vector will be denoted by (X_B, Y_B), M_B.

Then from 1.24 (11) we have

$$\begin{aligned} (X - iY) - (X_B - iY_B) &= \int_{AB}\{(\widehat{nx} - i\,\widehat{ny}) - (\widehat{nx}_B - i\,\widehat{ny}_B)\}\,ds \\ &= \int_{AB}\frac{d}{ds}\left(2\,i\frac{\partial\chi}{\partial z}\right)ds = \left[2\,i\frac{\partial\chi}{\partial z}\right]_{AB}, \end{aligned} \tag{1}$$

where the notation signifies the increment of the quantity within the square brackets on describing the arc AB in the sense from A to B.

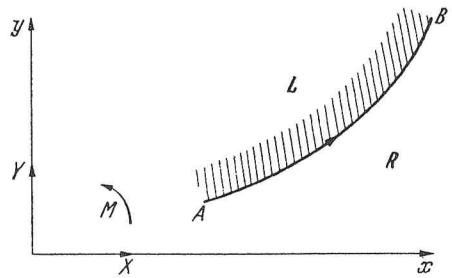

Fig. 1.26 (i)

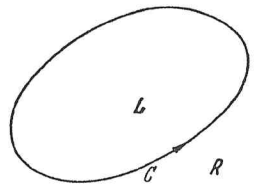

Fig. 1.26 (ii)

To find the moment about O we let Re denote "the real part of" and then

$$\begin{aligned} M - M_B &= \mathrm{Re}\int_{AB} i\,z\,\{(\widehat{nx} - i\,\widehat{ny}) - (\widehat{nx}_B - i\,\widehat{ny}_B)\}\,ds \\ &= \mathrm{Re}\int_{AB} -2\,z\frac{d}{ds}\left(\frac{\partial\chi}{\partial z}\right)ds \\ &= \mathrm{Re}\left[-2\,z\frac{\partial\chi}{\partial z}\right]_{AB} + \int_{AB}\left(dz\frac{\partial\chi}{\partial z} + d\bar{z}\frac{\partial\chi}{\partial\bar{z}}\right) \\ &= \mathrm{Re}\left[\chi - 2\,z\frac{\partial\chi}{\partial z}\right]_{AB}. \end{aligned} \tag{2}$$

This can also be written

$$M - M_B + iN = \left[\chi - 2\,z\frac{\partial\chi}{\partial z}\right]_{AB}, \tag{3}$$

where N is a real quantity whose value is irrelevant to the present investigation.

Formulae equivalent to (1) and (2) are

$$X - X_B = \left[\frac{\partial\chi}{\partial y}\right]_{AB}, \qquad Y - Y_B = \left[-\frac{\partial\chi}{\partial x}\right]_{AB}, \tag{4}$$

$$M - M_B = \left[\chi - x\frac{\partial\chi}{\partial x} - y\frac{\partial\chi}{\partial y}\right]_{AB}. \tag{5}$$

Formulae (1) and (2) remain valid when the open curve AB is replaced by a closed circuit C which encloses a simply connected region. In fig. 1.26 (ii) such a curve C is shown, described counterclockwise, and the formulae (1) and (2) with C written instead of AB give the force and moment exerted upon the boundary C by the material outside C. The region inside C may even be empty of material as, for example, a hole in a plate.

If in fig. 1.26 (ii) we imagine the sense of description to be reversed i. e. to be made clockwise, the same formulae will give the force and moment exerted on the boundary C in the reverse direction, i. e. from inside to outside. We note that if χ and $\partial\chi/dz$ are one-valued, their increments on describing C are zero and therefore

$$X = X_B\,, \quad Y = Y_B\,, \quad M = M_B \tag{6}$$

that is to say the resultant force is simply that due to the body-vector. Referring to 1.1 (3) we see that the body-vector consists of the mass-acceleration and the reversed body force. In the case where the acceleration is zero (6) expresses that the resultant force is equal to the resultant reversed body force. In the case where the material is a fluid this is no other than the theorem known as the principle of Archimedes. We can therefore regard (6) as the generalization of this principle to continuous material of any kind, and also to include mass-acceleration as well as reversed body-force.

As a direct verification of the above generalizations of the principle of Archimedes, we have from 1.24 (10)

$$X_B - i\,Y_B = \frac{1}{2}\int_C (i\,\Theta_0 d\bar{z} + i\,\Phi_0\,dz)\,.$$

Use the Area theorem 3,11 (5). Then

$$X_B - i\,Y_B = \int_S \left(\frac{\partial\Theta_0}{\partial z} - \frac{\partial\Phi_0}{\partial\bar{z}}\right) dS = \int_S (b_1 - i\,b_2)\,dS\,,$$

from 1.22 (6). Here S is the area enclosed by C. Thus we see that the force in question is indeed that exerted over the area S by the body-vector field.

This force will in general tend to zero when the area S tends to zero.

We now envisage the case where $\partial\chi/\partial z$ and χ have discontinuities at a point P of the material, of such a kind that for a sequence of simple circuits C surrounding P

$$\lim_{C\to 0}\left[2\,i\,\frac{\partial\chi}{\partial z}\right]_C = X_0 - i\,Y_0\,, \tag{7}$$

$$\lim_{C\to 0}\,[\chi]_C = M_0\,. \tag{8}$$

Consider first the case where $M_0 = 0$. When the circuit C is of sufficiently small maximum diameter, the body-vector force may be neglected so that (1) and (2) give with sufficient approximation

$$X - i\,Y = X_0 - i\,Y_0\,, \quad M = \mathrm{Re}\,i\,z_P(X_0 - i\,Y_0)$$

which indicates the presence of a *concentrated force* $X_0 + i\,Y_0$ acting at P.

Again if $X_0 - i\,Y_0 = 0$ and $M_0 \neq 0$, (1) and (2) give

$$X - i\,Y = 0\,, \quad M = M_0$$

so that there is a *concentrated couple* of moment M_0 at P.

If neither $X_0 - i\,Y_0$ nor M_0 vanishes we have at P both a concentrated force $X_0 + i\,Y_0$ and a concentrated couple of moment M_0.

1.3. Plane deformation

At any point of the material there exist three mutually perpendicular principal directions of deformation namely the axes of the deformation ellipsoid (Love). These directions in general change from point to point.

Definition 2. A state of *plane deformation* is said to exist if the following conditions are satisfied:

(i) One of the principal directions of deformation is the same at every point of the material.

(ii) Apart from a rigid body movement of the material as a whole, particles which occupy planes perpendicular to the fixed principal direction prior to the deformation continue to occupy the same planes after the deformation.

Thus there is no warping of material planes which are perpendicular to the invariable direction. Let us take the axis OR in the invariable direction. Let the particle which was at (x, y, R) before deformation occupy the position $(x + u,\ y + v,\ R + w)$ after deformation. Then (u, v, w) is the displacement.

Part (ii) of definition 2 states that, apart from a rigid body movement of the material as a whole, we must have

$$w = 0\,, \quad e_{zz} = 0 \tag{1}$$

where e_{zz} is the component of the deformation tensor appropriate to the direction OR. This deformation coefficient is given by, 1.1 (10),

$$e_{zz} = \frac{\partial w}{\partial R} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial R}\right)^2 + \left(\frac{\partial v}{\partial R}\right)^2 + \left(\frac{\partial w}{\partial R}\right)^2\right\}.$$

Thus from (1)

$$\left(\frac{\partial u}{\partial R}\right)^2 + \left(\frac{\partial v}{\partial R}\right)^2 = 0$$

which implies that

$$\frac{\partial u}{\partial R} = 0\,, \quad \frac{\partial v}{\partial R} = 0\,. \tag{2}$$

So that u, v are functions of x, y only. The conditions (2) together with $w = 0$ show that all partial derivatives with respect to R vanish and so

$$\left.\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} \\ e_{xy} &= \frac{1}{2}\left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\} + \frac{1}{2}\left\{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right\} \\ e_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \end{aligned}\right\} \tag{3}$$

$$e_{xz} = e_{yz} = e_{zz} = 0\,. \tag{4}$$

Insofar as the stress tensor is held to depend only on the deformation tensor and its rates of change, we see that a system undergoing plane deformation is a plane system in the sense of the definition. The foregoing considerations are of general application to material continuously distributed. No special hypotheses have been made concerning properties of the material such as isotropy or anisotropy nor has it been specified whether the response is elastic, plastic, or otherwise.

1.4. Plane infinitesimal elastic deformation

Deformation is said to be infinitesimal when the quadratic terms in the expressions for the components of the deformation tensor are negligible.

Thus for plane infinitesimal deformation we have from 1.3 (3)

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xz} = e_{yz} = e_{zz} = 0\,. \tag{1}$$

In the case of infinitesimal elastic deformation we have, in addition, Hooke's law* which states that the stress tensor S is a linear function of the deformation tensor D [MILNE-THOMSON (4)]

$$\mathsf{S} = \mathsf{H}_{(4)} \cdot\cdot\, \mathsf{D}\,, \tag{2}$$

where $\mathsf{H}_{(4)}$ is a tensor of the fourth rank, which we may call the *Hooke's tensor*, whose components are known as moduli of elasticity and where $(\cdot\cdot)$ denotes the double scalar or twice contracted product [MILNE-THOMSON (3), (5)]. Here, by definition the double scalar product of $\boldsymbol{i}_\alpha;\, \boldsymbol{i}_\beta$ and $\boldsymbol{i}_\gamma;\, \boldsymbol{i}_\delta$ is given by

$$\boldsymbol{i}_\alpha;\, \boldsymbol{i}_\beta \cdot\cdot\, \boldsymbol{i}_\gamma;\, \boldsymbol{i}_\delta = (\boldsymbol{i}_\beta \cdot \boldsymbol{i}_\gamma)\,(\boldsymbol{i}_\alpha \cdot \boldsymbol{i}_\delta)\,.$$

If $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ are orthogonal unit vectors this product vanishes unless both $\gamma = \beta$ and $\delta = \alpha$.

* ROBERT HOOKE, Professor of Geometry in Gresham College and Surveyor to the City of London, stated (1678) that for springy material the force is proportional to the extension it produces. On this all subsequent generalizations have been based.

It is a consequence of elastic response as given by Hooke's law that a strain energy function or elastic potential must exist, which measures the recoverable stored energy per unit volume (SOKOLNIKOFF). This function U is defined by

$$U = \frac{1}{2}\,(\mathsf{S} \cdot\cdot\, \mathsf{D}) = \frac{1}{2}\,\{(\mathsf{H}_{(4)} \cdot\cdot\, \mathsf{D}) \cdot\cdot\, \mathsf{D}\} \tag{3}$$

and is therefore a homogeneous quadratic function of the second rank tensor D. By this we mean that if t is a scalar parameter,

$$U(t\mathsf{D}) = t^2 U(\mathsf{D}) \tag{4}$$

and for such a function we have the tensor analogue of Euler's theorem on homogeneous functions namely.

Theorem

$$\mathsf{D} \cdot\cdot\, \frac{\partial U}{\partial \mathsf{D}} = 2\,U\,, \tag{5}$$

where the tensor derivative $\partial U/\partial \mathsf{D}$ is defined as the symmetric tensor of the second rank which satisfies the relation

$$dU = d\mathsf{D} \cdot\cdot\, \frac{\partial U}{\partial \mathsf{D}}\,. \tag{6}$$

Proof.

Let $\mathsf{E} = t\mathsf{D}$.

Then $U(\mathsf{E}) = U(t\mathsf{D}) = t^2 U(\mathsf{D})$ and therefore from (6)

$$\frac{\partial U(\mathsf{E})}{\partial t} = \frac{\partial \mathsf{E}}{\partial t} \cdot\cdot\, \frac{\partial U(\mathsf{E})}{\partial \mathsf{E}} = 2t\, U(\mathsf{D})\,.$$

Since $\partial \mathsf{E}/\partial t = \mathsf{D}$ we have the required theorem by putting $t = 1$.

Q. E. D.

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CASTIGLIANO's first theorem is a direct consequence of this result, for combining (5) and (3) we have

$$\mathsf{D} \cdot\cdot\, \frac{\partial U}{\partial \mathsf{D}} = \mathsf{S} \cdot\cdot\, \mathsf{D} = \mathsf{D} \cdot\cdot\, \mathsf{S}$$

since the double scalar product commutes, and therefore

$$\frac{\partial U}{\partial \mathsf{D}} = \mathsf{S}\,, \tag{7}$$

which is the theorem in question.

The existence of the strain energy function places a limitation on the tensor $\mathsf{H}_{(4)}$, for from (3) we have

$$dU = \frac{1}{2}\,(\mathsf{H}_{(4)} \cdot\cdot\, \mathsf{D}) \cdot\cdot\, d\mathsf{D} + \frac{1}{2}\,(\mathsf{H}_{(4)} \cdot\cdot\, d\mathsf{D}) \cdot\cdot\, \mathsf{D}$$

and for this to be compatible with (6) we must have

$$(\mathsf{H}_{(4)} \cdot\cdot\, d\mathsf{D}) \cdot\cdot\, \mathsf{D} = (\mathsf{H}_{(4)} \cdot\cdot\, \mathsf{D}) \cdot\cdot\, d\mathsf{D}\,. \tag{8}$$

This relation in conjunction with the symmetry of S shows that the cartesian representation of $\mathsf{H}_{(4)}$ with unit vectors $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ is of the type

$$\mathsf{H}_{(4)} = \{\alpha\,\beta\,\varrho\,\sigma\}\,(\boldsymbol{i}_\alpha;\, \boldsymbol{i}_\beta;\, \boldsymbol{i}_\varrho;\, \boldsymbol{i}_\sigma) \tag{9}$$

wherein the moduli of elasticity $\{\alpha\beta, \varrho\sigma\}$ depend solely on the pairs $\alpha\beta$ and $\varrho\sigma$ and not on the order of the pairs, nor on the order of the numbers in the pairs. Thus

$$\begin{aligned} \{\alpha\beta, \varrho\sigma\} &= \{\varrho\sigma, \alpha\beta\} \\ \{\alpha\beta, \varrho\sigma\} &= \{\beta\alpha, \varrho\sigma\} = \{\beta\alpha, \sigma\varrho\} = \{\alpha\beta, \sigma\varrho\}\,. \end{aligned} \tag{10}$$

This reduces the 81 moduli implied by (9) to 21 at most, so that 21 moduli of elasticity accompany only the highest degree of anisotropy. It is important to observe that the moduli of elasticity will themselves depend upon the frame of reference in which the tensor $\mathsf{H}_{(4)}$ is represented.

The material is said to be *elastically homogeneous* if the Hooke's tensor $\mathsf{H}_{(4)}$ is not a function of position. Hooke's law in the form (2) can be inverted to give

$$\mathsf{D} = \mathsf{K}_{(4)} \cdot\cdot\, \mathsf{S}\,, \tag{11}$$

where $\mathsf{K}_{(4)}$ is a tensor of the fourth rank such that

$$\mathsf{H}_{(4)} \cdot\cdot\, \mathsf{K}_{(4)} = \mathsf{J}_{(4)}\,, \tag{12}$$

where $\mathsf{J}_{(4)}$ is the fourth rank idemfactor for double scalar multiplication. Its Cartesian representation is

$$\mathsf{J}_{(4)} = \boldsymbol{i}_\alpha;\, \boldsymbol{i}_\beta;\, \boldsymbol{i}_\beta;\, \boldsymbol{i}_\alpha \tag{13}$$

and $\mathsf{J}_{(4)}$ has the property that for an arbitrary tensor $\mathsf{T}_{(n)}$ of rank $n \geqq 2$

$$\mathsf{T}_{(n)} \cdot\cdot\, \mathsf{J}_{(4)} = \mathsf{J}_{(4)} \cdot\cdot\, \mathsf{T}_{(n)} = \mathsf{T}_{(n)}\,. \tag{14}$$

An argument from

$$U = \frac{1}{2}\{\mathsf{S} \cdot\cdot\, (\mathsf{K}_{(4)} \cdot\cdot\, \mathsf{S})\} \tag{15}$$

similar to that used for (3), shows a reduction of $\mathsf{K}_{(4)}$ to 21 distinct components at most with symmetries of the types (10) and leads to Castigliano's second theorem

$$\frac{\partial U}{\partial \mathsf{S}} = \mathsf{D}\,. \tag{16}$$

1.5. Matrix expression of Hooke's law

Hooke's law may be conveniently expressed in matrix form as follows

$$\begin{bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{zz} \\ \widehat{yz} \\ \widehat{zx} \\ \widehat{xy} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \\ h_{41} & h_{42} & h_{43} & h_{44} & h_{45} & h_{46} \\ h_{51} & h_{52} & h_{53} & h_{54} & h_{55} & h_{56} \\ h_{61} & h_{62} & h_{63} & h_{64} & h_{65} & h_{66} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2\,e_{yz} \\ 2\,e_{zx} \\ 2\,e_{xy} \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2\,e_{yz} \\ 2\,e_{zx} \\ 2\,e_{xy} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{zz} \\ \widehat{yz} \\ \widehat{zx} \\ \widehat{xy} \end{bmatrix} \tag{2}$$

The symmetry is expressed by

$$h_{rs} = h_{sr}\,, \qquad k_{rs} = k_{sr}\,. \tag{3}$$

The relation between the h's and k's may be obtained thus. Let H denote the determinant of the h-matrix and let H_{rs} be the cofactor of h_{rs} in this determinant. Then

$$k_{rs} = k_{sr} = H_{rs}/H\,. \tag{4}$$

It is assumed here that $H \neq 0$, i. e. that the h-matrix is non-singular. This appears to be obvious on physical grounds.

In the plane case $e_{xz} = e_{yz} = e_{zz} = 0$ and so

$$\begin{aligned} \widehat{xx} &= h_{11}e_{xx} + h_{12}e_{yy} + 2\,h_{16}e_{xy} \\ \widehat{yy} &= h_{21}e_{xx} + h_{22}e_{yy} + 2\,h_{26}e_{xy} \\ \widehat{zz} &= h_{31}e_{xx} + h_{32}e_{yy} + 2\,h_{36}e_{xy} \\ \widehat{xy} &= h_{61}e_{xx} + h_{62}e_{yy} + 2\,h_{66}e_{xy}\,. \end{aligned} \tag{5}$$

If in addition the xy-plane is one of *elastic symmetry* [LOVE] we have

$$h_{41} = h_{42} = h_{43} = h_{46} = h_{51} = h_{52} = h_{53} = h_{56} = 0 \tag{6}$$

and then $\widehat{yz} = \widehat{zx} = 0$.

Another important case of a plane system is that engendered by Filon's notion of *generalized plane stress* in which the actual stresses and displacements are replaced by averages taken through the thickness of the material. The equations satisfied by these components of average stress and average displacement are of the same mathematical form as for plane deformation.

A material with two (and therefore three) perpendicular planes of elastic symmetry is said to be *orthotropic*. In this case, taking the planes of symmetry as coordinate planes, the square matrix of (2) is replaced by

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 0 & 0 \\ k_{31} & k_{32} & k_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{66} \end{bmatrix} \tag{7}$$

When every plane is a plane of elastic symmetry the material is said to be *isotropic* with the matrix

$$\begin{bmatrix} \frac{1}{E} & \frac{-\eta}{E} & \frac{-\eta}{E} & 0 & 0 & 0 \\ -\frac{\eta}{E} & \frac{1}{E} & -\frac{\eta}{E} & 0 & 0 & 0 \\ -\frac{\eta}{E} & -\frac{\eta}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{bmatrix} \tag{8}$$

Here E is *Young's modulus*, η is *Poisson's ratio* and μ is the *rigidity*. In terms of Lamé's moduli of elasticity λ, μ (see 1.7)

$$E = \frac{\mu\,(3\,\lambda + 2\,\mu)}{\lambda + \mu} = 2\,\mu\,(1 + \eta)\,. \tag{9}$$

1.6. Compatibility

If $\boldsymbol{q}$ is the displacement vector, we have

$$\mathsf{D} = \frac{1}{2}\,(\nabla;\, \boldsymbol{q} + \boldsymbol{q};\, \nabla)$$

so that, denoting vector multiplication by the symbol $\wedge$,

$$\nabla_\wedge \mathsf{D} = \frac{1}{2}\,\nabla_\wedge\,(\boldsymbol{q};\, \nabla)$$

and therefore

$$\nabla_\wedge \mathsf{D}_\wedge\, \nabla = 0\,. \tag{1}$$

This is the equation of compatibility [MILNE-THOMSON (1)]. In the case of a plane deformation in which $e_{xz} = e_{yz} = e_{zz} = 0$, the only surviving scalar equation of (1) is

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\,\frac{\partial^2 e_{xy}}{\partial x\,\partial y}\,. \tag{2}$$

If we now use the equivalence of operators given in 1.22 (2), the compatibility equation (2) reduces to

$$\frac{\partial^2 \overline{G}}{\partial z^2} + \frac{\partial^2 G}{\partial \overline{z}^2} + 2\,\frac{\partial^2 F}{\partial z\,\partial \overline{z}} = 0\,, \tag{3}$$

where

$$F = e_{xx} + e_{yy}\,, \quad G = e_{yy} - e_{xx} + 2\,i\,e_{xy} \tag{4}$$

1.61. Notation for indefinite integration

We now introduce a convenient notation. Just as we often write $dW(z)/dz = W'(z)$, thereby expressing differentiation with respect to the argument (in this case z) by a prime to the right of W, so we shall express *indefinite integration* with respect to the argument by a prime to the left of W. Thus

$$`W(z) = \int W(z)\,dz\,. \tag{1}$$

The operator can be repeated

$$``W(z) = \int `W(z)\,dz \tag{2}$$

and we note that, for example,

$$(`W(z))' = W(z)\,, \quad (``W(z))' = `W(z)\,. \tag{3}$$

1.62. The equation satisfied by Airy's stress function

We consider, in illustration, the case of plane deformation in which the xy-plane is one of elastic symmetry so that from 1.5 (2)

$$\begin{aligned} e_{xx} &= k_{11}\widehat{xx} + k_{12}\widehat{yy} + k_{13}\widehat{zz} + k_{16}\widehat{xy} \\ e_{yy} &= k_{21}\widehat{xx} + k_{22}\widehat{yy} + k_{23}\widehat{zz} + k_{26}\widehat{xy} \\ e_{zz} &= k_{31}\widehat{xx} + k_{32}\widehat{yy} + k_{33}\widehat{zz} + k_{36}\widehat{xy} = 0 \\ 2\,e_{xy} &= k_{61}\widehat{xx} + k_{62}\widehat{yy} + k_{63}\widehat{zz} + k_{66}\widehat{xy} \end{aligned}$$

Eliminating $\widehat{zz}$ by means of the third equation we get

$$\begin{aligned} e_{xx} &= l_{11}\widehat{xx} + l_{12}\widehat{yy} + l_{16}\widehat{xy} \\ e_{yy} &= l_{12}\widehat{xx} + l_{22}\widehat{yy} + l_{26}\widehat{xy} \\ 2\,e_{xy} &= l_{16}\widehat{xx} + l_{26}\widehat{yy} + l_{66}\widehat{xy} \end{aligned} \tag{1}$$

where

$$l_{rs} = k_{rs} - \frac{k_{r3}\,k_{s3}}{k_{33}} = l_{sr}\,, \tag{2}$$

since $k_{rs} = k_{sr}$.

Substitution in the compatibility equation 1.6 (2) gives

$$\left(l_{12}\frac{\partial^2}{\partial x^2} - l_{16}\frac{\partial^2}{\partial x\,\partial y} + l_{11}\frac{\partial^2}{\partial y^2}\right)\widehat{xx} + \left(l_{22}\frac{\partial^2}{\partial x^2} - l_{26}\frac{\partial^2}{\partial x\,\partial y} + l_{12}\frac{\partial^2}{\partial y^2}\right)\widehat{yy} + \left(l_{26}\frac{\partial^2}{\partial x^2} - l_{66}\frac{\partial^2}{\partial x\,\partial y} + l_{16}\frac{\partial^2}{\partial y^2}\right)\widehat{xy} = 0\,. \tag{3}$$

The required equation satisfied by Airy's stress function χ is now obtained by substituting the stresses as given by 1.22 (10) into (3). If

we denote by f_0 the result of replacing $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ by $\widehat{xx}_0$, $\widehat{yy}_0$, $\widehat{xy}_0$ in the left hand side of (3), we get

$$l_{22}\frac{\partial^4\chi}{\partial x^4}-2\,l_{26}\frac{\partial^4\chi}{\partial x^3\partial y}+(2\,l_{12}+l_{66})\frac{\partial^4\chi}{\partial x^2\partial y^2}-2\,l_{16}\frac{\partial^4\chi}{\partial x\,\partial y^3}+l_{11}\frac{\partial^4\chi}{\partial y^4}+f_0=0\,. \tag{4}$$

To solve (4), write

$$f(\lambda)=l_{11}\lambda^4-2\,l_{16}\lambda^3+(2\,l_{12}+l_{66})\,\lambda^2-2\,l_{26}\lambda+l_{22} \tag{5}$$

and let the roots of the *characteristic equation* $f(\lambda)=0$ be λ_1, λ_2, λ_3, λ_4. Then if we define the operator D_k by

$$D_k=\frac{\partial}{\partial y}-\lambda_k\frac{\partial}{\partial x}\,,\qquad k=1,2,3,4, \tag{6}$$

(4) can be written

$$l_{11}D_1D_2D_3D_4\,\chi+f_0=0. \tag{7}$$

If χ_0 is a particular solution of this, (4) becomes

$$D_1D_2D_3D_4\,(\chi-\chi_0)=0\,. \tag{8}$$

Now put

$$D_4\,(\chi-\chi_0)=\varphi_3\,,\quad D_3D_4\,(\chi-\chi_0)=\varphi_2\,,\quad D_2D_3D_4\,(\chi-\chi_0)=\varphi_1\,,$$

so that

$$D_1\varphi_1=0\,,\quad D_2\varphi_2=\varphi_1\,,\quad D_3\varphi_3=\varphi_2\,,\quad D_4\,(\chi-\chi_0)=\varphi_3\,. \tag{9}$$

The solution of

$$D_1\varphi_1=\left(\frac{\partial}{\partial y}-\lambda_1\frac{\partial}{\partial x}\right)\varphi_1=0$$

is $\varphi_1=f_1\,(x+\lambda_1 y)$, where f_1 is an arbitrary function, as is readily verified by differentiation, and therefore $D_2\varphi_2=f_1\,(x+\lambda_1 y)$ has the solution, using the notation of 1.61,

$$\varphi_2=\frac{f_1\,(x+\lambda_1 y)}{\lambda_1-\lambda_2}+f_2\,(x+\lambda_2 y)\,.$$

Proceeding in this way we find that

$$\chi-\chi_0=F_1(x+\lambda_1 y)+F_2(x+\lambda_2 y)+F_3(x+\lambda_3 y)+F_4(x+\lambda_4 y)\,, \tag{10}$$

where the F_k are arbitrary functions. This result is also readily verifiable by differentiation.

We shall show in the next section that the characteristic equation $f(\lambda)=0$ has no real roots, and, since its coefficients are real, the roots must be conjugate complex say λ_1, $\bar\lambda_1$, λ_2, $\bar\lambda_2$. Also $\chi-\chi_0$ is necessarily real. Therefore the general solution of (4) can be written in the form

$$\chi-\chi_0=\frac{1}{4}\,''W_1(z_1)+\frac{1}{4}\,''\overline{W}_1(\bar z_1)+\frac{1}{4}\,''W_2(z_2)+\frac{1}{4}\,''\overline{W}_2(\bar z_2) \tag{11}$$

where

$$z_1=x+\lambda_1 y\,,\quad \bar z_1=x+\bar\lambda_1 y\,,\quad z_2=x+\lambda_2 y\,,\quad \bar z_2=x+\bar\lambda_2 y\,. \tag{12}$$

If we change the axes of reference by turning xy through the angle α to become $x'y'$ so that

$$x'=x\cos\alpha+y\sin\alpha\,,\quad y'=-x\sin\alpha+y\cos\alpha\,,$$

the equation satisfied by the stress function will become

$$D_1^{(1)}D_2^{(1)}D_3^{(1)}D_4^{(1)}\,(\chi-\chi_0)=0\,, \tag{13}$$

where

$$D_k^{(1)}=\frac{\partial}{\partial y'}-\frac{\lambda_k\cos\alpha-\sin\alpha}{\cos\alpha+\lambda_k\sin\alpha}\,\frac{\partial}{\partial x'}\,,\quad k=1,2,3,4. \tag{14}$$

Comparing this with (6) we see that the roots of the characteristic equation say λ_1', λ_2', λ_3' λ_4' in the new axes will be

$$\lambda_k'=\frac{\lambda_k\cos\alpha-\sin\alpha}{\cos\alpha+\lambda_k\sin\alpha}\,,\qquad k=1,2,3,4. \tag{15}$$

1.63. The characteristic equation has no real roots

From 1.4 (3) we have for the strain energy function

$$U=\frac{1}{2}\,(\widehat{xx}\,e_{xx}+\widehat{xy}\,e_{xy}+\widehat{yx}\,e_{yx}+\widehat{yy}\,e_{yy})$$
$$=\frac{1}{2}\,[l_{11}\widehat{xx}^2+2\,l_{26}\,\widehat{xy}\,\widehat{yy}+2\,l_{12}\,\widehat{xx}\,\widehat{yy}+l_{66}\,\widehat{xy}^2+2\,l_{16}\,\widehat{xx}\,\widehat{xy}+l_{22}\,\widehat{yy}^2] \tag{1}$$

using 1.62 (1).

Inasmuch as the strain energy function represents stored or recoverable energy, when the system is in stable equilibrium, the strain energy function must be positive semi-definite, that is to say greater than zero for any system of values assigned to the stresses which are not simultaneously zero.

We shall assume that a system undergoing elastic deformation is in stable equilibrium provided none of the stress components exceeds some positive value ε.

To prove that the equation $f(\lambda)=0$, where $f(\lambda)$ is defined in 1.62 (5), has no real roots we write, in the strain energy function,

$$\widehat{xx}=\frac{k^2}{N}\,,\quad \widehat{yy}=\frac{1}{N}\,,\quad \widehat{xy}=-\frac{k}{N}\,,$$

where N can always be chosen large enough to prevent any component exceeding ε. Then

$$U=\frac{1}{2\,N^2}f(k)$$

and since $U>0$, $f(k)>0$ so that $f(k)$ can never vanish for any real k, or $f(\lambda)=0$ has no real root. Q. E. D.

1.7. The isotropic case

When the material is isotropic, Hooke's law assumes the form

$$\mathsf{S} = \lambda\, \mathsf{I}\, \mathsf{D}_I + 2\,\mu\, \mathsf{D}\,, \tag{1}$$

where λ, μ are Lamé's moduli of elasticity, I is the second rank idem-factor and D_I is the *first scalar invariant* [MILNE-THOMSON (5)] of the deformation tensor D.

Take the first scalar invariant of (1). Then

$$\mathsf{S}_I = (3\,\lambda + 2\,\mu)\, \mathsf{D}_I \tag{2}$$

and therefore if

$$\eta = \lambda/(2\,\lambda + 2\,\mu) \tag{3}$$

is Poisson's ratio, we find that

$$2\,\mu\, \mathsf{D} = \mathsf{S} - \frac{\eta}{1+\eta}\, \mathsf{I}\, \mathsf{S}_I\,. \tag{4}$$

In the case of plane deformation with $e_{xz} = e_{yz} = e_{zz} = 0$ we get from (4)

$$2\,\mu\, e_{xx} = \frac{\widehat{xx} - \eta\,(\widehat{yy} + \widehat{zz})}{1+\eta} \tag{5}$$

$$2\,\mu\, e_{yy} = \frac{\widehat{yy} - \eta\,(\widehat{xx} + \widehat{zz})}{1+\eta} \tag{6}$$

$$2\,\mu\, e_{xy} = \widehat{yx} \tag{7}$$

$$0 = e_{zz} = \frac{\widehat{zz} - \eta\,(\widehat{xx} + \widehat{yy})}{1+\eta}\,. \tag{8}$$

Eliminate $\widehat{zz}$. Then

$$2\,\mu\, e_{xx} = \widehat{xx} - \eta\,(\widehat{xx} + \widehat{yy}) \tag{9}$$

$$2\,\mu\, e_{yy} = \widehat{yy} - \eta\,(\widehat{xx} + \widehat{yy}) \tag{10}$$

and therefore using 1.6 (4)

$$2\,\mu F = 2\,\mu\,(e_{xx} + e_{yy}) = (1 - 2\,\eta)\,(\widehat{xx} + \widehat{yy}) = (1 - 2\,\eta)\,\Theta \tag{11}$$

$$2\,\mu G = 2\,\mu\,(e_{yy} - e_{xx} + 2\,i\,e_{xy}) = \widehat{yy} - \widehat{xx} + 2\,i\,\widehat{xy} = \Phi \tag{12}$$

and therefore from 1.6 (3) the compatibility equation is

$$\frac{\partial^2 \overline{\Phi}}{\partial z^2} + \frac{\partial^2 \Phi}{\partial \bar{z}^2} + 2\,(1 - 2\,\eta)\,\frac{\partial^2 \Theta}{\partial z\, \partial \bar{z}} = 0\,. \tag{13}$$

Using 1.22 (7) we obtain the equation satisfied by χ, namely

$$16\,(1-\eta)\,\frac{\partial^4 \chi}{\partial z^2\, \partial \bar{z}^2} + \frac{\partial^2 \overline{\Phi}_0}{\partial z^2} + \frac{\partial^2 \Phi_0}{\partial \bar{z}^2} + 2\,(1 - 2\,\eta)\,\frac{\partial^2 \Theta_0}{\partial z\, \partial \bar{z}} = 0\,. \tag{14}$$

Here the functions Θ_0, Φ_0 satisfy 1.22 (6).

As an alternative procedure we note from 1.62 (2) that

$$l_{11} = l_{22} = (1-\eta)/2\,\mu\,, \quad l_{12} = -\eta/2\,\mu\,, \quad l_{66} = 1/\mu\,, \quad l_{16} = l_{26} = 0\,.$$

Therefore, 1.62 (4) becomes

$$\frac{\partial^4 \chi}{\partial x^4} + 2\,\frac{\partial^4 \chi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} + 2\,\mu\, f_0/(1-\eta) = 0\,. \tag{15}$$

Since the first three terms on the left-hand side reduce to $16\,\partial^4 \chi/\partial z^2 \partial \bar{z}^2$ it is easy to show that (15) and (14) are equivalent.

1.72. Case where the body-vector derives from a potential

In many practically important cases (e. g. gravity, steady rotation) the body-vector $b_1 - i\,b_2$ is derivable from a real-valued scalar potential function V.

From now onwards we shall assume such to be the case so that we can write

$$b_1 - i\,b_2 = \frac{\partial V}{\partial x} - i\,\frac{\partial V}{\partial y} = 2\,\frac{\partial V}{\partial z}\,. \tag{1}$$

Then 1.22 (6) can be written in the form

$$\frac{\partial}{\partial z}\,(\Theta_0 - 2\,V) - \frac{\partial \Phi_0}{\partial \bar{z}} = 0 \tag{2}$$

so that we can take for our particular solution

$$\Theta_0 = 2\,V\,, \quad \Phi_0 = 0 \tag{3}$$

and 1.7 (14) becomes

$$16\,\frac{\partial^4 \chi}{\partial z^2\, \partial \bar{z}^2} + \frac{1 - 2\,\eta}{1-\eta}\,\frac{4\,\partial^2 V}{\partial z\, \partial \bar{z}} = 0\,. \tag{4}$$

Alternatively with

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z\, \partial \bar{z}} \tag{5}$$

(4) can be written

$$\nabla_1^4\, \chi + \frac{1 - 2\,\eta}{1-\eta}\,\nabla_1^2 V = 0\,. \tag{6}$$

The case of generalized plane stress is obtained from that of plane deformation by replacing η by η' where

$$(1 - \eta')\,(1 + \eta) = 1 \tag{7}$$

and (6) then becomes

$$\nabla_1^4\, \chi + (1 - \eta)\,\nabla_1^2 V = 0\,. \tag{8}$$

Both cases are included in the equation

$$\nabla_1^4\, \chi + \nu\,\nabla_1^2 V = 0 \tag{9}$$

where

$$\left.\begin{aligned} &\nu = 2\,(\varkappa - 1)/(\varkappa + 1)\,,\\ &\varkappa = 3 - 4\,\eta \text{ for plane deformation,}\\ &\varkappa = (3 - \eta)/(1 + \eta) \text{ for generalized plane stress.}\end{aligned}\right\} \tag{10}$$

1.74. Determination of Airy's stress function in the isotropic case

In the case where the body-vector is derivable from a potential function, the determination of χ depends on the solution of the equation 1.72 (9). This equation can be further transformed by introducing a real-valued function Q defined by

$$\nabla_1^2 Q = V\,, \quad \text{or} \quad 4\frac{\partial^2 Q}{\partial z\,\partial \bar z} = V\,. \tag{1}$$

It is always possible to determine Q by quadratures since only a particular solution of (1) will be required.

With the aid of Q equation 1.72 (9) becomes*

$$\nabla_1^4 (\chi + \nu Q) = 0 \tag{2}$$

or

$$16\frac{\partial^4}{\partial z^2\,\partial \bar z^2}(\chi + \nu Q) = 0\,. \tag{3}$$

An equation of the type

$$\nabla_1^4 \psi = 0 \tag{4}$$

is known as a *plane biharmonic equation* and its solutions are known as *plane biharmonic functions* in obvious analogy with the name plane harmonic function given to solutions of Laplace's equation $\nabla_1^2 \varphi = 0$.

Evidently every harmonic function is also a biharmonic function.

It now appears from (2) that $\chi + \nu Q$ is a plane biharmonic function.

It also appears from (2) that $\nabla_1^2(\chi + \nu Q)$ is a real-valued plane harmonic function and therefore we can write

$$\nabla_1^2(\chi + \nu Q) = \frac{4\,\partial^2(\chi + \nu Q)}{\partial z\,\partial \bar z} = W(z) + \overline{W}(\bar z)\,, \tag{5}$$

for provided that $W(z)$ is twice differentiable, the right-hand member of (5) is the most general real-valued solution of Laplace's equation in two dimensions.

Integrating with respect to $\bar z$ we obtain

$$\frac{4\,\partial(\chi + \nu Q)}{\partial z} = \bar z\, W(z) + {}'\overline{W}(\bar z) + {}'w(z) \tag{6}$$

where $'w(z)$ denotes an arbitrary differentiable function of z, written in this form for subsequent convenience.

A further integration with respect to z gives

$$4(\chi + \nu Q) = \bar z\, {}'W(z) + z\, {}'\overline{W}(\bar z) + {}''w(z) + {}''\overline{w}(\bar z)\,, \tag{7}$$

for the last arbitrary function $''\overline{w}(\bar z)$ must take this form in order to render the right-hand side of (7) real, since by hypothesis $\chi + \nu Q$ is real-valued. Therefore

$$\chi = \frac{1}{4}\{\bar z\, {}'W(z) + z\, {}'\overline{W}(\bar z) + {}''w(z) + {}''\overline{w}(\bar z)\} - \nu Q\,. \tag{8}$$

* Comparing (2) and 1.62 (8) we see that in this case $\chi_0 = -\nu Q$.

The right-hand side of (7) represents the form of the most general real-valued plane biharmonic function.

It is also obvious from (7) that any plane biharmonic function which is not at the same time a harmonic function is of the form $xH_1 + yH_2 + H_3$ where H_1, H_2, H_3 are harmonic functions, for example $x^3 = x(x^2 - y^2) + y(xy)$, and conversely that any function of this type is biharmonic.

1.75. Effective stress function

If in 1.74 (7) we write

$$\chi_e = \chi + \nu Q$$

so that

$$4\,\chi_e = \bar z\, {}'W(z) + z\, {}'\overline{W}(\bar z) + {}''w(z) + {}''\overline{w}(\bar z)\,,$$

we can call χ_e the *effective stress function* yielding the effective fundamental stress combinations

$$\Theta_e = 4\frac{\partial^2 \chi_e}{\partial z\,\partial \bar z}\,, \qquad \Phi_e = 4\frac{\partial^2 \chi_e}{\partial z^2}\,.$$

Comparing these with 1.22 (7) we have

$$\Theta_e = \Theta - \Theta_0 + \nu\frac{4\,\partial^2 Q}{\partial z\,\partial \bar z}$$

$$\Phi_e = \Phi - \Phi_0 + \nu\frac{4\,\partial^2 Q}{\partial z^2}\,.$$

Clearly, since all the terms of the right-hand sides axcept Θ and Φ can be regarded as known the problems of finding Θ, Φ and Θ_e, Φ_e are equivalent. This notation often enables a considerable simplification to be made in the algebra.

1.76. The displacement in the isotropic case

From 1.22 (7), 1.7 (12) and 1.72 (3) we have

$$4\frac{\partial^2 \chi}{\partial z^2} = \Phi = 2\mu\,(e_{yy} - e_{xx} + 2\,i\,e_{xy})$$

$$= 2\mu\left\{\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\} = -4\mu\,\frac{\partial(u - i\,v)}{\partial z}$$

and therefore

$$4\mu\,\frac{\partial(u + i\,v)}{\partial \bar z} = -4\frac{\partial^2 \chi}{\partial \bar z^2}$$

so that

$$4\mu\,(u + iv) = -4\frac{\partial \chi}{\partial \bar z} + f(z)\,, \tag{1}$$

where $f(z)$ is to be determined.

Again from 1.22 (7), 1.7 (11) and 1.72 (3)

$$(\varkappa - 1)\left(4\,\frac{\partial^2 \chi}{\partial z\,\partial \bar z} + 2V\right) = 4\,\mu\,(e_{xx} + e_{yy}) = 4\,\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
$$= 4\,\mu\left\{\frac{\partial\,(u + i\,v)}{\partial z} + \frac{\partial\,(u - i\,v)}{\partial \bar z}\right\} \tag{2}$$
$$= -\,8\,\frac{\partial^2 \chi}{\partial z\,\partial \bar z} + f'(z) + \bar f'(\bar z)$$

from (1).

Therefore using 1.72 (10), 1.74 (1) and 1.74 (6)

$$f'(z) + \bar f'(\bar z) = 4\,(\varkappa + 1)\,\frac{\partial^2 \chi}{\partial z\,\partial \bar z} + 2\,(\varkappa - 1)\,V$$
$$= (\varkappa + 1)\,\frac{4\,\partial^2\,(\chi + \nu Q)}{\partial z\,\partial \bar z}$$
$$= (\varkappa + 1)\,\{W(z) + \overline{W}(\bar z)\}\,.$$

Therefore

$$f'(z) = (\varkappa + 1)\,W(z) + 4\,\mu\,i\,\gamma\,,$$

where γ is real constant and therefore

$$f(z) = (\varkappa + 1)\,\grave{}W(z) + 4\,\mu\,(\alpha + i\,\beta + i\,\gamma\,z)\,, \tag{3}$$

where α, β are real constants.

Thus finally, using 1.74 (8)

$$4\,\mu\,(u + i\,v) = \varkappa\,\grave{}W(z) - z\,\overline{W}(\bar z) - \grave{}\overline{w}(\bar z) + 4\,\nu\,\frac{\partial Q}{\partial \bar z}$$
$$+\,4\,\mu\,(\alpha + i\,\beta + i\,\gamma\,z)\,. \tag{4}$$

Here $\alpha + i\,\beta + i\,\gamma\,z$ represents an arbitrary (small) rigid body displacement of translation and rotation.

Examples I

1. Obtain the formulae $\Theta_1 = \Theta$, $\Phi_1 = \Phi\,e^{2i\alpha}$ by rotation of axes from (x, y) to (x_1, y_1) through the angle α applied directly to the definitions of Θ and Φ.

2. Express the body-vector due (i) to gravity in a direction making angle α with the x-axis (ii) to rotation with constant angular speed n about the origin.

3. Show that the stress function

$$\chi = \frac{1}{4\,\pi}\,(X - i\,Y)\,(z - z_0)\ln(z - z_0) + \frac{1}{4\,\pi}\,(X + i\,Y)\,(\bar z - \bar z_0)\ln(\bar z - \bar z_0)$$

gives a concentrated force (X, Y) at the point z_0.

4. Construct a stress function which gives a concentrated couple of moment M at the point z_0.

5. Adjacent particles, of a contimuous body, at vector positions $\boldsymbol{P}_0$, $\boldsymbol{P}_0 + d\boldsymbol{P}_0$, arrive after displacement at positions $\boldsymbol{P}$, $\boldsymbol{P} + d\boldsymbol{P}$. Show that if the body is rigid, $d\boldsymbol{P}\cdot d\boldsymbol{P} - d\boldsymbol{P}_0\cdot d\boldsymbol{P}_0 = 0$.

6. In Ex. 5 write

$$\mathsf{E} = \frac{\partial;\boldsymbol{P}_0}{\partial\boldsymbol{P}}\,,\quad \mathsf{E}_c = \frac{\boldsymbol{P}_0;\partial}{\partial\boldsymbol{P}}\,,$$

where E, E_c are conjugate tensors of the second rank. Prove that

$$d\boldsymbol{P}\cdot d\boldsymbol{P} - d\boldsymbol{P}_0\cdot d\boldsymbol{P}_0 = d\boldsymbol{P}\cdot(\mathsf{I} - \mathsf{E}\cdot\mathsf{E}_c)\cdot d\boldsymbol{P}\,,$$

where I is the unit tensor of the second rank.

7. Defining the Eulerian deformation tensor by

$$\mathsf{D}^E = \frac{1}{2}\,(\mathsf{I} - \mathsf{E}\cdot\mathsf{E}_c)$$

in the notation of Ex. 6, and writing $\boldsymbol{q} = \boldsymbol{P} - \boldsymbol{P}_0$ for the displacement undergone by the particle originally at $\boldsymbol{P}_0$, prove that

$$\mathsf{D}^E = \frac{1}{2}\left(\frac{\partial;\boldsymbol{q}}{\partial\boldsymbol{P}} + \frac{\boldsymbol{q};\partial}{\partial\boldsymbol{P}}\right) - \frac{1}{2}\,\frac{\partial;\boldsymbol{q}}{\partial\boldsymbol{P}}\cdot\frac{\boldsymbol{q};\partial}{\partial\boldsymbol{P}}\,.$$

8. Show that in cartesian coordinates in which $\boldsymbol{P}$ is the point (x, y, z) and $\boldsymbol{q}$ is the vector (u, v, w) typical components of the deformation tensor D^E are

$$e_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right];$$
$$e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) - \frac{1}{2}\left[\frac{\partial u}{\partial y}\,\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\,\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\,\frac{\partial w}{\partial z}\right].$$

9. Writing

$$\mathsf{L} = \frac{\partial;\boldsymbol{P}}{\partial\boldsymbol{P}_0}\,,\quad \mathsf{L}_c = \frac{\boldsymbol{P};\partial}{\partial\boldsymbol{P}_0}$$

show that $d\boldsymbol{P}\cdot d\boldsymbol{P} - d\boldsymbol{P}_0\cdot d\boldsymbol{P}_0 = d\boldsymbol{P}_0\cdot(\mathsf{L}\cdot\mathsf{L}_c - \mathsf{I})\cdot d\boldsymbol{P}_0$, and so defines the Lagrangian deformation tensor

$$\mathsf{D}^L = \frac{1}{2}\,(\mathsf{L}\cdot\mathsf{L}_c - \mathsf{I})$$

and hence obtain the components of this tensor given in 1.1.

10. The deformable material which occupies the volume τ within a closed surface Σ undergoes a virtual displacement in which the displacement $\boldsymbol{q}$ is further increased to $\boldsymbol{q} + \delta\boldsymbol{q}$. Express the principle of conservation of energy in the form

$$\int_\Sigma d\boldsymbol{\Sigma}\cdot\mathsf{S}\cdot\delta\boldsymbol{q} + \int_\tau \varrho\boldsymbol{g}\,d\tau\cdot\delta\boldsymbol{q} + \int_\tau T\,\delta\,(\sigma\varrho\,d\tau)$$
$$= \int_\tau \delta\left(\frac{1}{2}\,\boldsymbol{v}\cdot\boldsymbol{v}\varrho\,d\tau\right) + \int_\tau \delta\,(e_i\varrho\,d\tau)$$

where S is the stress tensor, $\boldsymbol{g}$ the body force per unit mass, ϱ the density, T the temperature, σ the entropy per unit mass, $\boldsymbol{v}$ the velocity and e_i the internal energy per unit mass.

11. From Ex. 10 deduce that

$$\left(\mathsf{S}\cdot\frac{\partial}{\partial P}\right)\cdot\delta q + T\varrho\,\delta\sigma = \varrho\,\delta\,e_i$$

for any virtual displacement δq.

12. In Ex. 11 writing

$$\varrho v_s = \left(\mathsf{S}\cdot\frac{\partial}{\partial P}\right)\cdot\delta q = \frac{1}{2}\,\mathsf{S}\cdot\cdot\left(\frac{\partial;\,\delta q}{\partial P} + \frac{\delta q;\,\partial}{\partial P}\right),$$

where v_s is, by definition, the virtual work of the stress per unit mass, show that

$$v_s = \delta\,e_i - T\,\delta\sigma\,.$$

Hence prove that v_s is an exact differential $v_s = \delta u$, if and only if, $T\,\delta\sigma$ is an exact differential. When it exists the function u is called the strain energy density per unit mass.

13. In the notation of Ex. 12 show that a strain energy density exists if either $T =$ constant, isothermal deformation, or $\delta\sigma = 0$, adiabatic deformation.

14. Prove that the strain energy function is

$$U = \frac{1}{2}\,\widehat{x_\alpha x_\beta}\,e_{\alpha\beta}$$

whether the material is isotropic or not.

15. In the isotropic case express the strain energy function (i) in terms of stress components only (ii) in terms of strain components only.

16. Use the property that every solution of Laplace's equation is an analytic function to show that $\nabla^2(\chi + \nu Q)$ is an analytic function.

Chapter II

The complex stresses and their properties in the isotropic case

This chapter aims to describe and unify the properties of the two functions of the complex variable in terms of which the solution of plane problems may be expressed in the case of isotropic material undergoing infinitesimal elastic deformation. As we are in effect dealing with plane geometry it is more convenient and intuitive to refer to the material as if it were in the form of a thin plate. In so doing there is no loss of generality, for we can proceed from this case of generalized plane stress to the case of plane deformation by a mere change in the value of the elastic constant $\varkappa$.

2.10. The complex stresses in the isotropic case

We recall the solution of the equation satisfied by Airy's stress function when isotropic material is undergoing infinitesimal elastic deformation namely, 1.74 (7),

$$\chi = \frac{1}{4}\{\bar{z}\,W(z) + z\,\overline{W}(\bar{z}) + \bar{z}\,w(z) + z\,\overline{w}(\bar{z})\} - \nu Q\,, \tag{1}$$

where Q is any real-valued solution of

$$4\frac{\partial^2 Q}{\partial z\,\partial\bar{z}} = V\,,\quad b_1 - i b_2 = 2\frac{\partial V}{\partial z}\,, \tag{2}$$

and ν is defined in 1.72 (10).

Here V is the body-potential 1.72, whose existence we shall assume throughout, and $b_1 + i b_2$ is the mass-acceleration and reversed body-force vector in the plane.

Combining these with 1.22 (7) and 1.72 (3) we find for the fundamental stress combinations

$$\widehat{xx} + \widehat{yy} = \Theta = W(z) + \overline{W}(\bar{z}) + (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial\bar{z}} \tag{3}$$

$$\widehat{yy} - \widehat{xx} + 2i\,\widehat{xy} = \Phi = \bar{z}\,W'(z) + w(z) - 4\nu\frac{\partial^2 Q}{\partial z^2} \tag{4}$$

From these relations it is clear that the functions $W(z)$ and $w(z)$ have each the physical dimensions of stress. We shall therefore call $W(z)$ and $w(z)$ the *complex stresses*.

2.11. The body-potential

We investigate the forms of V and Q when the body-force is due to gravity and the acceleration is due to a steady rotation in the plane about a fixed point a in the plane. These are the most important practical cases.

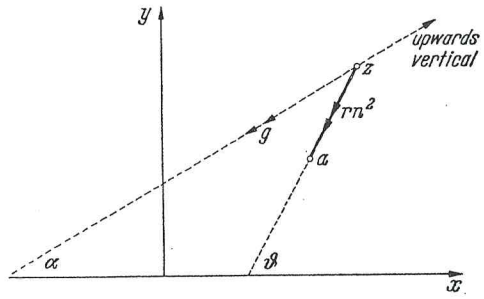

Fig. 2.11

Take right-handed axes in the xy-plane and let the x-axis make angle α with the upwards vertical. We denote by ϱ, n, g, the density, the angular speed of rotation, and the intensity of the gravitational field per unit mass.

Then if $(z-a) = r e^{i\theta}$, we have from 2.10 (2)

$$2\frac{\partial V}{\partial z} = \varrho\,(-rn^2\cos\theta + g\cos\alpha) - i\varrho\,(-rn^2\sin\theta + g\sin\alpha)$$
$$= -n^2\varrho\,(\bar z - \bar a) + g\,\varrho\,e^{-i\alpha}\,. \tag{1}$$

Integrating with respect to z we get

$$2\,V = -n^2\varrho\,z\,(\bar z - \bar a) + g\,\varrho\,e^{-i\alpha}z + f(\bar z)\,,$$

where $f(\bar z)$ is an arbitrary function of $\bar z$. But V is real-valued and therefore $f(\bar z)$ must be chosen to make the right-hand side real-valued. Therefore

$$2\,V = -n^2\varrho\,(z-a)\,(\bar z - \bar a) + g\,\varrho\,e^{-i\alpha}z + g\,\varrho\,e^{i\alpha}\bar z\,. \tag{2}$$

To find Q we have from 2.10 (2)

$$4\frac{\partial^2 Q}{\partial z\,\partial\bar z} = V\,.$$

Integrating successively with respect to $\bar z$ and z we find that a real-valued expression for Q is furnished by

$$4\,Q = -\frac{1}{8}n^2\varrho\,(z-a)^2\,(\bar z - \bar a)^2 + \frac{1}{4}g\,\varrho\,e^{-i\alpha}z^2\bar z + \frac{1}{4}g\,\varrho\,e^{i\alpha}\bar z^2 z\,. \tag{3}$$

From (3) we get

$$4\frac{\partial Q}{\partial z} = -\frac{1}{4}n^2\varrho\,(z-a)\,(\bar z - \bar a)^2 + \frac{1}{2}g\,\varrho\,e^{-i\alpha}z\bar z + \frac{1}{4}g\,\varrho\,e^{i\alpha}\bar z^2 \tag{4}$$

$$4\frac{\partial^2 Q}{\partial z^2} = -\frac{1}{4}n^2\varrho\,(\bar z - \bar a)^2 + \frac{1}{2}g\,\varrho\,e^{-i\alpha}\bar z\,. \tag{5}$$

Observe that Q and its partial derivatives are one-valued.

The method illustrated above may be used to find the function Q appropriate to other body-vector fields which are derivable from a potential.

2.12. The displacement

From 1.76 (4) the displacement

$$\upsilon = u + iv \tag{1}$$

is given by

$$4\,\mu\,\upsilon = 4\,\mu\,(u+iv) = \varkappa\,\grave{}W(z) - z\,\overline{W}(\bar z) - \grave{}\overline{w}(\bar z) + 4\,\nu\frac{\partial Q}{\partial\bar z}$$
$$+\,4\,\mu\,(\alpha + i\,\beta + i\,\gamma\,z)\,, \tag{2}$$

the last term corresponding to an arbitrary small translation $\alpha + i\,\beta$ and an arbitrary rotation through the small angle γ, in fact a displacement which is possible in a rigid body, or a rigid body displacement.

We note that while the fundamental stress combinations are expressible in terms of $W(z)$, $W'(z)$ and $w(z)$, the displacement requires for its expression the integrals $\grave{}W(z)$ and $\grave{}w(z)$.

2.13. Determinateness of the complex stresses

We first examine to what extent the complex stresses are determinate when the displacements are given at every point of the material.

Let $W(z)$, $w(z)$ and $W_1(z)$, $w_1(z)$ be two sets of complex stresses which lead to the same given displacements. From 2.12 we must then have

$$\varkappa\,\grave{}W_1(z) - z\,\overline{W}_1(\bar z) - \grave{}\overline{w}_1(\bar z) = \varkappa\,\grave{}W(z) - z\,\overline{W}(\bar z) - \grave{}\overline{w}(\bar z)\,. \tag{1}$$

Differentiate with respect to z. Then

$$\varkappa\{W_1(z) - W(z)\} = \overline{W}_1(\bar z) - \overline{W}(\bar z)$$

and therefore

$$W_1(z) = W(z)\,. \tag{2}$$

It then follows from (1) that

$$\varkappa\,\{\grave{}W_1(z) - \grave{}W(z)\} = \grave{}\overline{w}_1(\bar z) - \grave{}\overline{w}(\bar z)\,.$$

Since the left-hand side involves z only and the right-hand side $\bar z$ only each side must be equal to a constant (not necessarily real) say $\varkappa\,c$. It follows that

$$\left.\begin{aligned}\grave{}W_1(z) &= \grave{}W(z) + c\\ \grave{}w_1(z) &= \grave{}w(z) + \varkappa\,\bar c\,.\end{aligned}\right\} \tag{3}$$

It appears from (2) and (3) that, when the displacements are given, $W(z)$ and $w(z)$ are completely determinate. Moreover the displacements corresponding with $W(z)$ and $w(z)$ are unaltered if we simultaneously replace

$$\left.\begin{aligned}\grave{}W(z) &\quad\text{by}\quad \grave{}W(z) + c\\ \grave{}w(z) &\quad\text{by}\quad \grave{}w(z) + \varkappa\,\bar c\,.\end{aligned}\right\} \tag{4}$$

Suppose now that, instead of the displacements, the stresses are given.

Let $W(z)$, $w(z)$ and $W_2(z)$, $w_2(z)$ be two sets of complex stresses which lead to the same stress distribution. Then from 2.10 we must have

$$W_2(z) + \overline{W}_2(\bar z) = W(z) + \overline{W}(\bar z) \tag{5}$$

$$\bar z\,W'_2(z) + w_2(z) = \bar z\,W'(z) + w(z)\,. \tag{6}$$

From (5)

$$W_2(z) - W(z) = -\{\overline{W}_2(\bar z) - \overline{W}(\bar z)\} = i\,\gamma\,,$$

where γ is a necessarily real constant, for the first member is a function of z only and the second a function of $\bar{z}$ only and the conjugates have opposite signs. Thus

$$W_2(z) = W(z) + i\gamma\,, \quad \gamma \text{ real}, \tag{7}$$

and now from (6) we get, with the use of (7),

$$w_2(z) = w(z)\,. \tag{8}$$

Thus when the stresses are given $w(z)$ is completely determinate but $W(z)$ is determinate only up to an imaginary added constant.

Integrating (7) and (8) we have

$$\left.\begin{aligned} {}'W_2(z) &= {}'W(z) + e + i\gamma z\,, \ \gamma \text{ real} \\ {}'w_2(z) &= {}'w(z) + f \end{aligned}\right\} \tag{9}$$

where e and f are arbitrary complex constants.

Thus the state of stress is unaltered if we simultaneously replace

$$\left.\begin{aligned} {}'W(z) &\text{ by } {}'W(z) + e + i\gamma z\,, \ \gamma \text{ real} \\ {}'w(z) &\text{ by } {}'w(z) + f\,. \end{aligned}\right\} \tag{10}$$

If both the stress distribution and the displacements are prescribed, comparison of (3) with (9) shows that we must have

$$\gamma = 0\,, \quad e = c\,, \quad f = \varkappa\bar{c}\,.$$

Thus $W(z)$, $w(z)$ are completely determinate and (4) expresses the conditions that ${}'W(z)$ and ${}'w(z)$ must satisfy when both stress and displacement are prescribed.

2.14. The rotation

The rotation* is defined by

$$\varepsilon = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{1}$$

so that if $\upsilon = u + iv$ as in 2.12,

$$\varepsilon = -\frac{1}{2}i\left(\frac{\partial \upsilon}{\partial z} - \frac{\partial \bar{\upsilon}}{\partial \bar{z}}\right). \tag{2}$$

Therefore from 2.12 (2)

$$4\mu\varepsilon = -\frac{1}{2}i(\varkappa + 1)\{W(z) - \overline{W}(\bar{z})\} + 4\mu\gamma. \tag{3}$$

Observe that this expression does not contain either w or Q explicitly.

* As a vector the rotation is one-half of the curl of the displacement vector. [cf. the notion of vorticity in fluids. See e. g. MILNE-THOMSON (5)].

2.20. Change of direction of the axes of reference

If we turn the axes of reference x, y about the origin O through the angle α so that they become x_1, y_1, we have, as in 1.24,

$$\Theta_1 = \Theta \tag{1}$$

$$\Phi_1 = \Phi\, e^{2i\alpha}\,. \tag{2}$$

It follows that

$$\begin{aligned} 2\,(\widehat{x_1x_1} - i\,\widehat{x_1y_1}) &= \Theta - \Phi\, e^{2i\alpha} \\ &= W(z) + \overline{W}(\bar{z}) - \{\bar{z}W'(z) + w(z)\}\, e^{2i\alpha} \\ &\quad + (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}} + 4\nu\frac{\partial^2 Q}{\partial z^2}\, e^{2i\alpha}\,. \end{aligned} \tag{3}$$

The displacement vector $\upsilon = u + iv$ is related to its new expression $\upsilon_1 = u_1 + iv_1$ by

$$\upsilon_1 = u_1 + iv_1 = \upsilon\, e^{-i\alpha} = (u + iv)\, e^{-i\alpha}, \tag{4}$$

as is at once obvious from a diagram.

Formulae (3) and (4) are of more general application

Consider the arc ds at the point P of a curve C drawn in the material and let the normal to C at P make angle α with the x-axis (fig. 2.20). Then the normal component of stress $\widehat{nn}$ across the arc ds corresponds to $\widehat{x_1x_1}$ in (3) and the tangential component $\widehat{ns}$ corresponds to $\widehat{x_1y_1}$ and so (3) is applicable directly with this change of notation.

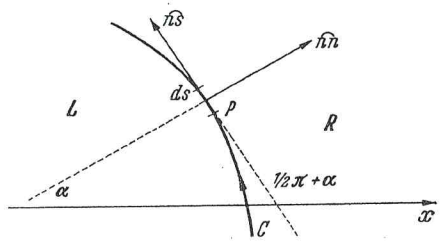

Fig. 2.20

Further at P, $dz = ds\, e^{i\left(\alpha + \frac{1}{2}\pi\right)}$ and so

$$\frac{dz}{ds} = i\, e^{i\alpha}\,, \quad \frac{d\bar{z}}{ds} = -i\, e^{-i\alpha} \tag{5}$$

and therefore

$$e^{2i\alpha} = -\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds}\,. \tag{6}$$

Therefore (3) becomes

$$\begin{aligned} 2\,(\widehat{nn} - i\,\widehat{ns}) &= W(z) + \overline{W}(\bar{z}) + \{\bar{z}\,W'(z) + w(z)\}\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds} \\ &\quad + (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}} - 4\nu\frac{\partial^2 Q}{\partial z^2}\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds}\,. \end{aligned} \tag{7}$$

If we denote by u_n, u_s the normal and tangential components of the displacement at P, (4) and (5) give

$$(u_n + i\,u_s) = (u + iv)\, i \frac{d\bar{z}}{ds}. \tag{8}$$

Now $ds = |d\bar{z}| = |dz|$. Therefore we have for the displacement

$$4\,\mu\,(u_n + i u_s) = i \frac{d\bar{z}}{|d\bar{z}|}\left\{\varkappa' W(z) - z\,\overline{W}(\bar{z}) - \overline{w}(\bar{z}) + 4\,\nu \frac{\partial Q}{\partial \bar{z}}\right\} + 4\,\mu\,(\alpha + i\beta + i\gamma z) \tag{9}$$

where the last term represents a small arbitrary rigid body displacement.

2.21. Change of origin

We suppose the complex stresses $W(z)$ and $w(z)$ to be given. If we transfer to parallel axes x_1, y_1 through the point $z = c$, we shall have

$$z = z_1 + c\,, \quad z_1 = x_1 + i y_1 \tag{1}$$

while the complex stresses referred to the new axes will become say $W_1(z_1)$, $w_1(z_1)$. It is required to deduce these from the given functions $W(z)$, $w(z)$.

The stress function χ, the body-potential V, and the function Q defined in 2.10 are real-valued scalar functions of position so that their values at a given point P and their derivates with respect to z or z_1 are the same in both sets of coordinates. Thus

$$\begin{aligned} \chi &= \chi(z, \bar{z}) = \chi(z_1 + c, \bar{z}_1 + \bar{c}) = \chi_1(z_1, \bar{z}_1) \\ Q &= Q(z, \bar{z}) = Q(z_1 + c, \bar{z}_1 + \bar{c}) = Q_1(z_1, \bar{z}_1) \\ \frac{\partial Q}{\partial z} &= \frac{\partial Q_1}{\partial z_1}, \quad \frac{\partial^2 Q}{\partial z^2} = \frac{\partial^2 Q_1}{\partial z_1^2} \end{aligned} \tag{2}$$

and so on.

Again it follows from (2) and the definitions that the fundamental stress combinations are the same at P when expressed in either set of coordinates and therefore

$$W_1(z_1) + \overline{W}_1(\bar{z}_1) = W(z) + \overline{W}(\bar{z}) \tag{3}$$

$$\bar{z}_1 W_1'(z_1) + w_1(z_1) = \bar{z}\, W'(z) + w(z)\,. \tag{4}$$

From (1) and (3)

$$\{W_1(z_1) - W(z_1 + c)\} + \{\overline{W}_1(\bar{z}_1) - \overline{W}(\bar{z}_1 + \bar{c})\} = 0$$

and therefore

$$W_1(z_1) - W(z_1 + c) = i\varkappa\,,$$

where $\varkappa$ is a real constant, which without loss of generality may be taken to be zero. Indeed the stresses derived from 2.10 (3), (4) are not affected by adding an imaginary constant to $W(z)$, while the displacement,

2.12 (2), is only affected by a rigid body movement which may be regarded as absorbed in the arbitrary part. Thus we have

$$W_1(z_1) = W(z_1 + c)\,. \tag{5}$$

Now from (4) and (1)

$$w_1(z_1) = w(z_1 + c) + (\bar{z}_1 + \bar{c})\, W'(z_1 + c) - \bar{z}_1 W_1'(z_1)\,.$$

Combine this with (5). Then

$$w_1(z_1) = w(z_1 + c) + \bar{c}\, W'(z_1 + c)\,. \tag{6}$$

2.30. The stress boundary condition

Let the normal and tangential components of the force per unit area applied at the point P of the boundary be N and S respectively (fig. 2.30). In the notation of fig. 2.20 the stress boundary conditions are $N = \widehat{nn}$ and $S = \widehat{ns}$ and therefore from 2.20 (7), at the boundary,

$$W(z) + \overline{W}(\bar{z}) + \{\bar{z}\, W'(z) + w(z)\} \frac{dz}{ds}\Big/\frac{d\bar{z}}{ds} = 2\,(N - iS) - B\,, \tag{1}$$

where

$$B = (8 - 4\,\nu) \frac{\partial^2 Q}{\partial z\, \partial \bar{z}} - 4\,\nu \frac{\partial^2 Q}{\partial z^2} \frac{dz}{ds}\Big/\frac{d\bar{z}}{ds}\,. \tag{2}$$

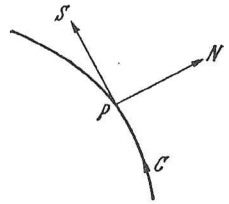

Fig. 2.30

Note (i). We see that $B = 0$ whenever the body-vector vanishes, that is, whenever the body-potential is zero.

Note (ii). If the equation of the boundary is $f(x, y) = 0$, we can use the relations $x = \frac{1}{2}(z + \bar{z})$, $y = -\frac{1}{2} i\,(z - \bar{z})$ to express the equation in the form $F(z, \bar{z}) = 0$. We then get

$$\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds} = -\frac{\partial F}{\partial \bar{z}}\Big/\frac{\partial F}{\partial z} = -\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\Big/\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right). \tag{3}$$

2.40. Resultant of the stress forces on an arc

The stress resultants, force and moment, (X, Y), M reduced to the origin as base-point are derivable directly from the formulae given in 1.26, which assumes right-handed axes, by substituting therein the expression for χ in terms of the complex stresses. We then get

$$X - iY = \frac{1}{2} i\,[\overline{W}(\bar{z}) + \bar{z}\, W(z) + w(z)]_{AB} - \nu i \left[\frac{2\,\partial Q}{\partial z}\right]_{AB} + i \int_{AB} \frac{4\,\partial^2 Q}{\partial z\, \partial \bar{z}}\, d\bar{z}\,. \tag{1}$$

Since $\chi = \mathrm{Re}\,\frac{1}{2}\{\bar{z}\,'W(z) + ''w(z)\} - \nu Q$ we have for the moment

$$M = \mathrm{Re}\,\frac{1}{2}\,[\{\bar{z}\,'W(z) - z\,'\overline{W}(\bar{z})\} - z\bar{z}\,W(z) + ''w(z) - z\,'w(z)]_{AB}$$
$$+ \mathrm{Re}\left\{-\nu\left[[Q - 2\,z\,\frac{\partial Q}{\partial z}\right]_{AB} - \int\limits_{AB} z\,\frac{4\,\partial^2 Q}{\partial z\,\partial \bar{z}}\,d\bar{z}\right\}$$

since $\bar{z}\,'W(z) - z\,'\overline{W}(\bar{z})$ is imaginary we have

$$M = \mathrm{Re}\left\{\frac{1}{2}\,[-z\bar{z}\,W(z) + ''w(z) - z\,'w(z)]_{AB}\right.$$
$$\left. - \nu\left[Q - 2\,z\,\frac{\partial Q}{\partial z}\right]_{AB} - \int\limits_{AB} z\,\frac{4\,\partial^2 Q}{\partial z\,\partial \bar{z}}\,d\bar{z}\right\}. \tag{2}$$

Note that if AB is part of a boundary at which the material terminates so that there is none to the right of AB, (1) and (2) above will give the resultant external force, if any, applied to AB, and therefore acting upon the material to the left of AB.

2.50. The cyclic function

By means of the relations $x = \frac{1}{2}(z + \bar{z})$, $y = -\frac{1}{2}\,i\,(z - \bar{z})$ any function of x and y can be expressed as a function of z and $\bar{z}$. Let

$$f = f(z, \bar{z}) \tag{1}$$

be such a function defined in the region D.
Let C be a simple closed curve lying in the region D and let z be a point of C which is not a singular point of $f(z, \bar{z})$. The function f maybe one-valued as for example $z^3\,\bar{z} + z^2$ or many-valued as for example $(z - 1)^{1/2}$ or $\bar{z}\ln z$.

Let us start from z with a definite determination of the function $f(z, \bar{z})$ say $f_1(z, \bar{z})$ and follow the variation of this as z describes the circuit C. After describing the circuit we shall arrive back at z with the determination $f_2(z, \bar{z})$ of the function.

With the notation of 1.26 we write

$$[f(z, \bar{z})]_C = f_2(z, \bar{z}) - f_1(z, \bar{z})\,. \tag{2}$$

We call the function

$$[f(z, \bar{z})]_C \tag{3}$$

the *cyclic function** of $f(z, \bar{z})$, relative to the circuit C and the initial determination of $f(z, \bar{z})$.

* The terminology is due to FILON.

The cyclic function of any function f for which

$$\frac{\partial^2 f}{\partial z\,\partial \bar{z}} = \frac{\partial^2 f}{\partial \bar{z}\,\partial z} \tag{4}$$

is the same for C and for any other curve C' which passes through z and is reconcilable with C without passing over a singularity of $f(z, \bar{z})$.

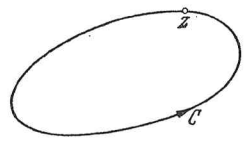

Fig. 2.50 (i)

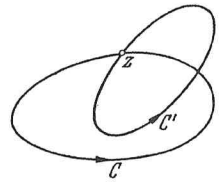

Fig. 2.50 (ii)

Proof. Suppose we describe C in its positive sense and C' in its negative sense. Then we can write

$$[f(z, \bar{z})]_{C-C'} = \int\limits_{C-C'} df(z, \bar{z}) = \int\limits_{C-C'}\left(\frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z}\right)$$
$$= 2\,i\int\limits_S\left\{\frac{\partial}{\partial \bar{z}}\,\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\,\frac{\partial f}{\partial \bar{z}}\right\}dS$$

by Stokes's complex theorem 3.11 (5)] taken over the area S "between" (suitably defined) C and C'. The last integrand vanishes by (4) and therefore

$$[f(z, \bar{z})]_C = [f(z, \bar{z})]_{C'}\,.$$

Q. E. D.

Thus the cyclic function of a given function is characteristic of a family of reconcilable circuits passing through a common point. The first three of the following theorems are direct consequences of the definition of the cyclic function.

Theorem (i). The cyclic function of the sum of two functions is equal to the sum of the cyclic functions of each separately.

Theorem (ii). If f and g are two functions such that $[g]_C = 0$, then $[fg]_C = g\,[f]_C$.

As an example of (ii) if C is a circle round the origin

$$[\bar{z}\ln z]_C = \bar{z}\,[\ln z]_C = 2\,\pi\,i\,\bar{z}\,.$$

Theorem (iii). If $f(z, \bar{z})$ is one-valued in D, then $[f(z, \bar{z})]_C = 0$ for any circuit C in D.

This is clearly the case for a function $F(z)$ holomorphic in D. When $f(z, \bar{z})$ is many-valued the cyclic function may vanish or not according to the circuit chosen. Thus for example, for a circle C, $[\bar{z} \ln z]_C = 2\pi i \bar{z}$ or 0 according as C does or does not enclose the origin.

Theorem (iv). The partial derivative of the cyclic function with respect to z (or $\bar{z}$) is equal to the cyclic function of the derivative.

Proof.

It follows from (2) that

$$\frac{\partial}{\partial z}[f(z, \bar{z})]_C = \frac{\partial f_2(z, \bar{z})}{\partial z} - \frac{\partial f_1(z, \bar{z})}{\partial z} = \left[\frac{\partial f(z, \bar{z})}{\partial z}\right]_C. \tag{5}$$

Q. E. D.

In the above proof we have assumed that given the determination of the function, the partial derivative is uniquely determined. For example if $f(z, \bar{z}) = z\bar{z} \ln z$, and we choose $f_1(z, \bar{z}) = z\bar{z} \ln_1 z$ where $\ln_1 z$ is that logarithm which reduces to zero when $z = 1$, then $\partial f_1(z, \bar{z})/\partial z = \bar{z}(1 + \ln_1 z)$ which is uniquely determined.

It follows from (5) that if $[\partial f/\partial z]_C = 0$, then $[f(z, \bar{z})]_C$ must be a function of $\bar{z}$ only, including zero or a constant as special cases.

If therefore f is independent of $\bar{z}$, we see that

$$\left[\frac{\partial f(z)}{\partial z}\right]_C = 0 \quad \text{implies} \quad [f(z)]_C = \alpha, \tag{6}$$

where α is a constant which may be zero.

Similarly we infer that

$$\left[\frac{\partial^2 f(z)}{\partial z^2}\right]_C = 0 \quad \text{implies} \quad [f(z)]_C = \alpha z + \beta \tag{7}$$

where α, β are constants.

Another way of looking at these results is

$$[F(z)]_C = 0 \quad \text{implies} \quad [`F(z)]_C = \alpha, \quad [``F(z)]_C = \alpha z + \beta. \tag{8}$$

2.60. Cyclic properties of the complex stresses

We proceed to make some deductions from the following hypotheses concerning the elastic systems which we shall consider.

(i) *In our elastic system the stresses are one-valued.*

(ii) *The body-potential V and the values of Q and its partial derivatives of the first and second orders with respect to z and $\bar{z}$ are one-valued.*

Since the stresses are one-valued so are the fundamental stress combinations Θ and Φ. Therefore for an arbitrary circuit C in the material

$$[\Theta]_C = 0, \quad [\Phi]_C = 0. \tag{1}$$

Also from (ii)

$$\left[\frac{\partial^2 Q}{\partial z\, \partial \bar{z}}\right]_C = 0, \quad \left[\frac{\partial^2 Q}{\partial z^2}\right]_C = 0. \tag{2}$$

Therefore from 2.50 theorems (i) and (iii) and 2.10 (3), (4)

$$[W(z) + \overline{W}(\bar{z})]_C = 0 \tag{3}$$

$$[\bar{z} W'(z) + w(z)]_C = 0. \tag{4}$$

Note that the circuit C is quite arbitrary; (3) and (4) hold independently of the connectivity of the region occupied by the material.

From (3), using 2.50 theorem (i), we see that

$$[W(z)]_C = -[\overline{W}(\bar{z})]_C = i\delta,$$

where δ is a real constant, for $[W(z)]_C$ is independent of $\bar{z}$, and its complex conjugate $[\overline{W}(\bar{z})]_C$ is independent of z. It is convenient to write $2\pi B$ instead of δ so that

$$[W(z)]_C = 2\pi i B, \quad B \text{ real}. \tag{5}$$

Here the constant B will, in general, depend upon the circuit C.

From 2.50 theorem (iv) it follows that

$$[W'(z)]_C = 0 \tag{6}$$

for every circuit C.

Again from (5) and 2.50 (8)

$$[`W(z)]_C = 2\pi i A + 2\pi i B z, \quad B \text{ real}, \tag{7}$$

where $2\pi i A$ denotes an arbitrary complex constant.

Thus (5), (6), (7) express the basic cyclic properties of $W(z)$. They are direct consequences of hypothesis (i) and (ii) above.

To find the cyclic properties of $w(z)$ we note that from (6) and 2.50 theorem (ii) $[\bar{z} W'(z)]_C = 0$ and so (4) and 2.50 theorem (i) give

$$[w(z)]_C = 0. \tag{8}$$

Therefore we have successively

$$[`w(z)] = 2\pi i F \tag{9}$$

$$[``w(z)] = 2\pi i E + 2\pi i F z \tag{10}$$

where E and F are arbitrary complex constants.

Thus (8), (9), (10) express the cyclic properties of $w(z)$.

From 2.12 (2) we obtain the cyclic function of the displacement. In fact

$$\begin{aligned}[4\mu(u + iv)]_C &= \varkappa [`W(z)]_C - [z\overline{W}(\bar{z})]_C - [`\overline{w}(\bar{z})]_C + \left[4\nu \frac{\partial Q}{\partial z}\right]_C \\ &= 2\pi i \{(\varkappa A + \overline{F}) + (\varkappa + 1) B z\}.\end{aligned} \tag{11}$$

Writing

$$2\pi i(\varkappa A + \bar{F}) = 4\mu\alpha\,, \quad 2\pi i(\varkappa + 1)B = 4\mu i\gamma\,,$$

we have

$$[u + iv]_C = \alpha + i\gamma z\,, \quad \gamma \text{ real.} \tag{12}$$

Thus, in general, the cyclic function of the displacement consists of a rigid body displacement, namely a translation α and a rotation through the (small) angle γ. This result means that, in contrast with the stresses, the displacement is, in general, not one-valued. The explanation is to be found in the concept of dislocations which will now be discussed.

2.61. Dislocations

Consider material which occupies a two-dimensional multiply-connected region. Draw a curve in the plane which regarded as an impassable barrier reduces the connectivity by one, give this curve a small rigid body displacement, and cut away the material between the two positions of the curve.

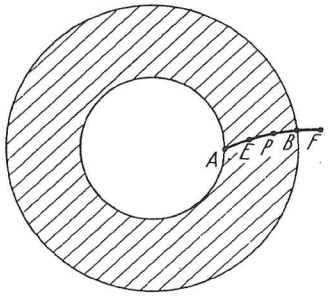

Fig. 2.61 (i)

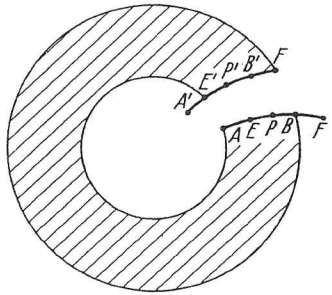

Fig. 2.61 (ii)

The idea is illustrated by an annulus in fig. 2.61 (i) where the curve $A\,E\,B\,F$ which crosses the inner and outer rims at the points A and B reduces the connectivity of the region occupied by the material from 2 to 1.

In fig. 2.61 (ii) the curve has been displaced, as if rigid, to occupy the position $A'E'B'F'$ where points with the same letters correspond. In the new position the curve crosses the inner and outer rims at E' and F'. The rigid displacement may be described* by a translation α and a rotation γ about the origin of coordinates, both small. If P is the point z, its correspondent P' will be the point $z' = z + \alpha + i\gamma z$ and therefore to bring about coincidence of P and P' we must give P the displacement $\alpha + i\gamma z$.

* If the rotation is about $z = c$, the same displacement could be written $\alpha + i\gamma c + i\gamma(z - c)$.

We now force the lips of the material in fig. 2.61 (ii) into coincidence so that A comes on A' and F on F' which is possible since each curve arises from the other by a rigid displacement. The lips so brought into coincidence are cemented together, the cement supplying the necessary forces to hold the material in place as shown in fig. 2.61 (iii) which must be regarded as enormously exaggerated.

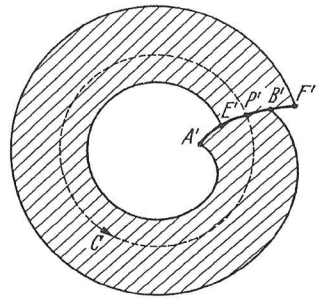

Fig. 261 (iii)

If we move counterclockwise along the curve C starting at and returning to P' [fig. 2.61 (iii)], the displacement will increase by $\alpha + i\gamma z$.

The type of deformation here described in terms of cutting and recementing is called *dislocation**. Deformation of this type may be described as dislocational.

If the material is originally unstressed, it will now be selfstressed. Observe that dislocation cannot occur in a simply connected region.

2.70. Analytic form of the complex stresses for finite regions

To fix our ideas suppose that the region L occupied by the material is bounded externally by a contour C and internally by non-intersecting simple contours C_1 and C_2, as for example in the case of a plate with two holes.

Let a_1 within C_1 and a_2 within C_2 be arbitrary fixed points. Let C_1' be a contour enclosing and reconcilable with C_1.

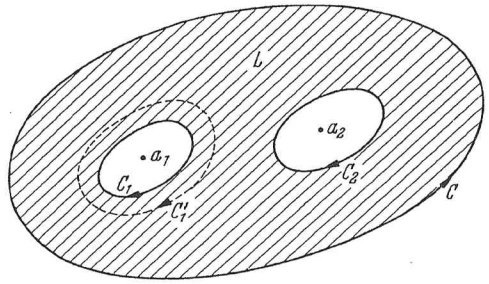

Fig. 2.70

Then from 2.60 (7) we have

$$[W(z)]_{C_1'} = 2\pi i(A_1 + B_1 z)\,, \quad B_1 \text{ real,} \tag{1}$$

where A_1 is a complex constant.

* The name is due to LOVE, the idea to WEINGARTEN. The detailed development was made by VOLTERRA.

Now from 2.50 (ii)

$$[(A_1 + B_1 z) \ln (z - a_1)]_{C_1'} = 2\pi i (A_1 + B_1 z) . \tag{2}$$

Therefore from 2.50 (i)

$$[W(z) - (A_1 + B_1 z) \ln (z - a_1)]_{C_1} = 0$$

from which we conclude that

$$W(z) = (A_1 + B_1 z) \ln (z - a_1) + h_1(z) , \tag{3}$$

where the function $h_1(z)$ satisfies the condition

$$[h_1(z)]_{C_1'} = 0 \tag{4}$$

for any contour reconcilable with C_1.

A similar argument applied to any contour C_2' enclosing, and reconcilable with, C_2 shows that

$$'W(z) = (A_2 + B_2 z) \ln (z - a_2) + h_2(z) , \tag{5}$$

where B_2 is real and where the function $h_2(z)$ satisfies the condition

$$[h_2(z)]_{C_2'} = 0 \tag{6}$$

for any contour C_2' reconcilable with C_2.

Since $[(A_2 + B_2 z) \ln (z - a_2)]_{C_1'} = 0$ for any contour C_1' reconcilable with C_1, and $[(A_1 + B_1 z) \ln (z - a_1)]_{C_2'} = 0$ for any contour reconcilable with C_2 it follows that

$$'W(z) = (A_1 + B_1 z) \ln (z - a_1) + (A_2 + B_2 z) \ln (z - a_2) + h(z) \tag{7}$$

where $h(z)$ is holomorphic in the region L, and B_1 and B_2 are real.

Differentiation of (7) leads to

$$W(z) = B_1 \ln (z - a_1) + B_2 \ln (z - a_2) + f(z) , \tag{8}$$

where $f(z)$ is holomorphic in the region L, and B_1 and B_2 are real.

It is now evident that starting from 2.60 (10) the same reasoning applied to $w(z)$ will lead to the analogue of (7) namely

$$''w(z) = (E_1 + F_1 z) \ln (z - a_1) + (E_2 + F_2 z) \ln (z - a_2) + k(z) , \tag{9}$$

where $k(z)$ is holomorphic in the region L and E_1, E_2, F_1, F_2 are complex constants.

Differentiation gives

$$'w(z) = F_1 \ln (z - a_1) + F_2 \ln (z - a_2) + g(z) \tag{10}$$

where $g(z)$ is holomorphic in the region L.

Further differentiation of (10) shows that $w(z)$ is holomorphic in L, the holes C_1, C_2 contributing no singularities at all at any point of the material.

The generalization of the foregoing argument to the case of n internal non-intersecting boundary contours C_s, $(s = 1, 2, \ldots, n)$ is trivial and leads, with obvious notation, to

$$'W(z) = \sum_{s=1}^{n} (A_s + B_s z) \ln (z - a_s) + h(z) , \tag{11}$$

$$W(z) = \sum_{s=1}^{n} B_s \ln (z - a_s) + f(z) , \tag{12}$$

$$''w(z) = \sum_{s=1}^{n} (E_s + F_s z) \ln (z - a_s) + k(z) , \tag{13}$$

$$'w(z) = \sum_{s=1}^{n} F_s \ln (z - a_s) + g(z) , \tag{14}$$

where the $B_s (s = 1, 2, \ldots, n)$ are real, and the functions $h(z), f(z), k(z), g(z)$ are all holomorphic in the region L occupied by the material. Moreover differentiation of (14) shows that $w(z)$ is holomorphic throughout L.

2.71. Non-dislocational solutions

We consider the same situation as in 2.70, material occupying a region L, defined by an exterior boundary C and n internal non-intersecting boundaries C_s $(s = 1, 2, \ldots, n)$.

For a circuit which embraces none of the contours C_s it is clear from 2.12 (2) and 2.70 (11) (12) and (14) that the cyclic function of the displacement $u + iv$ is zero. Now consider the circuit C_1. We have from the above mentioned formulae

$$\begin{aligned}[4\mu (u + iv)]_{C_1} &= 2\pi i \{\varkappa (A_1 + B_1 z) + z B_1 + \overline{F}_1\} \\ &= 2\pi i \{\varkappa A_1 + \overline{F}_1 + B_1 (\varkappa + 1) z\}\end{aligned} \tag{1}$$

and therefore the displacement field is, in general, dislocational, with dislocation (in respect of C_1)

$$D_1 = \alpha_1 + i\gamma_1 z , \quad \alpha_1 = \frac{\pi i}{2\mu} (\varkappa A_1 + \overline{F}_1) , \quad \gamma_1 = \frac{\pi (\varkappa + 1)}{2\mu} B_1 . \tag{2}$$

The condition that the circuit C_1 should yield no dislocation is therefore

$$B_1 = 0 , \quad \varkappa A_1 + \overline{F}_1 = 0 . \tag{3}$$

If the situation is to be entirely non-dislocational, a similar condition must hold for each of the other boundaries, so that

$$B_s = 0 , \quad \varkappa A_s + \overline{F}_s = 0 , \qquad (s = 1, 2, \ldots, n) \tag{4}$$

and in this case we have from 2.70 (11) and (14)

$$'W(z) = \sum_{s=1}^{n} A_s \ln(z - a_s) + h(z) \tag{5}$$

$$''w(z) = -\sum_{s=1}^{n} \varkappa \bar{A}_s \ln(z - a_s) + g(z)\,, \tag{6}$$

where $h(z)$ and $g(z)$ are holomorphic in L.

2.72. Stress resultants on a contour

A simple closed contour C will separate the plane into two regions each inaccessible from the other without crossing C. We call these regions the left-hand region L and the right-hand region R, the terms being used with respect to an observer who moves along C in a prearranged sense.

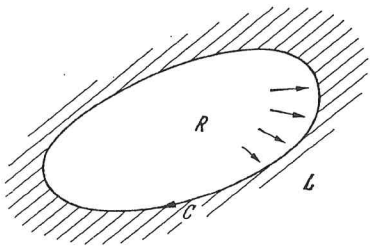

Fig. 2.72

Let us consider the case of a plate with a hole whose boundary is C.

Suppose that external forces are applied to the internal boundary of the hole and let their resultant be (X, Y), M referred to the origin of coordinates as base-point.

The resultant, which must balance the resultant of the stress forces exerted by the material on the right of C, can be found directly from 2.40 by replacing the arc AB by the closed contour C. Before doing so we make the following observations with regard to 2.40 (1) and (2). On account of 2.60 hypothesis (ii) the second terms on the right-hand side vanish when AB is replaced by the closed contour C.

$[\bar{z}W(z)]_C$ can be replaced by $[-\bar{z}\bar{W}(\bar{z})]_C$ on account of 2.60 (3). From 2.60 (5), $[z\bar{z}W(z)]_C$ is imaginary and therefore contributes nothing to the moment.

Therefore we can write

$$(X - iY) - (X_B - iY_B) = \frac{1}{2} i\,[\,'\bar{W}(\bar{z}) - \bar{z}\,\bar{W}(\bar{z}) + ''w(z)]_C \tag{1}$$

$$X_B - iY_B = i \oint_C \frac{4\,\partial^2 Q}{\partial z\,\partial \bar{z}}\, d\bar{z} \tag{2}$$

$$M - M_B = \mathrm{Re}\,\frac{1}{2}\,[\,''w(z) - z\,'w(z)]_C \tag{3}$$

$$M_B = -\,\mathrm{Re} \oint_C z\,\frac{4\,\partial^2 Q}{\partial z\,\partial \bar{z}}\, d\bar{z}\,. \tag{4}$$

Since by definition $4\,\partial^2 Q/\partial z\,\partial \bar{z} = V$ we have, using the Area theorem,

$$X_B - iY_B = -i \oint_C V\, d\bar{z} = -2 \int_S \frac{\partial V}{\partial z}\, dS = -\int_S (b_1 - ib_2)\, dS\,, \tag{5}$$

where S is the area within C.

$$M_B = \mathrm{Re} \oint_C zV\, d\bar{z} = -\,\mathrm{Re}\, 2i \int_S \frac{\partial}{\partial z}(zV)\, dS$$

$$= -\,\mathrm{Re}\, 2i \int_S z\,\frac{\partial V}{\partial z}\, dS \text{ since } V \text{ is real.}$$

Therefore

$$M_B = -\mathrm{Re} \int_S iz\,(b_1 - ib_2)\, dS\,. \tag{6}$$

Thus (X_B, Y_B), M_B is the resultant of the reversed body-vector field imagined to be acting over the area S within C, that is to say the resultant of the reversed mass acceleration and the body-force. For example if there is no acceleration and the body force is due to gravity only, (X_B, Y_B) is a force gS acting vertically down through the centroid of S and M_B is the moment of this force about the origin.

2.725. New expressions for the stress resultants

Let $f(z)$ be a function of z whose derivative $f'(z)$ is holomorphic within C except at isolated singularities at which a residue exists.

We define

$$(f(z))_{RC} = \text{sum of the residues of } f(z) \text{ at its singularities within } C. \tag{1}$$

Now for the cyclic function of $f(z)$ we have

$$[f(z)]_C = \int_C f'(z)\, dz = 2\pi i\,(f'(z))_{RC} \tag{2}$$

and therefore

$$[\bar{f}(\bar{z})]_C = -2\pi i\,\overline{(f'(z))_{RC}}\,, \tag{3}$$

where the bar as usual denotes the complex conjugate.

Note that if $f(z) = 'W(z) - zW(z)$, then $f'(z) = -zW'(z)$ and if $f(z) = ''w(z) - z\,'w(z)$, $f'(z) = -zw(z)$ and from 2.70 we see that these functions are holomorphic except at isolated singularities at which the function has a residue.

Therefore finally 2.72 (1) and (3) can be written in the form

$$(X - iY) - (X_B - iY_B) = \pi\,\{\overline{(zW'(z))_{RC}} + (w(z))_{RC}\} \tag{4}$$

$$M - M_B = \mathrm{Re}\,\pi i\,(zw(z))_{RC} \tag{5}$$

(see also Ex. II, 9) where the notation is defined by (2) and (X_B, Y_B), M_B are given by 2.72 (2), (4).

We note in view of (4), (5) and 2.10 (3), (4), that the values of the complex stresses $W(z)$, $w(z)$ alone suffice to calculate the stress distribution and the force across a contour. To calculate displacement and force across an open arc integrals of the complex stresses are needed.

2.73. Relation of the stress resultants and dislocations to the complex stresses

Consider first the case in which the material occupies the annular region L bounded externally by the contour C and internally by the contour C_1, as in the case of a plate with a hole, fig. 2.73.

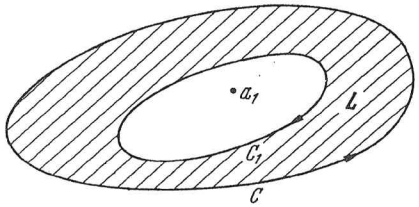

Fig. 2.73

Take a point a_1 within C_1. From 2.70 (11) and (13) we can write

$$'W(z) = (A_1 + B_1 z) \ln (z - a_1) + h(z), \quad B_1 \text{ real} \tag{1}$$

$$''w(z) = (E_1 + F_1 z) \ln (z - a_1) + k(z), \tag{2}$$

where $h(z)$ and $k(z)$ are holomorphic in L.

We propose to relate the constants A_1, B_1, E_1, F_1 to the external force applied to the boundary C_1 and to the dislocation

$$D_1 = \alpha_1 + i\gamma_1 z, \quad \gamma_1 \text{ real.} \tag{3}$$

Suppose that external forces are applied to the inner boundary C_1 as in fig. 2.72 and that reduced to the origin as base-point the resultant of these applied external forces is (X_1, Y_1), M_1. Let

$$\overline{P}_1 = X_1 - iY_1 - (X_B - iY_B)_{C_1}, \tag{4}$$

$$M^{(1)} = M_1 - (M_B)_{C_1}. \tag{5}$$

Here P_1 is the resultant of the force external to the boundary C_1 and reversed mass-acceleration and the body-force acting over the whole area within C_1, while $M^{(1)}$ is the moment of this system about the origin. Suffix C_1 in (4) and (5) denotes the body-vector contribution.

Then 2.725 (4), (5) give

$$\overline{P}_1 = \pi \{\overline{(zW'(z))_{RC_1}} + (w(z))_{RC_1}\}, \tag{6}$$

$$M^{(1)} - iN^{(1)} = \pi i (zw(z))_{RC_1}, \tag{7}$$

where $N^{(1)}$ is a real quantity whose value does not affect the force and moment.

Differentiate (1) und (2) and substitute in (6), (7). Then we find that

$$\overline{P}_1 = \pi(-\overline{A}_1 + F_1), \quad M^{(1)} - iN^{(1)} = -\pi i E_1 \tag{8}$$

Also from 2.71 (1)

$$4\mu D_1 = [4\mu(u + iv)]_{C_1} = 2\pi i \{\varkappa A_1 + \overline{F}_1 + B_1 z(\varkappa + 1)\} \tag{9}$$

so that using (3)

$$\pi(\varkappa A_1 + \overline{F}_1) = -2i\mu\alpha_1, \quad \pi B_1(\varkappa + 1) = 2\mu\gamma_1, \tag{10}$$

and therefore

$$\left.\begin{aligned} A_1 &= \frac{-P_1 - 2\mu\alpha_1 i}{\pi(\varkappa + 1)}, & F_1 &= \frac{\varkappa\overline{P}_1 + 2\mu\overline{\alpha}_1 i}{\pi(\varkappa + 1)}, \\ B_1 &= \frac{2\mu\gamma_1}{\pi(\varkappa + 1)}, & E_1 &= \frac{iM^{(1)} + N^{(1)}}{\pi}. \end{aligned}\right\} \tag{11}$$

When the plate has n-holes bounded by non-intersecting contours C_s, $(s = 1, 2, \ldots, n)$ we have at once for C_s

$$\left.\begin{aligned} A_s &= -\frac{P_s + 2\mu i\alpha_s}{\pi(\varkappa + 1)}, & F_s &= -\frac{\varkappa\overline{P}_s + 2\mu i\overline{\alpha}_s}{\pi(\varkappa + 1)}, \\ B_s &= \frac{2\mu\gamma_s}{\pi(\varkappa + 1)}, & E_s &= \frac{iM^{(s)} + N^{(s)}}{\pi}. \end{aligned}\right\} \tag{12}$$

Noting that on the model of (3) $-2\mu i\alpha_s + 2\mu\gamma_s z = -2\mu i D_s$, we can write instead of (1) and (2)

$$'W(z) = -\sum_{s=1}^{n} \frac{P_s + 2\mu i D_s}{\pi(\varkappa + 1)} \ln(z - a_s) + h(z), \tag{13}$$

$$''w(z) = \sum_{s=1}^{n} \frac{(\varkappa + 1)(iM^{(s)} + N^{(s)}) + (\varkappa\overline{P}_s + 2\mu i\overline{\alpha}_s) z}{\pi(\varkappa + 1)} \ln(z - a_s) + k(z). \tag{14}$$

Now let P_0 denote the resultant of external forces applied to the outer boundary C and the body force and reversed mass acceleration acting over the whole area enclosed by C. We then find from 2.725 (4) and (5), by considering the circuit round the outer boundary C, that

$$\overline{P}_0 + \sum_{s=1}^{n} \overline{P}_s = 0$$

$$M^{(0)} + \sum_{s=1}^{n} M^{(s)} = 0$$

in other words in order to maintain the state which has just been investigated an appropriate external force (which may be zero) must be applied at the outer boundary.

In the absence of acceleration, the external forces applied to the outer and inner must be in equilibrium with the resultant body-force acting on the material.

2.74. Analytic form of the complex stresses for infinite regions

We now consider the case where the region L occupied by the material extends to infinity in all directions. By this we mean that outside a circle K_r of radius r, described about the origin, there are always points of L on the continuation of every radius, however large r may be.

Referring to the notation of 2.70 let r be chosen so large that all the contours C_s and therefore all the points a_s, $(s = 1, 2, \ldots, n)$ lie inside the circle K_r.

Then for any point z outside K_r

$$\ln(z - a_s) = \ln z + \ln\left(1 - \frac{a_s}{z}\right)$$

and

$$\ln\left(1 - \frac{a_s}{z}\right) = -\frac{a_s}{z} - \frac{1}{2}\left(\frac{a_s}{z}\right)^2 - \frac{1}{3}\left(\frac{a_s}{z}\right)^3 - \cdots$$

is holomorphic outside K_r. It follows that $(A_s + B_s z)\ln\left(1 - \frac{a_s}{z}\right)$ is also holomorphic outside K_r and therefore from 2.70 (11), (13) if we put

$$A = \Sigma A_s\,, \quad B = \Sigma B_s\,, \quad E = \Sigma E_s\,, \quad F = \Sigma F_s\,, \tag{1}$$

we can write

$$W(z) = (A + Bz)\ln z + h(z)\,, \quad B \text{ real}, \tag{2}$$

$$w(z) = (E + Fz)\ln z + k(z)\,, \tag{3}$$

where $h(z)$ and $k(z)$ are certainly holomorphic when z is in that part of L which is comprised between the circle K_r of radius r and the concentric circle $K_{r'}$, where $r' > r$, for this is a finite region and so the considerations of 2.70 apply to it.

It follows that in this region between K_r and $K_{r'}$ we have Laurent expansions in positive and negative powers of z.

$$h(z) = \sum_{m=-\infty}^{\infty} \alpha_m^{(1)} z^m\,, \quad k(z) = \sum_{m=-\infty}^{\infty} \beta_m^{(1)} z^m\,. \tag{4}$$

From 2.10 we have

$$\begin{aligned} \widehat{xx} + \widehat{yy} &= 2B + B\ln z\bar{z} + \frac{A}{z} + \frac{\bar{A}}{\bar{z}} \\ &\quad + h'(z) + \bar{h}'(\bar{z}) + 4(2-\nu)\frac{\partial^2 Q}{\partial z\,\partial\bar{z}} \end{aligned} \tag{5}$$

$$\begin{aligned} \widehat{yy} - \widehat{xx} + 2i\,\widehat{xy} &= \bar{z}\left\{\frac{B}{z} - \frac{A}{z^2} + h''(z)\right\} + \frac{F}{z} - \frac{E}{z^2} \\ &\quad + k''(z) - 4\nu\frac{\partial^2 Q}{\partial z^2}\,. \end{aligned} \tag{6}$$

It is now evident that on account of the term $\ln z\bar{z}$ the stresses at infinity will in general be infinite. Since infinite stress would entail rupture of the material, we shall exclude infinite stress once for all by introducing the restriction that

All stresses are bounded at infinity. (7)

An immediate consequence is that

$$\Sigma B_s = B = 0\,, \tag{8}$$

which, while it does not exclude dislocations, means that the sum of the angles of rotation due to dislocations must vanish. In particular for an infinite plate with one hole rotational dislocation is impossible.

Apply (7) to (5) and (6) and we see further that the expressions S and T where

$$\left.\begin{aligned} S &= h'(z) + \bar{h}'(\bar{z}) + 4(2-\nu)\frac{\partial^2 Q}{\partial z\,\partial\bar{z}}\,, \\ T &= \bar{z}h''(z) + k''(z) - 4\nu\frac{\partial^2 Q}{\partial z^2} \end{aligned}\right\} \tag{9}$$

must be bounded at infinity.

Let us examine the implication of this in the case of gravity and steady rotation. From 2.11 we have

$$\frac{4\,\partial^2 Q}{\partial z\,\partial\bar{z}} = -\frac{1}{2}n^2\varrho\,(z-a)\,(\bar{z}-\bar{a}) + \frac{1}{2}g\,\varrho e^{-i\alpha}z + \frac{1}{2}g\,\varrho e^{i\alpha}\bar{z}$$

and if S is to be bounded at infinity, it is evident that we must have

$$n = 0\,, \tag{10}$$

since no term in S can cancel the term $n^2\varrho\, z\bar{z}$.

Thus (9) becomes

$$S = h'(z) + z\left(1 - \frac{1}{2}\nu\right)g\varrho e^{-i\alpha} + \bar{h}'(\bar{z}) + \bar{z}\left(1 - \frac{1}{2}\nu\right)g\varrho e^{i\alpha}$$

$$T = \bar{z}\left(h''(z) - \frac{1}{2}\nu g\varrho e^{-i\alpha}\right) + k''(z)$$

and therefore

$$h'(z) + z\left(1 - \frac{1}{2}\nu\right)g\varrho e^{-i\alpha}, \quad \bar{z}\left(h''(z) - \frac{1}{2}\nu g\varrho e^{-i\alpha}\right), \; k''(z) \tag{11}$$

must be separately bounded at infinity. Using the Laurent series (4) the first of (11) shows that

$\alpha_m^{(1)} = 0$, $m > 2$, and $\alpha_2^{(1)} + \left(1 - \frac{1}{2}\nu\right)g\varrho e^{-i\alpha} = 0$, while the second of (11) shows that $\alpha_2^{(1} - \frac{1}{2}\nu g\varrho e^{-i\alpha} = 0$, so that we must have $\alpha_2^{(1)} = 0$ and

$$g = 0\,. \tag{12}$$

Thus in the neighborhood of infinity $h(z)$ has an expansion of the form

$$h(z) = \alpha_1^{(1)}z + \alpha_0^{(1)} + \frac{\alpha_{-1}^{(1)}}{z} + \frac{\alpha_{-2}^{(1)}}{z^2} + \frac{\alpha_{-3}^{(1)}}{z^3} + \cdots \tag{13}$$

Also from (11) since $k''(z)$ is bounded in the same neighborhood we have

$$k(z) = \beta_2^{(1)} z^2 + \beta_1^{(1)} z + \beta_0^{(1)} + \frac{\beta_{-1}^{(1)}}{z} + \frac{\beta_{-2}^{(1)}}{z^2} + \frac{\beta_{-3}^{(1)}}{z^3} + \cdots \tag{14}$$

From this investigation it appears that when the material extends to infinity the stresses there cannot be bounded in the presence of steady rotation or of gravity acting in the plane of the material.

Thus in dealing with infinite regions we shall suppose that (7) is satisfied and there is no body-vector field so that the stresses and displacement are given by

$$\Theta = \widehat{xx} + \widehat{yy} = W(z) + \overline{W}(\bar{z}), \tag{15}$$

$$\Phi = \widehat{yy} - \widehat{xx} + 2i\,\widehat{xy} = \bar{z} W'(z) + w(z), \tag{16}$$

$$4\mu\,(u + iv) = \varkappa\, W(z) - z\overline{W}(\bar{z}) - \overline{w}(\bar{z}), \tag{17}$$

where in (17) we have omitted an arbitrary rigid body displacement. Combining (2) and (3) with (13) and (14), we have, for sufficiently large values of $|z|$, the expressions

$$W(z) = A \ln z + \alpha_1^{(1)} z + \alpha_0^{(1)} + \frac{\alpha_{-1}^{(1)}}{z} + \frac{\alpha_{-2}^{(1)}}{z^2} + \cdots$$

$$w(z) = F\,(1 + \ln z) + \frac{E}{z} + 2\beta_2^{(1)} z + \beta_1^{(1)} - \frac{\beta_{-1}^{(1)}}{z^2} - \frac{2\,\beta_{-2}^{(1)}}{z^3} - \cdots$$

From these by differentiation and re-naming the coefficients we get the definitive forms for the complex stresses in the neighborhood of infinity,

$$W(z) = \alpha_0 + \frac{A}{z} + \frac{\alpha_2}{z^2} + \frac{\alpha_3}{z^3} + \cdots, \tag{18}$$

$$w(z) = \beta_0 + \frac{F}{z} - \frac{E}{z^2} + \frac{\beta_3}{z^3} + \cdots, \tag{19}$$

We now proceed to interpret the constants α_0, β_0, A, E, F.

Let (X, Y), M be the resultant, referred to the origin as basepoint, of the external forces applied to the boundaries C_s and let α be the sum of the translational dislocations, so that in the notation of 2.73

$$(X + iY) = \sum_{s=1}^{n} (X_s + iY_s), \quad M = \sum_{s=1}^{n} M^{(s)}, \quad \alpha = \sum_{s=1}^{n} \alpha_s. \tag{20}$$

Then from 2.73 (12) we have

$$\left.\begin{aligned} A &= -\frac{X + iY + 2\mu i \alpha}{\pi(\varkappa + 1)}, \quad E = \frac{iM + N'}{\pi} \\ F &= \frac{\varkappa(X - iY) + 2\mu i \bar{\alpha}}{\pi(\varkappa + 1)} \end{aligned}\right\} \tag{21}$$

where N' is a real constant.

Again if ε_∞ denotes the rotation at infinity, we have from 2.14

$$\varepsilon_\infty = -\frac{1}{8\mu} i\,(\varkappa + 1)\,(W_\infty - \overline{W}_\infty) = -\frac{1}{8\mu} i\,(\varkappa + 1)\,(\alpha_0 - \bar{\alpha}_0),$$

where we denote by suffix ∞ the value at infinity, and where we have used (18). Thus we have

$$\alpha_0 = \lambda_0 + \frac{4\mu i \varepsilon_\infty}{\varkappa + 1}, \quad \lambda_0 \text{ real}. \tag{22}$$

Further let the state of stress at infinity be described by principal stresses T_∞ and N_∞ of which the direction of T_∞ makes the angle φ with the real axis. Then at infinity the fundamental stress combinations are

$$\Theta_\infty = N_\infty + T_\infty = W_\infty + \overline{W}_\infty = \alpha_0 + \bar{\alpha}_0 \tag{23}$$

$$\Phi_\infty = (N_\infty - T_\infty)\, e^{-2i\varphi} = (\bar{z} W'(z))_\infty + w_\infty = \beta_0. \tag{24}$$

From (22) and (23) we find that

$$\alpha_0 = \frac{1}{2}(N_\infty + T_\infty) + \frac{4\mu i \varepsilon_\infty}{\varkappa + 1}. \tag{25}$$

Thus we finally have for $W(z)$, $w(z)$ the expressions

$$W(z) = \frac{1}{2}(N_\infty + T_\infty) + \frac{4\mu i \varepsilon_\infty}{\varkappa + 1} - \frac{X + iY + 2\mu i \alpha}{\pi(\varkappa + 1)} \cdot \frac{1}{z} + \frac{\alpha_2}{z^2} + \cdots, \tag{26}$$

$$\begin{aligned} w(z) = (N_\infty - T_\infty)\, e^{-2i\varphi} &+ \frac{\varkappa(X - iY) + 2\mu i \bar{\alpha}}{\pi(\varkappa + 1)} \cdot \frac{1}{z} \\ &- \frac{iM + N'}{\pi} \cdot \frac{1}{z^2} + \cdots, \end{aligned} \tag{27}$$

which are valid outside the circle K_r, that is to say in the neighbourhood of infinity.

2.80. Behaviour of the displacement in the neighbourhood of infinity

From 2.12 and 2.74 (18), (19) we have for the displacement outside the circle K_r, that is to say in the neighbourhood of infinity

$$4\mu\,(u + iv) = (\varkappa\alpha_0 - \bar{\alpha}_0)\, z - \bar{\beta}_0 \bar{z} + \varkappa A \ln z - \overline{F} \ln \bar{z} - \overline{A}\,\frac{z}{\bar{z}} - \cdots, \tag{1}$$

where the dots represent terms which tend to zero when $|z| \to \infty$, and where an arbitrary rigid body displacement has been omitted. The interpretation of the constants in (1) has been given in 2.74.

It is at once evident that the displacement at infinity will be bounded only if, simultaneously,

$$\varkappa\alpha_0 - \bar{\alpha}_0 = 0, \quad \beta_0 = 0, \quad \varkappa A = 0, \quad F = 0. \tag{2}$$

From 2.74 (21) the conditions $\varkappa A = 0$, $F = 0$ give

$$X + iY + 2\mu i\alpha = 0\,, \quad \varkappa(X + iY) - 2\mu i\alpha = 0$$

and therefore

$$X + iY = 0\,, \quad \alpha = 0\,. \tag{3}$$

The condition $X + iY = 0$ means that the vector sum of the external forces applied to the internal boundaries C_s, $s = 1, 2, \ldots, n$, must vanish. This does not preclude the existence of couples applied to these boundaries.

The condition $\alpha = 0$ means that the sum of the translational dislocations must vanish. (The sum of the angles of rotational dislocations has already been shown to vanish 2.74 (8).)

The conditions $\beta_0 = 0$, $\varkappa\alpha_0 - \bar{\alpha}_0 = 0$ lead to

$$N_\infty - T_\infty = 0\,, \quad \frac{1}{2}(\varkappa - 1)(N_\infty + T_\infty) + 4\mu i\varepsilon_\infty = 0$$

from which it follows that

$$N_\infty = 0\,, \quad T_\infty = 0\,, \quad \varepsilon_\infty = 0 \tag{4}$$

the stress and rotation at infinity must be zero.

Thus if the displacement is to be bounded at infinity, we must have, outside K_r,

$$W(z) = \frac{\alpha_2}{z^2} + \frac{\alpha_3}{z^3} + \cdots, \tag{5}$$

$$w(z) = -\frac{iM + N'}{\pi} \cdot \frac{1}{z^2} + \frac{\beta_3}{z^3} + \cdots \tag{6}$$

The leading term of $w(z)$ puts in evidence the resultant of the external couples which are applied to the internal boundaries

$$C_s, \ (s = 1, 2, \ldots, n)\,.$$

Examples II

1. Obtain the effective fundamental stress combinations Θ_e, Φ_e in terms of Θ and Φ when the body vector field is due to gravity and steady rotation.

2. Discuss the expression of the displacement in terms of the effective stress function in the case cited in Ex. 1.

3. If $\widehat{rr}, \widehat{r\theta}, \widehat{\theta\theta}$ are the stress components in plane polar coordinates, prove that

$$2(\widehat{rr} - i\widehat{r\theta}) = W(z) - zW'(z) + \overline{W}(\bar{z}) - \frac{z}{\bar{z}}w(z) + \frac{4V}{\varkappa + 1} + \frac{8(\varkappa - 1)}{(\varkappa + 1)}\frac{z}{\bar{z}}\frac{\partial^2 Q}{\partial z^2}\,,$$

and find an expression for the hoop stress $\widehat{\theta\theta}$.

4. The boundary of an elastic body is acted on by normal pressure p and shear t at the element ds. Obtain the boundary condition in the form

$$\frac{d}{ds}\left\{\overline{W}(\bar{z}) + \bar{z}W(z) + w(z) - \frac{8(\varkappa - 1)}{\varkappa + 1}\frac{\partial Q}{\partial z}\right\} = -2(p + it + V)\frac{d\bar{z}}{ds}$$

and find the condition at an unstressed boundary in the absence of a body-vector field.

5. Determine the cyclic functions of

$$z\ln z\,, \quad \bar{z}\ln z\,, \quad z\ln\bar{z}\,, \quad z\ln z + \bar{z}\ln\bar{z}$$

with respect to a circuit C which encloses the origin.

6. Elastic material is in the form of an annulus. Discuss the terms in the complex stresses $W(z)$, $w(z)$ which will lead to (i) a translational dislocation t (ii) a rotational dislocation ε, and illustrate by a diagram.

7. Verify that the complex stresses

$$W(z) = -\frac{F}{\pi(\varkappa + 1)}\frac{1}{z - z_0}\,,$$

$$w(z) = \frac{\varkappa F}{\pi(\varkappa + 1)}\frac{1}{z - z_0} - \frac{\bar{z}_0 F}{\pi(\varkappa + 1)}\frac{1}{(z - z_0)^2}\,,$$

where $F = X + iY$ yield a concentrated force F acting at the point z_0, and find the moment about the origin.

8. Prove that the situation in Ex. 7 is non-dislocational.

9. Use the fact, that when the situation is non-dislocational the cyclic function of $W(z)$ is zero to obtain the formula

$$(X - iY) - (X_B - iY_B) = -\pi\{\overline{(W(z))_{RC}} - (w(z))_{RC}\}$$

alternative to 2.725 (4) in this case.

10. Prove that in the absence of a body-vector field, the complex stresses $W(z)$, $w(z)$ determined when given forces are applied to the boundary are independent of the elastic constants of the material composing the body, which is, however, assumed to be homogeneous and isotropic.

11. Prove that the complex stresses $W(z)$, $w(z)$ determined when the displacement is given at every point of the boundary, are, in general, dependent upon the elastic constants of the material.

12. An infinite plate is in a state of all-round tension T and is otherwise unstressed. Prove that

$$W(z) = T\,, \quad w(z) = 0\,.$$

13. An infinite plate is in a state of simple tension T at an angle φ to the real axis and is otherwise unstressed. Prove that

$$W(z) = \frac{1}{2}T\,, \quad w(z) = -Te^{-2i\varphi}\,.$$

14. Prove that a state of all round tension T in an infinite plate can be obtained by superposition of two states due to simple tension T in perpendicular directions.

15. Prove that the stress distribution due to either simple or all round tension is unaffected by supposing a non-zero rotation at infinity to be operative.

Chapter III

Mathematical preliminaries to the boundary value problems

In this chapter we summarize the mathematical tools required to tackle the four fundamental problems stated at the end.

3.10. Convention regarding sense of description of a contour

When the region occupied by the material is bounded by one or more non-intersecting simple contours, we shall agree that the positive sense of description of each contour shall be that which leaves the material on the left.

Thus in the case of a plate bounded externally by the contour C_0 and internally by contours C_1 and C_2 [fig. 3.10 (i)], the positive sense of description is counterclockwise for C_0 and clockwise for C_1 and C_2.

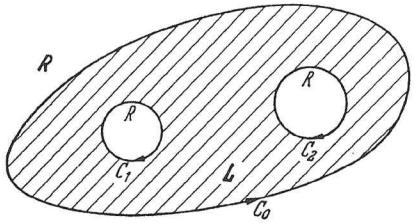

Fig. 3.10 (i)

We shall designate by L the left-hand region occupied by the material and by R the right-hand region free of the material.

Note that in the above example the region L is connected but that the region R is disconnected.

Should the contour C_0 lie entirely at infinity the positive sense of description of the accessible boundaries C_1 and C_2 is clockwise.

More generally we shall designate by L the region on the left of an observer who describes C in a prescribed sense.

If the closed contour C is replaced by an open arc BD described in the sense B to D, the region L can often be conveniently defined by considering the arc as prolonged indefinitely by drawing tangents to it at B and D fig. 3.10 (ii), or by joining B to D by an arc to form a closed curve figs. 3.10 (iii), (iv).

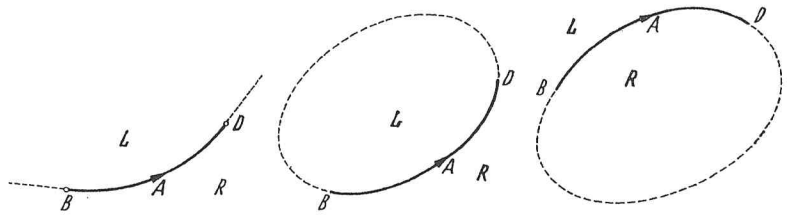

Fig. 3.10 (ii) Fig. 3.10 (iii) Fig. 3.10 (iv)

3.11. The Area theorem

Stokes's theorem for a curve C closed by a diaphragm S can be enunciated in the form [MILNE-THOMSON (5)]

$$\oint_C d\boldsymbol{C} \circ X = \int_S \left(d\boldsymbol{S}_\wedge \frac{\partial}{\partial \boldsymbol{P}} \right) \circ X , \tag{1}$$

where X is any continuous function of position, $d\boldsymbol{C}$, $d\boldsymbol{S}$ are properly directed elements of arc of C and surface of S and the small circle represents multiplication which may be scalar, vector or dyadic. The symbol $\wedge$ denotes vector multiplication. The situation is shown in fig. 3.11.

Now let C be a simple closed curve in the xy-plane, and let the diaphragm S be the portion of the xy-plane enclosed by C. Then if $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are orthogonal unit vectors along the axes and X is a scalar function $F(x, y)$, Stokes's theorem becomes

$$\oint_C (\boldsymbol{i}\, dx + \boldsymbol{j}\, dy)\, F(x, y) =$$

$$= \int_S \boldsymbol{k}_\wedge \left(\boldsymbol{i} \frac{\partial F}{\partial x} + \boldsymbol{j} \frac{\partial F}{\partial y} \right) dS . \tag{2}$$

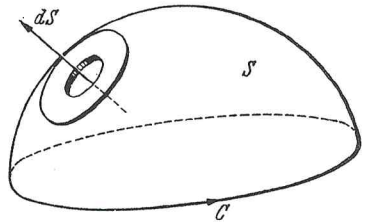

Fig. 3.11

Observing that $\boldsymbol{j} = \boldsymbol{k}_\wedge \boldsymbol{i}$ and that the operational equivalence

$$\boldsymbol{k}_\wedge = i \tag{3}$$

yields the complex variable, (2) becomes

$$\oint_C (dx + i\,dy)\,F(x, y) = i \int_S \left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right) dS = 2i \int_S \frac{\partial F}{\partial \bar z}\, dS\,, \quad (4)$$

where $z = x + iy$, $\bar z = x - iy$.

Also $F(x, y) = F\left(\frac{1}{2}(z + \bar z), -\frac{1}{2} i\,(z - \bar z)\right) = f(z, \bar z)$ say. Therefore finally we obtain the theorem which we shall designate as the Area theorem

$$\oint_C f(z, \bar z)\, dz = 2i \int_S \frac{\partial f}{\partial \bar z}\, dS\,. \quad (5)$$

We observe in passing that if in (4) we write $F = P - iQ$, we get the two well-known theorems

$$\begin{aligned} \int_C P\,dx + Q\,dy &= \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dS\,, \\ \int_C P\,dy - Q\,dx &= \int_S \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dS\,. \end{aligned} \quad (6)$$

Corollary (i). Let f be a function of z holomorphic inside and upon the contour C. Then $\partial f/\partial \bar z = 0$ and

$$\oint_C f(z)\, dz = 0\,. \quad (7)$$

This is Cauchy's theorem.

Corollary (ii). Let (7) hold for every contour C within a region D. Then for an arbitrary area S within this region

$$2i \int_S \frac{\partial f}{\partial \bar z}\, dS = 0\,.$$

Therefore $\partial f/\partial \bar z = 0$ and the function f is a holomorphic function of z in R. This is Morera's theorem.

Although the demonstrations in these corollaries are of a frankly heuristic character, they are nevertheless enlightening.

3.12. Cauchy's integral formula

Consider a function $f(t)$ of the complex variable t which is holomorphic inside and upon the simple closed contour C. Then by Cauchy's theorem

$$\oint_C f(t)\, dt = 0\,. \quad (1)$$

This theorem remains true if from the interior L of C we omit one point z (or a finite number of such points), provided that

$$\lim_{t \to z} (t - z)\, f(t) = 0\,, \quad (2)$$

when t tends to z by any path lying entirely inside C. For given a positive number ε we can choose $r = |t - z|$ so small that $|f(t)| < \varepsilon/r$, and so the modulus of the integral of $f(t)$ taken round the circle $|t - z| = r$ is less than $2\pi\,\varepsilon$.

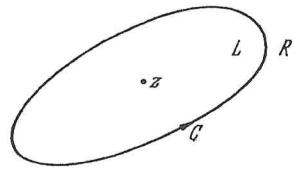

Fig. 3.12

Now consider the function

$$f(t) = \frac{F(t) - F(z)}{t - z}\,, \quad (3)$$

where $F(t)$ is holomorphic inside and upon C. The function $f(t)$ satisfies (2) and is holomorphic everywhere within C except at z, where it is undefined. Thus from (1)

$$\oint_C \frac{F(t) - F(z)}{t - z}\, dt = 0\,,$$

or since

$$\oint_C dt/(t - z) = 2\pi\, i$$

or zero according as z is inside or outside C.

$$\frac{1}{2\pi i} \oint_C \frac{F(t)\, dt}{t - z} = F(z) \text{ or } 0 \quad (4)$$

according as z is inside or outside C which is *Cauchy's integral formula*, expressing the value of a holomorphic function at any point interior to C in terms of its values on C.

Let $F(z)$ be holomorphic inside and upon C except at the point $z = a$, and there let

$$\lim_{z \to a} (z - a)\, F(z) = 0\,. \quad (5)$$

The point $z = a$ is said to be a *removable singularity* of $F(z)$, for from (4), the function

$$F^*(z) = \frac{1}{2\pi i} \oint_C \frac{F(t)\, dt}{t - z}$$

coincides with $F(z)$ at every interior point including $z = a$, provided that we alter the value of $F(z)$ at $z = a$ to

$$\frac{1}{2\pi i} \oint_C \frac{F(t)\, dt}{t - a}\,.$$

The condition (5) is thus sufficient for the singularity to be removable. It is also necessary, for $F^*(z)$ being holomorphic is finite and continuous and so

$$\lim_{z \to a} (z - a) F^*(z) = 0$$

which implies (5).

3.13. Cauchy's integral formula for the infinite region exterior to a closed contour

Let $f(z)$ be holomorphic in the region exterior to the simple closed contour C. Draw a circle Γ of radius R' large enough to include C in its interior. Let z be a point interior to Γ. Then by Cauchy's integral formula applied to the region between C and Γ

$$\lambda f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z} + \frac{1}{2\pi i} \oint_\Gamma \frac{f(t)\,dt}{t - z} \tag{1}$$

where

$$\left.\begin{aligned} \lambda &= 1 \quad \text{if } z \text{ is between } C \text{ and } \Gamma \\ \lambda &= 0 \quad \text{if } z \text{ is inside } C. \end{aligned}\right\} \tag{2}$$

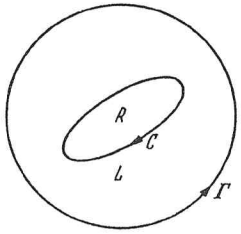

Fig. 3.13

Since the first two terms of (1) are independent of R' the last term

$$I = \frac{1}{2\pi i} \oint_\Gamma \frac{f(t)\,dt}{t - z} \tag{3}$$

must be independent of R'. Therefore

$$I = \lim_{R' \to \infty} I. \tag{4}$$

On Γ write $t = R' e^{i\theta}$. Then

$$\lim_{R' \to \infty} I = \lim_{R' \to \infty} \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(R' e^{i\theta})\, i\, d\theta}{1 - \dfrac{z}{R' e^{i\theta}}} = \lim_{R' \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(R' e^{i\theta})\, d\theta. \tag{5}$$

But by hypothesis $f(t)$ is holomorphic and therefore bounded at infinity. Thus from (3) and (5)

$$I = f(\infty)$$

and therefore from (1)

$$\left.\begin{aligned} f(z) &= f(\infty) + \frac{1}{2\pi i} \oint_C \frac{f(t)}{t - z}\, dt, \quad z \text{ outside } C \\ 0 &= f(\infty) + \frac{1}{2\pi i} \oint_C \frac{f(t)}{t - z}\, dt, \quad z \text{ inside } C. \end{aligned}\right\} \tag{6}$$

These constitute Cauchy's integral formula.

Now suppose that $f(z)$ is holomorphic inside and upon C except at the point a where it has a pole whose principal part is

$$g(z) = \sum_{s=1}^{n} A_s (z - a)^{-s}. \tag{7}$$

The function $g(z)$ is holomorphic outside C and vanishes at infinity. Therefore, from (6)

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{g(t)\,dt}{t - z} &= g(z), \quad \text{when } z \text{ is outside } C \\ &= 0, \quad \text{when } z \text{ is inside } C. \end{aligned} \tag{8}$$

Now by our hypothesis $f(z) - g(z)$ is holomorphic inside C and therefore to it we can apply Cauchy's integral formula which, using (8), gives

$$\left.\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z} &= f(z) - g(z), \quad z \text{ inside } C \\ \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z} &= -g(z), \quad z \text{ outside } C. \end{aligned}\right. \tag{9}$$

Similarly if $f(z)$ is holomorphic outside and upon C except at infinity where it has a pole whose principal part is

$$g(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

we see that $f(z) - g(z)$ is holomorphic outside C and vanishes at infinity. Therefore by similar steps

$$\left.\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z} &= f(z) - g(z), \quad z \text{ outside } C \\ \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z} &= -g(z), \quad z \text{ inside } C. \end{aligned}\right. \tag{10}$$

(9) and (10) are equivalent to the single statement

$$\frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z} = f(z) - g(z) \quad \text{or} \quad -g(z) \tag{11}$$

according as z is in the region L or the region R determined by the positive sense of description of C.

3.14. To find the function whose real part is given on a circumference

Let a circumference Γ, $|t| = a$, and a positive sense of description of Γ be given, so that left- and right-hand regions L and R are determined.

We now prove the theorem of SCHWARZ, that the function $f(z)$ which is holomorphic in L and whose real part takes the value $\varphi(t)$ on Γ is

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \left\{ \frac{2\varphi(t)}{t-z} - \frac{\varphi(t)}{t} \right\} dt + i\lambda, \tag{1}$$

where λ is an arbitrary real constant.

Proof. On the circumference $t\bar{t} = a^2$ and therefore

$$f(t) + \bar{f}\left(\frac{a^2}{t}\right) = 2\varphi(t) \tag{2}$$

so that

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)\,dt}{t-z} + \frac{1}{2\pi i} \oint_{\Gamma} \frac{\bar{f}\left(\frac{a^2}{t}\right) dt}{t-z} = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2\varphi(t)\,dt}{t-z}. \tag{3}$$

Case (i). Let the positive sense of description of Γ be counterclockwise. Then L is the region $|t| < a$ interior to Γ, and from (3) and 3.13 (10)

$$f(z) + \bar{f}(0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2\varphi(t)\,dt}{t-z}, \quad (z \text{ in } L). \tag{4}$$

Put $z = 0$. Then

$$f(0) + \bar{f}(0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2\varphi(t)\,dt}{t}. \tag{5}$$

Let $f(0) = \alpha + i\lambda$. Then $\bar{f}(0) = \alpha - i\lambda$ and so (4) and (5) give

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \left[\frac{2\varphi(t)}{t-z} - \frac{\varphi(t)}{t} \right] dt + \frac{1}{2}\left[f(0) - \bar{f}(0)\right] \tag{6}$$

which is equivalent to (1).

Case (ii). Let the positive sense of description of Γ be clockwise. Then L is the region $|t| > a$ exterior to Γ, and from (3) and 3.13 (6)

$$f(z) - f(\infty) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2\varphi(t)\,dt}{t-z}, \quad (z \text{ in } L), \tag{7}$$

$$-f(\infty) - \bar{f}\left(\frac{a^2}{z}\right) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2\varphi(t)\,dt}{t-z}. \quad (z \text{ in } R) \tag{8}$$

In (8) put $z = 0$. Then

$$-f(\infty) - \bar{f}(\infty) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2\varphi(t)\,dt}{t}. \tag{9}$$

Let $f(\infty) = \alpha + i\lambda$. Then (7) and (9) combine to give

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \left[\frac{2\varphi(t)}{t-z} - \frac{\varphi(t)}{t} \right] dt + \frac{1}{2}\left[f(\infty) - \bar{f}(\infty)\right],$$

which is equivalent to (1).

Q. E. D.

Note that (1) can also be written in the form

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \varphi(t) \frac{t+z}{t-z} \frac{dt}{t} + i\lambda. \tag{10}$$

Note also that

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{\varphi(t)}{t}\,dt = f(0) - i\lambda \text{ or } i\lambda - f(\infty) \tag{11}$$

according as $z = 0$ is in L or R.

3.16. Principal value of an integral

Let t_0 be a given point on an arc A (which may be a closed contour). Consider the line integral

$$\int_A \frac{\varphi(t)}{t-t_0}\,dt. \tag{1}$$

In general the integrand becomes infinite when $t = t_0$ and the integral is indeterminate. Describe a circle centre t_0 of radius ε so small that the circle cuts the arc A in exactly two points B and D, say. Denote by a the part of the arc A inside the circle, i. e. the arc BD, and denote by $A - a$ the rest of the arc A.

The integral (1) is said to exist as a *Cauchy principal value* if

$$\lim_{\varepsilon \to 0} \int_{A-a} \frac{\varphi(t)}{t-t_0}\,dt \quad \text{exists.} \tag{2}$$

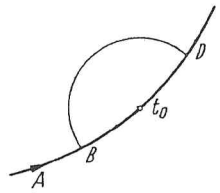

Fig. 3.16

Observe that an integral which exists in the ordinary sense, exists also as a Cauchy principal value. The converse is false.

When occasion demands, every integral will be considered as meaning its principal value without explicit attention being called to the fact.

Consider in particular, for a closed curve C,

$$\int_{C-a} \frac{dt}{t-t_0} = \left[\ln(t-t_0)\right]_D^B = i\left[\arg(t-t_0)\right]_D^B.$$

The limit of this when $\varepsilon \to 0$ is $i\pi$, and therefore

$$\oint \frac{dt}{t-t_0} = i\pi. \tag{3}$$

3.20. Cauchy integrals

Let $\varphi(t)$ be a function of t, not necessarily holomorphic, which is defined at every point t of a simple closed contour C and is such that

$\int_C |\varphi(t)|\,|dt|$ is bounded. Let z be any point of the plane not on C. The function $\Phi(z)$ defined by

$$\Phi(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{t-z}, \tag{1}$$

where it is assumed that the integral exists, will be called a *Cauchy integral.*

To fix our ideas let us consider the contour C of fig. 3.12 described in the counterclockwise sense so that of the two regions L and R into

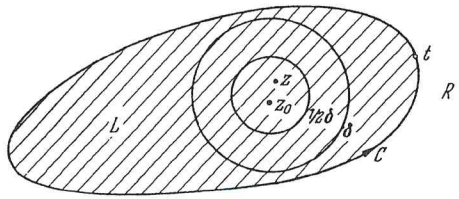

Fig. 3.20

which C separates the plane, the region L is interior to C. Then if the point z is in L, we define $\Phi^L(z)$ by

$$\Phi^L(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{t-z}. \tag{2}$$

Similarly if z is in the region R we define

$$\Phi^R(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{t-z}. \tag{3}$$

We proceed to show that each of the functions $\Phi^L(z)$, $\Phi^R(z)$ is continuous.

Proof. Consider $\Phi^L(z)$. Let z_0 be any point in L. With z_0 as centre describe a circle of radius δ so chosen that its circumference does not meet C at any point. Describe a concentric circle of radius $\frac{1}{2}\delta$. Then if z lies within this latter circle it is clear that

$$|t-z| > \frac{1}{2}\delta$$

where t is any point of C.

Then from (2)

$$\Phi^L(z) - \Phi^L(z_0) = (z-z_0)\frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{(t-z)(t-z_0)}. \tag{4}$$

Therefore

$$|\Phi^L(z) - \Phi^L(z_0)| < |z-z_0|\frac{1}{\delta^2}\int_C |\varphi(t)|\,|dt|$$

and since by hypothesis the last integral is bounded, we see that the left-hand side tends to zero as $z - z_0$ tends to zero. In other words $\Phi^L(z)$ is continuous at z_0 and so at any point of L.

Q. E. D.

If in (2) we replace $\varphi(t)$ by $\varphi(t)/(t-z_0)$, the same argument proves that

$$\frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{(t-z)(t-z_0)}$$

is a continuous function of z and therefore tends to a limit, say $L(z_0)$ when $z \to z_0$.

Therefore from (4)

$$\lim_{z\to z_0}\frac{\Phi^L(z)-\Phi^L(z_0)}{z-z_0} = L(z_0)$$

and therefore $\Phi^L(z)$ has a derivative. Thus we have proved that

$$\Phi^L(z) \quad \text{is holomorphic when } z \text{ is in } L. \tag{5}$$

An exactly similar proof will show that

$$\Phi^R(z) \quad \text{is holomorphic when } z \text{ is in } R. \tag{6}$$

The foregoing definitions and demonstrations continue to be valid when C is replaced by the union of several non-intersecting contours (cf. 3.10). More generally we can replace C by an open arc A or by the union of several such non-intersecting arcs for which the regions L and R can be defined.

If, as here, R is the region which contains the point infinity, we see from (3) that

$$\Phi^R(z) \to 0 \quad \text{as} \quad |z| \to \infty. \tag{7}$$

More generally, if we write

$$\Phi_n(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{(t-z)^n}, \tag{8}$$

we have the identity

$$\begin{aligned}\Phi_n(z) - \Phi_n(z_0) = &\left\{\frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{(t-z)^{n-1}(t-z_0)} - \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{(t-z_0)^n}\right\} \\ &+ (z-z_0)\frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{(t-z)^n(t-z_0)}.\end{aligned} \tag{9}$$

The induction hypothesis

$$\Phi'_{n-1}(z) = (n-1)\,\Phi_n(z) \tag{10}$$

then leads, by (9) and by what we have already proved, to the truth of (10) for all positive integral values of $n-1$.

Inferences:

(i) From Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(t)\,dt}{t-z},$$

where $f(t)$ is holomorphic inside C, we infer the existence of derivatives of $f(z)$ of every order

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(t)\,dt}{(t-z)^{n+1}}.$$

(ii) In (2) and (3) we can replace the closed contour C by an open arc A, see figs. 3. 10 (iii) and (iv), for such a line can be closed to form a circuit by ascribing to $\varphi(t)$ the value zero on the closing arc.

(iii) *Cauchy's estimate.* Consider a circle C of radius r about the point $z = a$ and let $|f(z)| \leqq M$ on C. Then from (i)

$$|f^{(n)}(a)| \leqq n!\,\frac{M}{r^n}. \tag{11}$$

(iv) *Liouville's theorem.* A function $f(z)$ which is holomorphic and bounded in the whole plane, including the point infinity, must reduce to a constant.

This means that there is a fixed quantity M such that on all circles centre a, $|f(z)| \leqq M$ and so by (11)

$$|f'(a)| \leqq \frac{M}{r}$$

however large r may be. Thus $f'(a) = 0$ for all a and so $f(a)$ is a constant.

(v) *Generalized Liouville's theorem.* A function $f(z)$ which is holomorphic in the whole plane and satisfies the inequality $|f(z)| < |z^n|$, n a fixed positive integer, $|z|$ sufficiently large, must reduce to a polynomial whose degree cannot exceed n.

The method of proof used in (iv) shows that $f^{(n+1)}(a) = 0$, whence we infer the result.

(vi) *The fundamental theorem of algebra.* Every polynomial equation $P(z) = 0$ of degree greater than zero must have a root.

For the contrary supposition shows that $1/P(z)$ is holomorphic in the whole plane and is zero at infinity. Thus $1/P(z)$ is bounded in the whole plane including the point infinity and so by (iv) $P(z)$ is a constant which is a contradiction.

3.22. Sectionally holomorphic functions

Consider an open arc A drawn from the point B to the point D in the z-plane. When the positive sense of description has been assigned, this arc A will define regions L and R. Denote by Ω the region consisting of the whole of the z-plane, and by $\Omega - A$ the region consisting of the z-plane cut along the arc A.

Def. A function $f(z)$ is said to be *sectionally holomorphic** in the entire plane if it satisfies the following conditions

(a) $f(z)$ is holomorphic everywhere in $\Omega - A$.

(b) It is continuous from the left and from the right at all points of A other than the ends.

(c) If $f(z)$ has a singularity at an end of A, the singularity is removable (3.12).

The arc A is, in general, a *line of discontinuity* or *a boundary* for $f(z)$.

The arc A can be replaced by a union of non-intersecting arcs or contours.

The condition (b) in the definition means that if t_0 is a point of A, $\lim\limits_{z\to t_0} f(z)$ exists when z tends to t_0 remaining in the region L (or in the region R) and we then define respectively

$$f^L(t_0) = \lim_{z\to t_0} f(z), \quad f^R(t_0) = \lim_{z\to t_0} f(z).$$

The Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_A \frac{\varphi(t)\,dt}{t-z}$$

offers an important example of a sectionally holomorphic function $\Phi(z)$. Into the conditions which must be satisfied by $\varphi(t)$ we shall not enter.

3.30. The formulae of PLEMELJ

Let t_0 be a point on a simple closed curve C which thereby defines the regions L and R. Fig. 3.30 shows these regions when the positive sense of description of C is taken to be counterclockwise. Consider the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\oint \frac{\varphi(t)\,dt}{t-z}. \tag{1}$$

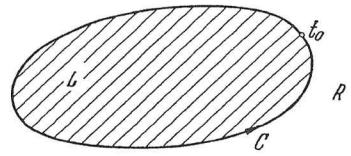

Fig. 3.30

By our previous results $\Phi(z)$ is holomorphic in the whole plane, including the point infinity, except perhaps on C. By Liouville's theorem, if $\Phi(z)$ were also holomorphic on C, it would reduce to a constant.

* This convenient term is due to MUSKHELISHVILI.

We have identically

$$\Phi(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)-\varphi(z)}{t-z}\,dt + \frac{1}{2\pi i}\oint_C \frac{\varphi(z)}{t-z}\,dt\,. \tag{2}$$

Therefore, if z is in L, we have

$$\Phi^L(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)-\varphi(z)}{t-z}\,dt + \varphi(z)\,. \tag{3}$$

Now let $z \to t_0$ a point of C. Then, 3.22, we *define* $\Phi^L(t_0)$ by

$$\Phi^L(t_0) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)-\varphi(t_0)}{t-t_0}\,dt + \varphi(t_0)\,. \tag{4}$$

When z is in R, $\oint_C dt/(t-z) = 0$ and therefore (2) gives

$$\Phi^R(z) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)-\varphi(z)}{t-z}\,dt \tag{5}$$

and, 3.22, we then *define* $\Phi^R(t_0)$ by

$$\Phi^R(t_0) = \frac{1}{2\pi i}\oint_C \frac{\varphi(t)-\varphi(t_0)}{t-t_0}\,dt\,. \tag{6}$$

Thus $\Phi^L(t_0)$, $\Phi^R(t_0)$ are the values of $\Phi(z)$ when $z \to t_0$ from L and from R respectively.

Now from 3.16 (3)

$$\frac{1}{2\pi i}\oint_C \frac{dt}{t-t_0} = \frac{1}{2}\,,\quad \frac{1}{2\pi i}\oint_C \frac{\varphi(t_0)\,dt}{t-t_0} = \frac{1}{2}\,\varphi(t_0)$$

and therefore (4) and (6) may be written

$$\Phi^L(t_0) = \frac{1}{2}\,\varphi(t_0) + \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{t-t_0} \tag{7}$$

$$\Phi^R(t_0) = -\frac{1}{2}\,\varphi(t_0) + \frac{1}{2\pi i}\oint_C \frac{\varphi(t)\,dt}{t-t_0}\,. \tag{8}$$

These formulae were discovered by J. Plemelj as were also the two following combinations of them by subtraction and addition, which will be more important than (7) and (8) for our purposes,

$$\Phi^L(t_0) - \Phi^R(t_0) = \varphi(t_0) \qquad \text{(First Plemelj formula)}$$

$$\Phi^L(t_0) + \Phi^R(t_0) = \frac{1}{\pi i}\oint_C \frac{\varphi(t)\,dt}{t-t_0} \qquad \text{(Second Plemelj formula)}$$

These formulae have been demonstrated to fix our ideas on the assumption that the positive sense of description of C is counterclockwise. Since the sense of description also defines the regions L and R, if we alter the sense of description, the first Plemelj formula is unaltered in form while on the second the arrow on the integral points the other way.

The formulae continue to hold when the closed contour C is replaced by an open arc A, for we may close the arc to form a contour, figs. 3.10 (iii), (iv) and on the closing part we ascribe to $\varphi(t)$ the value zero. It is assumed, however, that t_0 does not coincide with an end of the arc at which $\varphi(t_0) \neq 0$.

More generally C can consist of the union of several non-intersecting contours or open arcs.

3.31. A boundary value problem

Let B be a boundary, either an arc or a contour. Let it be required to find the sectionally holomorphic function $\Phi(z)$, which is zero at infinity and which satisfies on the boundary B the condition

$$\Phi^L(t_0) - \Phi^R(t_0) = \varphi(t_0)\,, \tag{1}$$

where $\varphi(t_0)$ is a function given on B.

The solution is given at once by the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_B \frac{\varphi(t)\,dt}{t-z}\,, \tag{2}$$

for by the first Plemelj formula this satisfies (1), is sectionally holomorphic, and vanishes at infinity.

Moreover, the solution is unique for if $\Phi_1(z)$ is a second solution, consider the difference $\psi(z) = \Phi(z) - \Phi_1(z)$. We see that $\psi(z) = 0$ for points not on B, and $\psi^L(t_2) - \psi^R(t_2) = 0$ for points on B. Thus $\psi(z)$ is holomorphic in the whole plane and is zero at infinity. Therefore by Liouville's theorem $\psi(z) \equiv 0$ and so $\Phi_1(z) \equiv \Phi(z)$.

Corollary (*i*). If the condition at infinity is

$$\Phi(z) = O(z^n)\,,$$

where n is zero or a positive integer the solution is

$$\Phi(z) = \frac{1}{2\pi i}\int_B \frac{\varphi(t)\,dt}{t-z} + P_n(z)\,, \tag{3}$$

where $P_n(z)$ is an arbitrary polynomial of degree n.

Corollary (*ii*). If in addition to the pole at infinity of corollary (i) $\Phi(z)$ is to have a pole of order m at $z = a$, the solution is

$$\Phi(z) = \frac{1}{2\pi i}\int_B \frac{\varphi(t)\,dt}{t-z} + P_n(z) + \frac{\alpha_m}{(z-a)^m} + \frac{\alpha_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{\alpha_1}{z-a}\,.$$

3.40. The Plemelj function $\chi(z)$

Let A be an arc whose positive sense of description is from the end point a to the end point b.

Then if τ is any constant such that

$$0 \leqq \operatorname{Re} \tau < 1 , \tag{1}$$

we define the Plemelj function $\chi(z)$ with respect to the arc A by

$$\chi(z) = \frac{(z-b)^{\tau-1}}{(z-a)^{\tau}} . \tag{2}$$

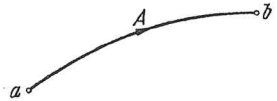

Fig. 3.40 (i)

More generally if, instead of a single arc, A consists of the union of n disjoint arcs A_s $(s = 1, 2, \ldots, n)$, whose end points are a_s, b_s, we can generalize (2) to

$$\chi(z) = \frac{\{(z-b_1)(z-b_2)\ldots(z-b_n)\}^{\tau-1}}{\{(z-a_1)(z-a_2)\ldots(z-a_n)\}^{\tau}} . \tag{3}$$

The function $\chi(z)$ has branch points at the ends of A and is therefore, in general, many-valued. Each branch is, however, one-valued in the plane cut along A. For definiteness we shall always use that branch for which

$$\lim_{z\to\infty} z^n \chi(z) = 1 . \tag{4}$$

On account of (1) $\chi(z)$ has no zeros in the finite part of the plane.

Near the end point a_s, $(z-a_s)\,\chi(z)$ behaves like $(z-a_s)^{1-\tau}$ and near b_s, $(z-b_s)\,\chi(z)$ behaves like $(z-b_s)^{\tau}$. Thus a_s and b_s are removable singularities for $\chi(z)$ unless $\operatorname{Re}\tau = 0$. Therefore $\chi(z)$ is sectionally holomorphic in the plane cut along A.

In the case $\operatorname{Re}\tau = 0$, so that $\tau = i\tau_0$, where τ_0 is real, we can replace the function $\chi(z)$ by $\chi^*(z)$, where

$$\chi^*(z) = \chi(z) \prod_{s=1}^{n} (z-b_s) = \prod_{s=1}^{n} \left(\frac{z-b_s}{z-a_s}\right)^{i\tau_0} \tag{5}$$

which is bounded at ends b_s where it vanishes. Thus $\chi^*(z)$ is sectionally holomorphic, and we replace (4) by

$$\lim_{z\to\infty} \chi^*(z) = 1 . \tag{6}$$

We now prove the following lemma.

Lemma.

The Plemelj function $\chi(z)$ satisfies the relation

$$\chi^L(t) = e^{2\pi i\tau}\,\chi^R(t) \tag{7}$$

at every point t of the arc A.

For simplicity of writing we prove this for a single arc A.

Proof. Let M be a point t on the arc A. We denote this point by M^L or M^R according as we regard it as approached from the left or right of the arc.

We can go from M^L to M^R via a simple curve γ_1 which encloses a or via a simple curve γ_2 which encloses b, fig. 3.40 (ii).

Now

$$(t-a)^{-\tau} = \exp\{-\tau \log|t-a| - i\tau \arg(t-a)\}$$

and if we take t round γ_1, arg $(t-a)$ increases by 2π and therefore

$$\{(t-a)^{-\tau}\}^L = \{(t-a)^{-\tau}\}^R e^{2\pi i\tau} . \tag{8}$$

Similarly we prove that

$$\begin{aligned}\{(t-b)^{\tau-1}\}^L &= \\ \{(t-b)^{\tau-1}\}^R e^{2\pi i(\tau-1)} &\\ = \{(t-b)^{\tau-1}\}^R e^{2\pi i\tau} &\end{aligned} \tag{9}$$

if we take t round γ_2.

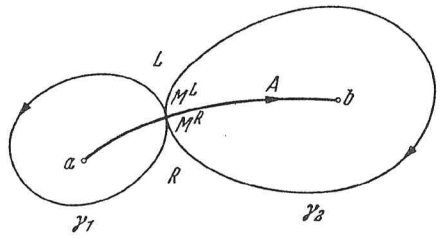

Fig. 3.40 (ii)

From (8) or (9), (7) follows.

Q. E. D.

Corollary. The same property (7) holds for $\chi^*(z)$ defined in (5).

Theorem 3.40. The homogeneous Hilbert functional equation

$$\varphi^L(t) = c\,\varphi^R(t) , \quad t \text{ on } A , \tag{10}$$

where c is a constant other than a positive real number, has the particular sectionally holomorphic solution

$$\varphi(z) = \chi(z) , \tag{11}$$

where $\chi(z)$ is given by (3) and

$$\tau = \frac{1}{2\pi i} \ln c = \frac{\beta}{2\pi} - i\frac{\ln|c|}{2\pi} , \; 0 < \beta < 2\pi . \tag{12}$$

When c is a positive real number the corresponding solution is $\chi^*(z)$ defined by (5).

This theorem is a direct consequence of the Lemma just proved and its corollary, for by (12) $e^{2\pi i\tau} = c$.

3.41. Integrals involving the Plemelj function $\chi(t)$

Consider

$$\frac{1}{2\pi i} \int_A \frac{f(t)\,dt}{\chi^L(t)\,(t-z)} \tag{1}$$

where $\chi(z)$ is defined by 3.40 (3), and A is the simple arc depicted in fig. 3.40 (ii).

Consider the double lacet M surrounding A, which consists of lines joining the ends of an arc of a circle about a to the ends of an arc of a circle about b (without crossing A), and the arcs themselves, fig. 3.41. We describe M clockwise and suppose z to be outside M.

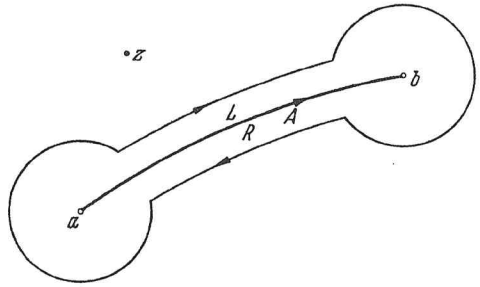

Fig. 3.41

If we let M shrink to A, we have

$$\frac{1}{2\pi i}\oint_M \frac{f(t)\,dt}{\chi(t)\,(t-z)} = \frac{1}{2\pi i}\int_{ab} \frac{f(t)\,dt}{\chi^L(t)\,(t-z)} + \frac{1}{2\pi i}\int_{ba} \frac{f(t)\,dt}{\chi^R(t)\,(t-z)}\,.$$

But from 3.40 (7) $\chi^R(t) = e^{-2\pi i\tau}\chi^L(t)$.

Therefore

$$\frac{1}{2\pi i}\int_A \frac{f(t)\,dt}{\chi^L(t)\,(t-z)} = \frac{1}{1-e^{2\pi i\tau}}\,\frac{1}{2\pi i}\oint_M \frac{f(t)\,dt}{\chi(t)\,(t-z)} \tag{2}$$

and the integral on the right may, in many cases, be calculated by the method of 3.13 (11).

Similarly we show that

$$\frac{1}{2\pi i}\int_A \frac{f(t)\,\chi^L(t)\,dt}{t-z} = \frac{1}{1-e^{-2\pi i\tau}}\,\frac{1}{2\pi i}\oint_M \frac{f(t)\,\chi(t)\,dt}{t-z}\,. \tag{3}$$

3.42. Solution of a homogeneous Hilbert functional equation

We seek a sectionally holomorphic solution of the Hilbert equation

$$\varphi^L(t) = c\,\varphi^R(t)\,, \quad t \text{ on A} \tag{1}$$

which has a pole at infinity, where A is the union of disjoint arcs as defined in 3.40.

We shall use the Plemelj function $\chi(z)$ of 3.40 (3), (12), when c is not a positive real number. When c is a positive real number we use the function $\chi^*(z)$ of 3.40 (5), (12). Therefore in what follows the function $\chi(z)$ will be supposed replaced by $\chi^*(z)$ whenever c is real and positive.

From Theorem 3.40, (1) has the particular sectionally holomorphic solution $\chi(z)$, which vanishes nowhere in the finite part of the plane, and therefore

$$\chi^L(t) = c\,\chi^R(t)\,, \quad \text{or} \quad c = \chi^L(t)/\chi^R(t)\,, \quad t \text{ on } A\,. \tag{2}$$

Therefore if $\psi(z) = \varphi(z)/\chi(z)$, we have

$$\psi^L(t) = \psi^R(t)\,, \quad t \text{ on } A\,.$$

If therefore we attribute to $\psi(z)$ suitable values on A, it follows that $\psi(z)$ is holomorphic in the whole plane.

Also at infinity $\varphi(z)$ has a pole and therefore $\psi(z)$ has a pole and thus $\psi(z) = O(z^m)$. Therefore by the generalized Liouville theorem (3.20 (v)) $\psi(z)$ is a polynomial of degree m and therefore

$$\varphi(z) = P_m(z)\,\chi(z)\,. \tag{3}$$

Notes. (i) If the order of the pole of $\varphi(z)$ is assigned beforehand to be l, we have $m = l + n$, or $m = l$ according as $\chi(z)$ or $\chi^*(z)$ is used.

(ii) If the solution is required to be bounded at infinity we must have $m \leqq n$ for $\chi(z)$ and $m = 0$ for $\chi^*(z)$.

(iii) If the solution is to be bounded near certain ends say a_1, b_2, b_3 we choose the polynomial $P_m(z)$ to have the appropriate factors in this case $(z-a_1)\,(z-b_2)\,(z-b_3)$ and write

$$P_m(z) = (z-a_1)\,(z-b_2)\,(z-b_3)\,P_{m-3}(z)$$

so that the solution is of the form $\chi_3(z)\,P_{m-3}(z)$, where

$$\chi_3(z) = (z-a_1)\,(z-b_2)\,(z-b_3)\,\chi(z)\,. \tag{4}$$

(iv) The function $\chi_3(z)$ of (4) is of the same form as (3) and is therefore itself a solution of (1) as is indeed any other function formed by multiplication of $\chi(z)$ by a polynomial.

In particular

$$\chi_a(z) = \chi(z)\prod_{s=1}^{n}(z-a_s)\,(z-b_s) = \prod_{s=1}^{n}(z-a_s)^{1-\tau}(z-b_s)^{\tau}$$

is a solution which vanishes at all ends and is therefore bounded at all ends.

3.43. Solution of an inhomogeneous equation

We now turn to the equation

$$\varphi^L(t) - c\,\varphi^R(t) = f(t)\,, \quad t \text{ on } A\,, \tag{1}$$

which reduces to 3.42 (1) when $f(t) \equiv 0$.

As before c is a constant and A is the union of disjoint arcs.

We seek a solution which has a pole at infinity. We define $\chi(z)$ by 3.40 (3), (12) and use 3.42 (2) to write the equation in the form

$$\left(\frac{\varphi(t)}{\chi(t)}\right)^L - \left(\frac{\varphi(t)}{\chi(t)}\right)^R = \frac{f(t)}{\chi^L(t)} \tag{2}$$

whose solution by use of 3.31 (3) is

$$\varphi(z) = \frac{\chi(z)}{2\pi i}\int_A \frac{f(t)\,dt}{\chi^L(t)\,(t-z)} + \chi(z)\,P_m(z)\,, \tag{3}$$

where $P_m(z)$ is an arbitrary polynomial.

This is the general solution when c is not a positive real number. If it is, we replace $\chi(z)$ by $\chi^*(z)$ defined in 3.40 (5).

Should a solution be required which has poles of orders not exceeding $p_1, p_2, \ldots, p_m, p$ at the points $z_1, z_2, \ldots, z_m, \infty$ we replace $P_m(z)$ in (3) by a rational function $R(z)$ which has poles of the required orders with the proper principal parts.

The Plemelj function has been so named because it converts the Hilbert equation into one adapted to the First Plemelj formula.

3.44. A mixed equation

Consider a *contour* divided into several arcs $A_1, A_2, \ldots, A_n$ on which

$$\varphi^L(t) - c\,\varphi^R(t) = f(t)\,, \tag{1}$$

and arcs $A_1', A_2', \ldots, A_n'$ on which

$$\varphi^L(t) - \varphi^R(t) = g(t)\,, \tag{2}$$

where c is a constant and $f(t)$, $g(t)$ are given functions.

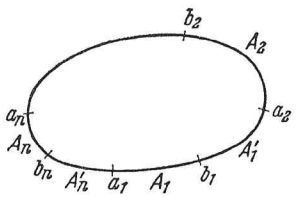

Fig. 3.44

Consider first the homogeneous problem

$$\left.\begin{aligned} \varphi^L(t) - c\,\varphi^R(t) &= 0 \quad \text{on} \quad A_1 + A_2 + \cdots + A_n \\ \varphi^L(t) - \varphi^R(t) &= 0 \quad \text{on} \quad A_1' + A_2' + \cdots + A_n'. \end{aligned}\right\} \tag{3}$$

If we define $\chi(z)$ as in 3.40 (3), (12), we have

$$\left.\begin{aligned} c &= \frac{\chi^L(t)}{\chi^R(t)} \quad \text{on} \quad A_1 + A_2 + \cdots + A_n\,, \\ 1 &= \frac{\chi^L(t)}{\chi^R(t)} \quad \text{on} \quad A_1' + A_2' + \cdots + A_n'\,, \end{aligned}\right\} \tag{4}$$

since by the second of (3), the arcs $A_1', \ldots, A_n'$ constitute no boundary for $\chi(z)$.

Therefore we can write (1) and (2) as the single equation

$$\left(\frac{\varphi(t)}{\chi(t)}\right)^L - \left(\frac{\varphi(t)}{\chi(t)}\right)^R = \frac{F(t)}{\chi^L(t)}, t \quad \text{on} \quad A = A_1 + A_1' + \cdots + A_n + A_n'\,, \tag{5}$$

where

$$\begin{aligned} F(t) &= f(t) \quad \text{on} \quad A_1 + \cdots + A_n \\ F(t) &= g(t) \quad \text{on} \quad A_1' + \cdots + A_n'\,. \end{aligned}$$

Thus the equations (3) with mixed boundary conditions have been reduced to the single equation (5) whose solution having a pole at infinity is as in 3.43

$$\varphi(z) = \left(\frac{\chi(z)}{2\pi i}\right)\int_A \frac{F(t)\,dt}{\chi^L(t)\,(t-z)} + \chi(z)\,P(z)\,,$$

where $P(z)$ is a polynomial.

3.50. Statement of four fundamental boundary value problems

The solution of problems concerning two-dimensional elastic infinitesimal deformation of isotropic material, reduces essentially to finding the complex stresses $W(z)$, $w(z)$. This will be effected by establishing certain relations into which they enter, perhaps also accompanied by their integrals, derivatives and complex conjugates, which are satisfied at the boundary of the material. We shall single out four basic problems for discussion:

Problem I — The stress is given at every point of the boundary;

Problem II — The displacement is given at every point of the boundary;

Problem III — At every point of the boundary *either* the displacement *or* the stress is given;

Problem IV — At every point of the boundary the normal displacement *and* the tangential stress are given.

Statement of Problem I.

From 2.20 (7) if N, S are the given normal and shearing (tangential) components of stress we have, on the boundary

$$\begin{aligned} &W(z) + \overline{W}(\bar{z}) + (\bar{z}W'(z) + w(z))\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds} \\ &= 2\,(N - iS) - (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}} + \frac{\partial^2 Q}{\partial z^2}\,4\nu\,\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds}\,. \end{aligned} \tag{1}$$

Statement of Problem II.

Let U_n, U_s be the given normal and tangential components of displacement. Then from 2.20 (9) we have, on the boundary

$$\varkappa\,W(z) - z\overline{W}(\bar{z}) - \overline{w}(\bar{z}) = -4i\mu\,(U_n + iU_s)\Big/\frac{d\bar{z}}{ds} - 4\nu\,\frac{\partial Q}{\partial \bar{z}} \tag{2}$$

where a rigid body displacement has been omitted without loss of generality.

Statement of Problem III.

We suppose the boundary to be marked out in non-intersecting arcs $A_1, A_2, \ldots, A_n$ on which the displacement $(U_n + iU_s)$ is given. This set of arcs collectively will be denoted by A. On the complementary set of arcs $A'_1, A'_2, \ldots, A'_n$ which constitute the rest of the boundary and which will be collectively denoted by A', the stress $(N + iS)$ is given.

$$\begin{aligned} &\text{Then on } A' \text{ condition (1) is satisfied,} \\ &\qquad\text{on } A \text{ condition (2) is satisfied} \end{aligned} \tag{3}$$

and these constitute the boundary conditions.

Statement of Problem IV.

Here at every point of the boundary U_n and S are given. Subtract from (1) its conjugate complex. Then at the boundary

$$\begin{aligned} &\{\bar{z}W'(z) + w(z)\}\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds} - \{z\overline{W}'(\bar{z}) + \bar{w}(\bar{z})\}\frac{d\bar{z}}{ds}\Big/\frac{dz}{ds} \\ &= -4iS + 4\nu\frac{\partial^2 Q}{\partial z^2}\frac{dz}{ds}\Big/\frac{d\bar{z}}{ds} - 4\nu\frac{\partial^2 Q}{\partial \bar{z}^2}\frac{d\bar{z}}{ds}\Big/\frac{dz}{ds}. \end{aligned} \tag{4}$$

Now multiply (2) by $d\bar{z}/ds$ and from the resulting expression subtract its conjugate complex. Then at the boundary

$$\begin{aligned} &\frac{d\bar{z}}{ds}\{\varkappa' W(z) - z\overline{W}(\bar{z}) - \bar{w}(\bar{z})\} - \frac{dz}{ds}\{\varkappa' \overline{W}(\bar{z}) - \bar{z}W(z) - w(z)\} \\ &= -8i\mu U_n - 4\nu\left\{\frac{\partial Q}{\partial \bar{z}}\frac{d\bar{z}}{ds} - \frac{\partial Q}{\partial z}\cdot\frac{dz}{ds}\right\}. \end{aligned} \tag{5}$$

Examples III

1. If $f(z)$ is holomorphic in the annulus comprised between the circumference C_1 and the concentric circumference C_2 of smaller radius, prove that

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{t-z} + \frac{1}{2\pi i}\oint_{C_2}\frac{f(t)\,dt}{t-z}$$

for every point z inside the annulus.

2. Prove that every function $f(z)$ holomorphic in an annulus about the origin has a LAURENT expansion of the form

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n,$$

where z is a point within the annulus, and obtain expressions for the a_n.

3. Let C be a simple closed curve with an assigned positive sense of description thus separating the plane into a left hand region L and a right hand region R. Let $f(z)$ be holomorphic in L except at a point $z = a$ where there is a pole whose principal part is $g(z)$. Prove that

$$g(z) + \frac{1}{2\pi i}\oint_C \frac{f(t)\,dt}{t-z} = f(z) \text{ or } 0$$

according as z is in L or R.

4. Let σ be a point on the unit circumference γ, $|\zeta| = 1$, and let c, ζ be two points inside γ. Prove that each of the following integrals vanishes

$$\oint_\gamma \frac{d\sigma}{(\sigma-c)(\sigma-\zeta)}, \quad \oint_\gamma \frac{d\sigma}{\sigma(\sigma-c)(\sigma-\zeta)}, \quad \oint_\gamma \frac{d\sigma}{\sigma^2(\sigma-\zeta)}.$$

5. Let c, ζ be points within the unit circumference γ, $|\zeta| = 1$, and let σ be a point on this circumference. If

$$J = \frac{1}{2\pi i}\oint_\gamma \frac{\ln(\sigma-c)}{\sigma-\zeta}\,d\sigma,$$

prove that $\frac{\partial J}{\partial c} = 0$ and that $J = \ln(\sigma_0 - \zeta) + \text{constant}$, where σ_0 is an arbitrary fixed point on γ.

6. If c is a point inside, and ζ a point outside, the unit circumference γ, prove that

$$\frac{1}{2\pi i}\oint_\gamma \frac{\ln(\sigma-\zeta)\,d\sigma}{\sigma-\zeta} = \ln(\sigma_0 - \zeta) - \ln\zeta + \text{constant},$$

where σ_0 is an arbitrary fixed point on γ.

7. If c is a point inside the unit circumference γ, prove that

$$\oint_\gamma \frac{\ln\sigma - \ln(\sigma-c)}{\sigma^2}\,d\sigma = 0.$$

8. Let $f(z)$ be holomorphic within the circumference $|z| = R$ and let the real part of $f(z)$ have the value $\varphi(\theta)$ on the circumference, where $z = Re^{i\theta}$. Prove the formula of SCHWARZ that, for any internal point z,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\,d\theta + ib,$$

where b is an arbitrary real constant.

9. Let $\omega(z) = \varphi + i\psi$, where φ and ψ are real valued, be a holomorphic function of z outside the unit circumference γ, $|z| = 1$, and on γ, let

$$2i\psi = B(\sigma)$$

where σ denotes a point of γ, and $B(\sigma)$ is a given function which can be expressed in the form

$$B(\sigma) = B_1(\sigma) + B_2(\sigma),$$

where $B_1(\sigma)$ contains all the negative powers of σ and no non-negative powers. Prove that $\omega(z) = B_1(z)$.

Restate the theorem when $\omega(z)$ is holomorphic inside γ.

10. Let $W_0(z)$, $w_0(z)$ be given complex stresses. Prove the *first circle theorem* that the complex stresses defined by

$$`W(z) = `W_0(z) - z\,\overline{W}_0\left(\frac{a^2}{z}\right) - `\overline{w}_0\left(\frac{a^2}{z}\right),$$

$$``w(z) = ``w_0(z) - \frac{d}{dz}\left[z``\overline{w}_0\left(\frac{a^2}{z}\right)\right] - z\frac{d}{dz}\left[z`\overline{W}_0\left(\frac{a^2}{z}\right)\right]$$

give zero stress on the circumference $|z| = a$.

11. Let $W_0(z)$, $w_0(z)$ be given complex stresses. Prove the *second circle theorem* that the complex stresses defined by

$$`W(z) = `W_0(z) - `\overline{w}_0\left(\frac{a^2}{z}\right),$$

$$``w(z) = ``w_0(z) - \frac{d}{dz}\left\{z``\overline{w}_0\left(\frac{a^2}{z}\right)\right\} - \int^z \left\{`\overline{W}_0\left(\frac{a^2}{z}\right) + \frac{a^2}{z}W_0(z)\right\}dz$$

give zero stress on the circumference $|z| = a$.

12. An infinite plate is in a state of all round tension T given by $W_0(z) = T$, $w_0(z) = 0$. Apply the second circle theorem (Ex. 11) to find the complex stresses when a circular hole $|z| = a$ is made in the plate and prove that the radial stress $\widehat{rr}$ and the hoop stress $\widehat{\theta\theta}$ are given by

$$\widehat{rr} = T\left(1 - \frac{a^2}{r^2}\right), \quad \widehat{\theta\theta} = T\left(1 + \frac{a^2}{r^2}\right),$$

and that on the boundary $\widehat{\theta\theta}_{\max} = 2T$.

13. If in Ex. 12 we start from a state of simple tension use the second circle theorem to show that

$$\widehat{\theta\theta}_{\max} = 3T$$

at the boundary of a circular hole $|z| = a$.

[Observe that this and the preceding result are independent of the radius of the hole, so that a plate could be seriously weakened by the presence of even a minute circular hole.]

14. Let A be a fixed point on the boundary of an elastic body and P, z, a variable point. If given external forces are applied to the boundary, use 2.40 (1) to show that

$$`\overline{W}(\bar{z}) + \bar{z}W(z) + `w(z) = k(z, \bar{z})$$

where $k(z, \bar{z})$ is a known function, and find an expression for it.

15. Use Ex. 14 to show that in the case of a circular hole $|z| = 1$ in an infinite plate the boundary conditions may be written in the forms

$$`\overline{W}\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}W(\sigma) + `w(\sigma) = k\left(\sigma, \frac{1}{\sigma}\right),$$

$$`W(\sigma) + \sigma\overline{W}\left(\frac{1}{\sigma}\right) + `\overline{w}\left(\frac{1}{\sigma}\right) = \bar{k}\left(\frac{1}{\sigma}, \sigma\right),$$

where $\sigma = e^{i\theta}$ is any point of the boundary.

16. A circular hole $|z| = 1$ is made in an infinite plate in which the stress at infinity is zero and there is no body force. By multiplying the boundary conditions of Ex. 15 by $d\sigma/\{2\pi i(\sigma - z)\}$ and integrating round γ, the boundary of the hole, show that

$$`W(z) = -\frac{1}{2\pi i}\int_\gamma \frac{\bar{k}\left(\frac{1}{\sigma}, \sigma\right)d\sigma}{\sigma - z},$$

$$`w(z) = -\frac{1}{z}W(z) - \frac{1}{2\pi i}\int_\gamma \frac{k\left(\sigma, \frac{1}{\sigma}\right)d\sigma}{\sigma - z}.$$

17. If in Ex. 16 we suppose that there is uniform stress at infinity given by complex stress $W_0(z)$, $w_0(z)$, and if we write

$$W(z) = W_H(z) + W_0(z), \quad w(z) = w_H(z) + w_0(z),$$

prove that by suitable modification of the function $k(z, \bar{z})$ the formulae of Ex. 16 will still apply to give $`W_H(z)$ and $`w_H(z)$.

Chapter IV

Planes and half-planes

In this chapter we consider the action of forces applied to isotropic material which occupies the whole plane or a half-plane. In the latter case the forces can be applied by a body in contact with the half-plane, as in the case of buildings on elastic foundations and similar situations.

4.10. The plane

We suppose the material to occupy the whole plane and that the deformation is non-dislocational. Then in the notation of 2.74 (26), (27), for sufficiently large values of $|z|$ we can write

$$W(z) = \frac{1}{2}(N_\infty + T_\infty) + \frac{4\mu i\varepsilon_\infty}{\varkappa + 1} - \frac{X + iY}{\pi(\varkappa + 1)}\cdot\frac{1}{z} + O\left(\frac{1}{z^2}\right) \tag{1}$$

$$w(z) = (N_\infty - T_\infty)e^{-2i\varphi} + \frac{\varkappa(X - iY)}{\pi(\varkappa + 1)}\cdot\frac{1}{z} + O\left(\frac{1}{z^2}\right), \tag{2}$$

where the axes are assumed to be right-handed, N_∞, T_∞ are the principal stresses at infinity, the direction of T_∞ making the angle φ with the x-axis, and ε_∞ is the rotation at infinity. The components along the axes of the resultant external force applied to the plane are denoted by (X, Y).

For brevity we can write for (1) and (2)

$$W(z) = A_\infty - \frac{F}{z} + O\left(\frac{1}{z^2}\right) \tag{3}$$

$$w(z) = B_\infty + \frac{\varkappa F}{z} + O\left(\frac{1}{z^2}\right), \tag{4}$$

where in particular

$$F = \frac{X + iY}{\pi(\varkappa + 1)}. \tag{5}$$

4.11. Standard concentrated force

The form of 4.10 (1), (2) suggests consideration of the complex stresses

$$W_0(z) = -\frac{X + iY}{\pi(\varkappa + 1)} \cdot \frac{1}{z}, \quad w_0(z) = \frac{\varkappa(X - iY)}{\pi(\varkappa + 1)} \cdot \frac{1}{z} \tag{1}$$

which lead to zero stress at infinity.

Let C be a circumference of radius a about the origin. By 2.40 (1) we see that the force exerted by the material within C upon the material outside C i. e. upon the rest of the plane is

$$\frac{1}{2} i \left[-\frac{X + iY}{\pi(\varkappa + 1)} \ln z + \frac{\varkappa(X + iY)}{\pi(\varkappa + 1)} \ln \bar{z}\right]_C = X + iY,$$

independently of the radius a however small it may be.

We describe this situation by saying that a *concentrated force* is acting at the origin.

In a small neighbourhood of the origin the stresses given by (1) are unbounded. Physically this difficulty is resolved by plastic yielding. This means that to preserve elastic deformation the force $X + iY$ instead of being concentrated at a single point must be applied as a distribution of stress over a small area round the point. The situation is similar to that of a vortex in hydrodynamics, [MILNE-THOMSON (5)].

Further we observe that the same result would be obtained by starting from the complex stresses

$$W(z) = W_0(z) + f(z), \quad w(z) = w_0(z) + g(z) \tag{2}$$

where $f(z)$, $g(z)$ are of order $1/z^2$ at the origin, and have no singularities other than at $z = 0$.

Thus it appears that the concept of concentrated force is somewhat arbitrary, or, put in another way, the complex stresses will be definite only when the concentrated force is regarded as the limit of a well defined finite distribution.

In 4.5 it will be shown that the complex stresses $W_0(z)$, $w_0(z)$ do indeed arise as the limit of a constant distribution round the circumference of a circle. In our further applications we shall always suppose that we are dealing with a concentrated force assumed to have arisen in this way. We can distinguish this by the term *standard concentrated force*. If, instead of at the origin, the standard concentrated force is applied at the point z_0, the corresponding complex stresses are, (2.21),

$$W(z) = -\frac{X + iY}{\pi(\varkappa + 1)} \cdot \frac{1}{z - z_0}. \tag{3}$$

$$w(z) = \frac{\varkappa(X - iY)}{\pi(\varkappa + 1)} \cdot \frac{1}{z - z_0} - \frac{\bar{z}_0(X + iY)}{\pi(\varkappa + 1)} \cdot \frac{1}{(z - z_0)^2} \tag{4}$$

It is characteristic of the standard concentrated force that the complex stress $W(z)$ has a simple pole whereas $w(z)$ has a pole of the second order at the point of application of the force.

In the case where the stress is due solely to the application of n standard concentrated forces, $X_s + iY_s$ at the points z_s, $s = 1, 2, \ldots, n$, the complex stresses are got by replacing in (3) and (4) (X, Y) by (X_s, Y_s), z_0 by z_s and summing from $s = 1$ to n.

4.12. Plate with straight collinear cracks

We suppose the plate to occupy the whole of the xy-plane and to have a finite number of straight cracks, slits, slots, or cuts, all collinear.

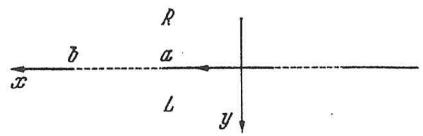

Fig. 4.12

We choose right-handed axes with the x-axis in the line of cracks. Proceeding in the positive sense of the x-axis let the sth crack $a_s b_s$ be delimited by the points $(a_s, 0)$, $(b_s, 0)$, $s = 1, 2, \ldots, n$.

It will be assumed that body-force is absent, that the stresses are bounded at infinity, that the deformation is non-dislocational, and that if the complex stresses have singularities at the ends of the cracks, these singularities are removable.

The union of the segments occupied by the cracks will be denoted by A. The whole plane is divided by the positive direction of the x-axis into left and right-hand regions L and R.

The complex stresses $W(z)$, $w(z)$ are sectionally holomorphic in the whole plane cut along A, and for large values of $|z|$ have the forms 4.10 (3), (4).

From 2.10 (3) and (4) we have

$$\begin{aligned} 2\,(\widehat{yy} - i\,\widehat{xy}) &= W(z) + \overline{W}(\bar{z}) + z\overline{W}'(\bar{z}) + \overline{w}(\bar{z}) \\ &= W(z) - \Omega(\bar{z}) + (z - \bar{z})\,\overline{W}'(\bar{z})\,, \end{aligned} \tag{1}$$

where

$$\Omega(z) = -\,\overline{W}(z) - z\,\overline{W}'(z) - \overline{w}(z) \tag{2}$$

so that

$$w(z) = -\,\overline{\Omega}(z) - W(z) - z\,W'(z)\,. \tag{3}$$

Thus instead of $W(z)$, $w(z)$ we can use $W(z)$, $\Omega(z)$ to express stress and displacement.

From (2) and 4.10 (3) and (4), the form of $\Omega(z)$ for large values of $|z|$ is

$$\Omega(z) = -\,\overline{A}_\infty - \overline{B}_\infty - \frac{\varkappa F}{z} + O\left(\frac{1}{z^2}\right) \tag{4}$$

Since

$$'\Omega(z) = -\,z\,\overline{W}(z) - '\overline{w}(z) + \text{constant} \tag{5}$$

we have for the displacement

$$4\mu\,\upsilon = 4\mu\,(u + iv) = \varkappa\,'W(z) + '\Omega(\bar{z}) - (z - \bar{z})\,\overline{W}(\bar{z}) + \text{constant}. \tag{6}$$

Let $z = t$ be a point on the x-axis. We denote the stress components at the point t by $\widehat{yy}^L$ and $\widehat{xy}^L$ when t is approached from L and by $\widehat{yy}^R$ and $\widehat{yx}^R$, when t is approached from R. Similarly we denote by $\upsilon^L(t)$, $\upsilon^R(t)$ the displacement $u + iv$ according as t is approached from L or R. Equations (1) and (6) then give the boundary values

$$2\,(\widehat{yy}^L - i\,\widehat{yx}^L) = W^L(t) - \Omega^R(t) \tag{7}$$

$$2\,(\widehat{yy}^R - i\,\widehat{yx}^R) = W^R(t) - \Omega^L(t) \tag{8}$$

$$4\mu\,\upsilon^L(t) = \varkappa\,'W^L(t) + '\Omega^R(t) + \text{constant} \tag{9}$$

$$4\mu\,\upsilon^R(t) = \varkappa\,'W^R(t) + '\Omega^L(t) + \text{constant}. \tag{10}$$

If we differentiate (9) and (10) with respect to t, we get

$$(4\mu\,\upsilon^L(t))' = \varkappa\,W^L(t) + \Omega^R(t) \tag{11}$$

$$(4\mu\,\upsilon^R(t))' = \varkappa\,W^R(t) + \Omega^L(t)\,. \tag{12}$$

In relation to fig. 4.12 each crack has an upper edge labelled R and a lower edge labelled L. Problem I arises when we are given the stress on the upper and lower edges, (7), (8); Problem II when we are given the displacement on the upper and lower edges, (9), (10) or generally more conveniently, (11), (12). Mixed problem arise when mixed conditions are given, for example stress on the upper, displacement on the lower edge.

Our primary aim in every case will be to determine the complex stresses, that is to say $W(z)$ and $\Omega(z)$, from the boundary conditions. The complex stresses will be sectionally holomorphic functions of z in the plane cut along A.

4.13. Problem I for the plate with collinear cracks

Here we are given the stresses on the upper and lower edges of each crack. We therefore use 4.12 (7) and (8) which give by addition and subtraction

$$\begin{aligned} &\{W(t) - \Omega(t)\}^L + \{W(t) - \Omega(t)\}^R \\ &= 2\,\{\widehat{yy}^L - i\,\widehat{xy}^L\} + 2\,\{\widehat{yy}^R - i\,\widehat{xy}^R\} = 2\,q_1(t) \text{ on } A. \end{aligned} \tag{1}$$

$$\begin{aligned} &\{W(t) + \Omega(t)\}^L - \{W(t) + \Omega(t)\}^R \\ &= 2\,\{\widehat{yy}^L - i\,\widehat{xy}^L\} - 2\,\{\widehat{yy}^R - i\,\widehat{xy}^R\} = 2\,q_2(t) \text{ on } A. \end{aligned} \tag{2}$$

Let us assume that the stresses at infinity N_∞, T_∞ and the angle φ are given and that the rotation at infinity is zero, $\varepsilon_\infty = 0$. Then $A_\infty = \overline{A}_\infty$ is real and therefore 4.10 (3) and 4.12 (4) give

$$W(\infty) + \Omega(\infty) = -\,\overline{B}_\infty = (T_\infty - N_\infty)\,e^{2i\varphi} \tag{3}$$

and therefore from (2) and 3.31

$$W(z) + \Omega(z) = \frac{1}{2\pi i}\int_A \frac{2\,q_2(t)\,dt}{t - z} + (T_\infty - N_\infty)\,e^{2i\varphi}\,, \tag{4}$$

for the first term on the right vanishes at infinity and so condition (3) is satisfied.

To solve (1) we first note that it is of the form 3.43 (1) with $c = -1$ and therefore from 3.40 (12)

$$e^{2\pi i\tau} = -1 \quad \text{or} \quad \tau = \frac{1}{2}\,.$$

The function $\chi(z)$ of 3.40 (3) is therefore

$$\chi(z) = \{(z - b_1)\,(z - b_2)\,\ldots\,(z - b_n)\,(z - a_1)\,(z - a_2)\,\ldots\,(z - a_n)\}^{-1/2}, \tag{5}$$

where we choose the branch of this two-valued function for which

$$\lim_{z\to\infty} z^n\,\chi(z) = 1\,. \tag{6}$$

Then the solution of (1) which is bounded at infinity is, as in 3.43 (3),

$$W(z) - \Omega(z) = \frac{\chi(z)}{2\pi i}\int_A \frac{2\,q_1(t)\,dt}{\chi^L(t)\,(t - z)} + 2\,P_n(z)\,\chi(z)\,, \tag{7}$$

where $P_n(z)$ is a polynomial of degree n at most,

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\,. \tag{8}$$

Now on account of (6) and (8), $2P_n(\infty)\,\chi(\infty) = 2a_n$ while from 4.10 (3) and 4.12 (4) $W(\infty) - \Omega(\infty) = 2A_\infty + \overline{B}_\infty$.

Therefore from (7)

$$a_n = A_\infty + \frac{1}{2}\overline{B}_\infty = e^{i\varphi}\{N_\infty \cos\varphi - iT_\infty \sin\varphi\}. \tag{9}$$

To determine the remaining coefficients $a_0, a_1, \ldots, a_{n-1}$ we use the fact that there are no dislocations and therefore the displacements are one-valued. Since $W(z)$ is holomorphic and therefore one-valued, it follows from 4.12 (6) that if C_s is any circuit round the crack $a_s b_s$,

$$[\varkappa W(z) + \Omega(\bar{z})]_{C_s} = 0$$

or

$$\varkappa \oint_{C_s} W(z)\,dz + \oint_{C_s} \Omega(\bar{z})\,d\bar{z} = 0.$$

If we now shrink C_s so as to fit glove-like on $a_s b_s$, so that C_s then consists of the segment $a_s b_s$ described along the upper edge of the crack in one sense and along the lower edge in the opposite sense, we get

$$\int_{a_s b_s} \{\varkappa\,[W^L(t) - W^R(t)] - \Omega^L(t) + \Omega^R(t)\}\,dt = 0. \tag{10}$$

Now by adding (4) and (7) we find separate expressions for $W(z)$, $\Omega(z)$ and from these we get $W^L(t)$, $W^R(t)$, $\Omega^L(t)$, $\Omega^R(t)$ by use of the Plemelj formulae. Thus (10) leads to a linear relation between the coefficients $a_0, a_1, \ldots, a_n$. We get n such relations, and since a_n is already known from (9), all the coefficients can be found.

Problem I can also be solved in the dislocational case provided we are given the dislocation due to a circuit round each crack separately. Equation (10) would then have a non-zero term on the right-hand side.

4.14. Plate weakened by linear cracks

If the edges of the cracks are unstressed, we have $q_1(t) = q_2(t) = 0$ and therefore 4.13 (4) and (7) give

$$W(z) + \Omega(z) = (T_\infty - N_\infty)\,e^{2i\varphi},$$
$$W(z) - \Omega(z) = 2P_n(z)\,\chi(z),$$

where $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ and from 4.13 (9)

$$a_n = e^{i\varphi}\{N_\infty \cos\varphi - iT_\infty \sin\varphi\}.$$

The remaining coefficients are determined by 4.13 (10) which in this case, using 4.13 (4), (7), 3.40 (7), reduces to

$$\int_{a_s b_s} P_n(t)\,\chi^L(t)\,dt = 0$$

and this solves the problem of a stretched plate weakened by linear cracks.

Consider, in particular, a single crack, $n = 1$, extending from $(-l, 0)$ to $(l, 0)$. Then

$$\chi(z) = (z^2 - l^2)^{-1/2}, \quad P_1(z) = a_0 + a_1 z,$$

where

$$a_1 = e^{i\varphi}\{N_\infty \cos\varphi - iT_\infty \sin\varphi\}.$$

To determine a_0 we have

$$0 = \int_{-l}^{l} \frac{(a_0 + a_1 t)\,dt}{\sqrt{(t^2 - l^2)}} = a_0 \pi i.$$

so that $a_0 = 0$. Thus

$$W(z) = a_1 z\,(z^2 - l^2)^{-1/2} + \frac{1}{2}(T_\infty - N_\infty)\,e^{2i\varphi}$$

$$\Omega(z) = -a_1 z\,(z^2 - l^2)^{-1/2} + \frac{1}{2}(T_\infty - N_\infty)\,e^{2i\varphi}.$$

4.15. Problem II for the plate with collinear cracks

Here we assume that the displacements $\upsilon^L(t)$, $\upsilon^R(t)$ are given on the edges of each crack. To preserve continuity of the material it is obviously necessary that at the ends a_s, b_s of each crack we must have

$$\upsilon^L(a_s) = \upsilon^R(a_s), \quad \upsilon^L(b_s) = \upsilon^R(b_s). \tag{1}$$

Then the boundary conditions given by 4.12 (11), (12) are, for each crack,

$$\varkappa W^L(t) + \Omega^R(t) = (4\mu\,\upsilon'(t))^L, \tag{2}$$

$$\varkappa W^R(t) + \Omega^L(t) = (4\mu\,\upsilon'(t))^R, \tag{3}$$

where the dash denotes differentiation with respect to t.

Combining (2) and (3) by addition and subtraction we get

$$\{\varkappa W(t) + \Omega(t)\}^L + \{\varkappa W(t) + \Omega(t)\}^R = 4\mu\,(\upsilon'(t))^L + 4\mu\,(\upsilon'(t))^R = 2q_1(t) \tag{4}$$

$$\{\varkappa W(t) - \Omega(t)\}^L - \{\varkappa W(t) - \Omega(t)\}^R = 4\mu\,(\upsilon'(t))^L - 4\mu\,(\upsilon'(t))^R = 2q_2(t) \tag{5}$$

This is formally the same problem as that posed by 4.13 (1) and (2) with $\varkappa W(t)$ written for $W(t)$. Following the same steps we obtain

$$\varkappa W(z) - \Omega(z)$$
$$= \frac{1}{2\pi i}\int_A \frac{2q_2(t)\,dt}{t - z} + \frac{1}{2}(\varkappa + 1)(T_\infty + N_\infty) + (T_\infty - N_\infty)\,e^{2i\varphi} \tag{6}$$

$$\varkappa W(z) + \Omega(z) = \frac{\chi(z)}{2\pi i}\int_A \frac{2q_1(t)\,dt}{\chi^L(t)\,(t - z)} + 2P_n(z)\,\chi(z), \tag{7}$$

where $P_n(z)$ is a polynomial of degree n at most,

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \tag{8}$$

and $\chi(z)$ is given by 4.13 (5). Expanding $\chi(z)$ we get

$$z^n \chi(z) = 1 + \frac{p}{z} + \cdots, \quad \text{where } p = \frac{1}{2}\sum (a_s + b_s) . \tag{9}$$

Multiply (7) by z^n and use 4.10 (3), 4.12 (4), 4.13 (5). Then for large values of $|z|$, writing only the highest powers of z,

$$((\varkappa - 1) A_\infty - \overline{B}_\infty) z^n - 2\varkappa F z^{n-1} + \cdots = 2 a_n z^n + (2 a_{n-1} + 2 p a_n) z^{n-1} + .$$

and so equating the coefficients of z^n and z^{n-1} we obtain a_n and a_{n-1}.

To obtain the remaining coefficients $a_{n-2}, \ldots, a_0$ we proceed as follows.

Across the line joining the end b_s of the crack $a_s b_s$ to the beginning a_{s+1} of the next crack $W(z)$ and $\Omega(z)$ are holomorphic. Therefore

$$\int_{b_s}^{a_{s+1}} \{\varkappa W(t) + \Omega(t)\}\, dt = [4\mu\, \upsilon(t)]_{b_s}^{a_{s+1}} = 4\mu \{\upsilon(a_{s+1}) - \upsilon(b_s)\} , \tag{10}$$

where the quantities on the right are known by (1). If therefore in (10) we substitute $\varkappa W(t) + \Omega(t)$ as given by (7), we obtain $n - 1$ linear equations to determine $a_0, a_1, \ldots, a_{n-2}$.

4.16. A mixed problem for a plate with a slit

Suppose a straight rigid bar of length $2l$ is welded to the lower edge of a slit in a plate and is pulled with a downwards force D at right angles to the bar and applied at the middle point, while the upper edge of the slit is acted upon by no external force. Let us investigate the complex stresses.

Taking the slit to be between $(-l, 0)$ and $(l, 0)$ we are given the stress distribution namely zero on the upper edge and the displacement namely a constant on the lower edge. Thus from 4.12 (8), (11)

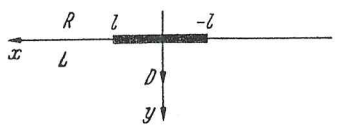

Fig. 4.16

$$W^R(t) - \Omega^L(t) = 0 \tag{1}$$

$$\varkappa W^L(t) + \Omega^R = 0 . \tag{2}$$

We combine these equations by the following general device. Multiply (1) by λ and then add to and subtract from (2). Then

$$\left[W^L(t) - \frac{\lambda}{\varkappa}\Omega^L(t)\right] + \frac{\lambda}{\varkappa}\left[W^R(t) + \frac{1}{\lambda}\Omega^R(t)\right] = 0$$

$$\left[W^L(t) + \frac{\lambda}{\varkappa}\Omega^L(t)\right] - \frac{\lambda}{\varkappa}\left[W^R(t) - \frac{1}{\lambda}\Omega^R(t)\right] = 0 .$$

If we now choose λ so that $\lambda/\varkappa = -1/\lambda$ i. e. $\lambda = i\sqrt{\varkappa}$ we get

$$\left[W^L(t) - \frac{i}{\sqrt{\varkappa}}\Omega^L(t)\right] + \frac{i}{\sqrt{\varkappa}}\left[W^R(t) - \frac{i}{\sqrt{\varkappa}}\Omega^R(t)\right] = 0 \tag{3}$$

$$\left[W^L(t) + \frac{i}{\sqrt{\varkappa}}\Omega^L(t)\right] - \frac{i}{\sqrt{\varkappa}}\left[W^R(t) + \frac{i}{\sqrt{\varkappa}}\Omega^R(t)\right] = 0 . \tag{4}$$

Put

$$e^{2\pi i \tau_1} = -\frac{i}{\sqrt{\varkappa}}, \quad e^{2\pi i \tau_2} = \frac{i}{\sqrt{\varkappa}} \tag{5}$$

$$\tau_1 = \frac{3}{4} + i\frac{\ln \varkappa}{4\pi}, \quad \tau_2 = \frac{1}{4} + i\frac{\ln \varkappa}{4\pi} \tag{6}$$

Then if we write

$$\chi_1(z) = (z + l)^{-\tau_1}(z - l)^{\tau_1 - 1}, \quad \chi_2(z) = (z + l)^{-\tau_2}(z - l)^{\tau_2 - 1} \tag{7}$$

we have from 3.43 the solutions of (3) and (4) which vanish at infinity

$$W(z) - \frac{i}{\sqrt{\varkappa}}\Omega(z) = 2A_1\chi_1(z), \quad W(z) + \frac{i}{\sqrt{\varkappa}}\Omega(z) = 2A_2\chi_2(z), \tag{8}$$

where A_1 and A_2 are constants. Thus

$$W(z) = A_1\chi_1(z) + A_2\chi_2(z), \quad \Omega(z) = i\sqrt{\varkappa}\{A_1\chi_1(z) - A_2\chi_2(z)\} . \tag{9}$$

Now from 3.40 (4) $z\chi_1(z)$ and $z\chi_2(z) \to 1$ when $z \to \infty$. Also by hypothesis $W(z)$, $\Omega(z)$ vanish at infinity. Therefore when $z \to \infty$, $zW(z) \to -iD/[\pi(\varkappa + 1)]$ from 4.10 (1), while $z\Omega(z) \to -i\varkappa D/[\pi(\varkappa + 1)]$.

Therefore from (8)

$$A_1 = -\frac{D(i + \sqrt{\varkappa})}{2\pi(\varkappa + 1)}, \quad A_2 = -\frac{D(i - \sqrt{\varkappa})}{2\pi(\varkappa + 1)}, \tag{10}$$

which with (9) complete the solution.

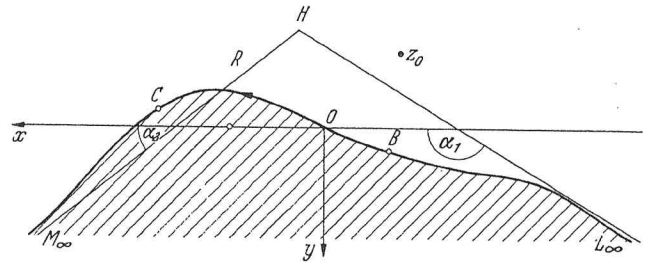

Fig. 4.20

4.20. Semi-infinite regions

We call semi-infinite a region such that points of it lie outside the circle K_r of radius r about the origin however great r may be, but such that points of the region do not lie on the continuation of *every* radius of K_r. Examples; a half-plane, an infinite strip bounded by two parallel lines.

We consider in particular the region bounded by a single line A which has two asymptotes HL_∞, HM_∞ the points L_∞, M_∞ being thought of as infinitely distant. On this line A we fix a sense of description thus distinguishing a left-hand region L from a right-hand region R.

We suppose the material to occupy the left-hand region L.

We assume that body-force is absent, that the stress at infinity is zero and that there is no rotation there. The region L being simply-connected there is no dislocation.

Take right-handed axes Ox, Oy with the origin O on A. It then follows that for large enough values of $|z|$ we can write

$$W(z) = \frac{A_1}{z} + O\left(\frac{1}{z^2}\right), \quad w(z) = \frac{B_1}{z} + O\left(\frac{1}{z^2}\right), \tag{1}$$

or alternatively, taking an arbitrary fixed point z_0 in the region R

$$W(z) = \frac{A_1}{z - z_0} + O\left(\frac{1}{z^2}\right), \quad w(z) = \frac{B_1}{z - z_0} + O\left(\frac{1}{z^2}\right). \tag{2}$$

From (2) we deduce

$$'W(z) = A_0 + A_1 \ln(z - z_0) + O\left(\frac{1}{z}\right), \tag{3}$$

$$'w(z) = B_0 + B_1 \ln(z - z_0) + O\left(\frac{1}{z}\right), \tag{4}$$

where A_0 and B_0 are constants.

Let $X' + iY'$ be the resultant of the external force applied to the portion BC of the boundary A. We assume that, as B, C tend to L_∞ and M_∞ respectively (where the sense of description of A is from L_∞ to M_∞), $X' + iY' \to X + iY$, where X and Y are finite.

From 2.40 (1) we have

$$X' - iY' = \frac{1}{2} i \left[{}'\overline{W}(\bar{z}) + \bar{z} W(z) + {}'w(z)\right]_{BC}.$$

If we write

$$z - z_0 = r_1 e^{i\theta_1} \text{ at } B, \quad z - z_0 = r_2 e^{i\theta_2} \text{ at } C$$

we find from (3) and (4) that

$$X' - iY' = \frac{1}{2} i \left\{\bar{A}_1 \left(\ln \frac{r_2}{r_1} - i(\theta_2 - \theta_1)\right) + A_1 \left(e^{-2i\theta_2} - e^{-2i\theta_1}\right)\right.$$
$$\left. + B_1 \left(\ln \frac{r_2}{r_1} + i(\theta_2 - \theta_1)\right) + \varepsilon\right\}$$

where $\varepsilon \to 0$ as $B, C \to L_\infty, M_\infty$ respectively.

Since the ratio r_2/r_1 is arbitrary, $X' - iY'$ can not have a definite limit unless

$$\bar{A}_1 + B_1 = 0 \tag{5}$$

and then

$$X - iY = \frac{1}{2}(\bar{A}_1 - B_1)(\alpha_2 - \alpha_1) + \frac{1}{2} i A_1 \left(e^{-2i\alpha_2} - e^{-2i\alpha_1}\right), \tag{6}$$

where α_1, α_2 are the limits of θ_1, θ_2 respectively. In fact, (see fig. 4.20) α_1, α_2 are the angles between the x-axis and the asymptotes HL_∞, HM_∞ respectively.

From (5) and (6) we can find A_1, B_1 in terms of X, Y, α_1, α_2.

4.21. The half-plane

When the elastic material occupies a half-plane the boundary line A is straight and the results of 4.20 simplify. We take the origin in A and the axis Oy normal to the boundary and into the material. The x-axis, which is along the boundary, will then be chosen so that the system xy is right-handed and the sense of description of A is that of Ox, fig. 4.21. The material, which occupies the region L, will then be in the half-plane $y \geqq 0$. In the notation of 4.20 we shall have $\alpha_1 = \pi$, $\alpha_2 = 0$ and therefore 4.20 (5) and (6) become

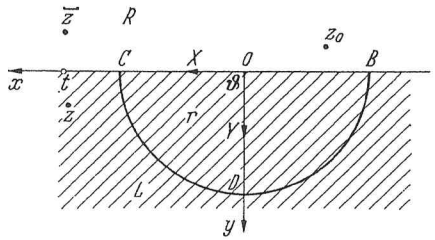

Fig. 4.21

$$\bar{A}_1 + B_1 = 0, \quad X - iY = -\frac{1}{2}\pi(\bar{A}_1 - B_1),$$

whence

$$A_1 = -\frac{1}{\pi}(X + iY), \quad B_1 = \frac{1}{\pi}(X - iY)$$

and so for large values of $|z|$

$$W(z) = -\frac{X + iY}{\pi z} + O\left(\frac{1}{z^2}\right) \tag{1}$$

$$w(z) = \frac{X - iY}{\pi z} + O\left(\frac{1}{z^2}\right). \tag{2}$$

These equations should be contrasted with 2.74 (26), (27).

4.22. Standard concentrated force acting on the boundary of a half-plane

If in 4.21 (1) and (2) we consider only the first terms on the right, the complex stresses given by

$$W(z) = -\frac{X + iY}{\pi z}, \quad w(z) = \frac{X - iY}{\pi z} \tag{1}$$

correspond to a standard concentrated force $X + iY$ applied at the origin O on the boundary. To see this describe a semicircle BCD of radius r about O, D lying in the material, fig. 4.21. The force F exerted by the material within BDC on the rest of the material is, by 2.40,

$$F = -\frac{1}{2} i \left[W(z) + z\overline{W}(\bar{z}) + \overline{w}(\bar{z})\right]_{BDC}$$

$$= -\frac{1}{2} i \left[-\frac{X+iY}{\pi} \ln z - z\frac{X-iY}{\pi \bar{z}} + \frac{X+iY}{\pi} \ln \bar{z}\right]_{BDC} = X + iY$$

since $z_B = r\, e^{i\pi}$, $z_C = r$.

Again from 2.40 (2) we find that the moment M about the origin is zero. Therefore the material within the semicircle exerts on the rest of the material a resultant force $X + iY$ through O and this resultant is independent of r. Thus when $r \to 0$ we get a concentrated force $X + iY$ at O (cf. 4.11).

We can use 1.24 (2) and (3) to calculate the stress in polar coordinates (r, θ). In fact

$$\widehat{rr} + \widehat{\theta\theta} = W(z) + \overline{W}(\bar{z}) = -\frac{2}{\pi r}(X \cos\theta + Y \sin\theta)$$

$$\widehat{\theta\theta} - \widehat{rr} + 2i\,\widehat{r\theta} = [\bar{z}W'(z) + w(z)]\, e^{2i\theta} = \frac{2}{\pi r}(X \cos\theta + Y \sin\theta)\,,$$

whence

$$\widehat{rr} = -\frac{2}{\pi r}(X \cos\theta + Y \sin\theta)\,, \quad \widehat{\theta\theta} = 0\,, \quad \widehat{r\theta} = 0.$$

Thus the well known but remarkable fact emerges that a standard concentrated force, with arbitrary components, applied at a point of the boundary of an elastic half plane produces purely radial stress from the point, and this stress has the same magnitude at all points distant r from the point of application of the concentrated force, cf. 7.41.

4.23. Analytic continuation of the complex stress $W(z)$

We consider material occupying the lower half-plane $y > 0$ as described in 4.21 and depicted in fig. 4.21.

The following assumptions will be made

(i) No body-force is operative,

(ii) Stress and rotation vanish at infinity,

(iii) The deformation is non-dislocational.

Assumption (iii) is, of course, satisfied automatically for a simply connected region, 2.61.

The forms of the complex stresses for large values of $|z|$ are then given by 4.20 (1), (2).

Now $W(z)$, $w(z)$ are holomorphic functions whose values are known everywhere in the region L. In the region R where the material is absent

these functions are undefined. We can therefore define $W(z)$ for z in R in any way we please. It will appear shortly that the mathematically useful definition will be that obtained as follows.

From 4.12 (1) we have

$$2\,(\widehat{yy} - i\,\widehat{xy}) = W(z) + \overline{W}(\bar{z}) + z\overline{W}'(\bar{z}) + \overline{w}(\bar{z})\,, \quad (z \text{ in } L) \tag{1}$$

In this write zero for $2\,(\widehat{yy} - i\,\widehat{xy})$ and z for $\bar{z}$ wherever $\bar{z}$ occurs. We then get

$$W(z) = -\overline{\{W(\bar{z}) + \bar{z}W'(\bar{z}) + w(\bar{z})\}}\,, \quad (z \text{ in } R) \tag{2}$$

where the bar, as usual, denotes the conjugate complex, and this formula we take as our definition of $W(z)$ for z in R. For when z is in R, $\bar{z}$ is in L and therefore the quantities in the curled brackets are known, and so the right-hand side of (2) is a well-defined holomorphic function and therefore determines $W(z)$ as such.

We note that in (2) $W(z) = \Omega(z)$ where $\Omega(z)$ is the function defined in 4.12 (2), so that we could alternatively develop the foregoing by defining $W(z)$ to be the same as $\Omega(z)$ when z is in R.

With this definition if z is in L (and therefore $\bar{z}$ in R), we have

$$W(\bar{z}) = -\overline{\{W(z) + zW'(z) + w(z)\}} \qquad (z \text{ in } L) \tag{3}$$

and therefore from (1)

$$2\,(\widehat{yy} - i\,\widehat{xy}) = W(z) - W(\bar{z}) + (z - \bar{z})\,\overline{W}'(\bar{z}) \qquad (z \text{ in } L) \tag{4}$$

If in this result we let $z \to t$, a point of the boundary, from the region L, then $\bar{z} \to t$ from the region R and therefore

$$2\,(\widehat{yy} - i\,\widehat{xy})^L = W^L(t) - W^R(t)\,. \tag{5}$$

Thus it appears that at an unloaded part of the boundary, $W^L(t) = W^R(t)$, in other words the function $W(z)$ as defined by (2) continues $W(z)$ analytically from the region L into the region R through the unloaded parts (if any) of the boundary.

If we take the conjugate of both sides of (2) and then write z for $\bar{z}$, we get

$$w(z) = -\overline{W}(z) - W(z) - zW'(z) \qquad (z \text{ in } L)\,. \tag{6}$$

We note that in this result $\overline{W}(z) = \overline{W(\bar{z})}$ which is defined by (2) when z is in L and therefore $\bar{z}$ is in R.

Thus it appears that the stress combinations can be expressed in terms of the single complex stress $W(z)$ extended to the whole plane by (2), for $w(z)$ can be eliminated by the use of (6). A similar remark applies to the displacements.

4.30. Boundary conditions for the half-plane

We denote by $p(t)$ the normal pressure and $s(t)$ the shear in the positive sense of the x-axis applied at the point t of the boundary, so that

$$\widehat{yy}^L = -p(t)\,, \quad \widehat{xy}^L = -s(t)\,.$$

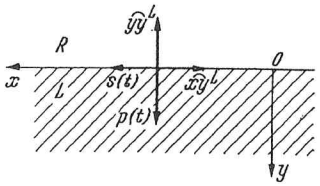

Fig. 4.30

Then from 4.23 (5)

$$W^L(t) - W^R(t) = -2\{p(t) - is(t)\}\,. \tag{1}$$

This is the first or *stress boundary condition*

From 2.12 (2) for the displacement $\upsilon = u + iv$ we have

$$4\mu\,\upsilon = \varkappa W(z) - z\overline{W}(\bar z) - \overline{w}(\bar z) + \text{constant}. \tag{2}$$

Differentiate with respect to x. Then $\partial z/\partial x = \partial \bar z/\partial x = 1$ and so

$$4\mu\frac{\partial \upsilon}{\partial x} = 4\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \varkappa W(z) - \overline{W}(\bar z) - z\overline{W}'(\bar z) - \overline{w}(\bar z)\,. \tag{3}$$

Eliminate $w(z)$ with the aid of 4.23 (3). Then

$$4\mu\frac{\partial \upsilon}{\partial x} = \varkappa W(z) + W(\bar z) - (z - \bar z)\,\overline{W}'(\bar z)\,, \qquad (z \text{ in } L)\,. \tag{4}$$

Now let

$$\upsilon(t) = u(t) + iv(t) \tag{5}$$

be the displacement at the point t of the boundary. In (4) let $z \to t$ from L, so that $\bar z \to t$ from R and we get the second or *displacement boundary condition*

$$4\mu\frac{\partial \upsilon(t)}{\partial(t)} = \varkappa W^L(t) + W^R(t)\,. \tag{6}$$

These two boundary conditions enable us to deal with situations in which either stress or displacement or combinations of them are given on the boundary.

Note that (6) can be integrated to give

$$4\mu\,\upsilon(t) = \varkappa W^L(t) + W^R(t) + \text{constant}. \tag{7}$$

We note also that by definition, for any analytic function, $\overline{W}(\bar z) = \overline{W(z)}$ and therefore if we let $z \to t$ from L or from R, we get respectively

$$\overline{W}^R(t) = \overline{W^L(t)}\,, \quad \overline{W}^L(t) = \overline{W^R(t)}\,. \tag{8}$$

4.32. Solution of Problem I for the half-plane

Here we are given the *pressure* $p(t) = -\widehat{yy}^L$ and the shear $s(t) = -\widehat{xy}^L$ at every point t of the boundary Ox or A. Then from 4.30 (1) the stress boundary condition is

$$W^L(t) - W^R(t) = -2\{p(t) - is(t)\} \tag{1}$$

and assuming the stress to vanish at infinity the solution is given by the first Plemelj formula as

$$W(z) = \frac{1}{2\pi i}\int_A \frac{-2\{p(t) - is(t)\}\,dt}{t - z}\,. \tag{2}$$

For another method of approach to this problem see MILNE-THOMSON (2).

4.34. Solution of Problem II for the half-plane

Here we are given the displacement

$$\upsilon(t) = u^L + iv^L \tag{1}$$

at every point of the boundary Ox or A.

From 4.30 (6) the displacement boundary condition is

$$\varkappa W^L(t) + W^R(t) = 4\mu\,\upsilon'(t)\,. \tag{2}$$

We introduce an auxiliary function $F(z)$ defined by

$$F(z) = W(z)\,, \qquad (z \text{ in } L) \tag{3}$$

$$\varkappa F(z) = -W(z) \qquad (z \text{ in } R) \tag{4}$$

Then (2) can be written in the form

$$F^L(t) - F^R(t) = 4\mu\,\upsilon'(t)/\varkappa \tag{5}$$

and therefore by the first Plemelj formula the solution which vanishes at infinity is

$$F(z) = \frac{1}{2\pi i\varkappa}\int_A \frac{4\mu\upsilon'(t)\,dt}{t - z} \tag{6}$$

and therefore from (3) and (4)

$$W(z) = \frac{1}{2\pi i\varkappa}\int_A \frac{4\mu\upsilon'(t)\,dt}{t - z} \quad \text{for } z \text{ in } L \tag{7}$$

$$W(z) = -\frac{1}{2\pi i}\int_A \frac{4\mu\upsilon'(t)}{t - z} \quad \text{for } z \text{ in } R. \tag{8}$$

4.35. An application of Problem II

Let the displacement at the boundary be

$$\upsilon(t) = i\,h\tan^{-1} t\,, \tag{1}$$

where h is a small constant and $-\pi/2 < \tan^{-1} t < \pi/2$. Then as t varies from $-\infty$ to ∞ the displacement ranges from $-\pi h/2$ to $\pi h/2$. From the

boundary condition

$$\varkappa W^L(t) + W^R(t) = \frac{4\mu h i}{1+t^2},$$

$$W(z) = \frac{2\mu h}{\varkappa}\left\{\frac{i-z}{1+z^2}\right\}, \quad (z \text{ in } L) \tag{2}$$

$$W(z) = -2\mu h\left\{\frac{-i-z}{1+z^2}\right\}, \quad (z \text{ in } R). \tag{3}$$

These results are obtained by evaluating

$$F(z) = \frac{1}{2\pi\varkappa}\int_{-\infty}^{\infty}\frac{4\mu i h\,dt}{(1+t^2)(t-z)}.$$

The integral can be replaced by one round the infinite semicircle in the region L. Since the integral round the curved portion $\to 0$, we have the required result by evaluating the residues at the internal poles $t = i$, $t = z$. For (3) the only pole is $t = i$.

4.40. Solution of Problem III for the half-plane

Let the material be bounded by the x-axis A and occupy the half-plane $y > 0$. Let

$$A' = A_1 + A_2 + \cdots + A_n \tag{1}$$

be the union of a finite number of segments $A_s = a_s b_s$, $(s = 1, 2, \ldots, n)$ of the real axis numbered so that the end points are encountered in the order $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ when moving in the positive sense along the x-axis, and on A' let the displacement be given

$$\upsilon(t) = u^L + i v^L \quad \text{on } A'. \tag{2}$$

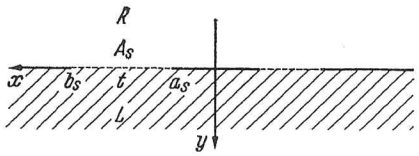

Fig. 4.40

On the complementary segments whose union will be denoted by A'', let the pressure and shear be given

$$\widehat{yy}^L - i\,\widehat{xy}^L = -p(t) + i\,s(t) \quad \text{on } A''. \tag{3}$$

Then from 4.30 (4) and (1) we have the boundary conditions

$$\varkappa W^L(t) + W^R(t) = 4\mu\,\upsilon'(t) \quad \text{on } A' \tag{4}$$

$$W^L(t) - W^R(t) = -2\{p(t) - i\,s(t)\} \quad \text{on } A'' \tag{5}$$

and these may be written as the single condition

$$W^L(t) - c\,W^R(t) = f(t), \tag{6}$$

where

$$c = -\frac{1}{\varkappa} \text{ and } f(t) = \frac{4\mu\upsilon'(t)}{\varkappa} \text{ on } A' \tag{7}$$

$$c = 1 \quad \text{and} \quad f(t) = -2\{p(t) - i\,s(t)\} \quad \text{on } A''. \tag{8}$$

As in 3.40 we write

$$\chi(z) = \frac{\{(z-b_1)(z-b_2)\ldots(z-b_n)\}^{\tau-1}}{\{(z-a_1)(z-a_2)\ldots(z-a_n)\}^{\tau}}, \tag{9}$$

where the branch of $\chi(z)$ is determined by

$$\lim_{z\to\infty} z^n \chi(z) = 1 \tag{10}$$

and where τ is given by

$$e^{2\pi i\tau} = -1/\varkappa \quad \text{or} \quad \tau = \frac{1}{2} + i\frac{\ln\varkappa}{2\pi}. \tag{11}$$

Also

$$\chi^L(t) = e^{2\pi i\tau}\chi^R(t) \quad \text{on } A' \tag{12}$$

$$\chi^L(t) = \chi^R(t) \quad \text{on } A''. \tag{13}$$

With these preliminaries the solution of Problem III which vanishes at infinity is

$$W(z) = \frac{\chi(z)}{2\pi i}\int_A \frac{f(t)\,dt}{\chi^L(t)(t-z)} + P_{n-1}(z)\,\chi(z), \tag{14}$$

wherein $P_{n-1}(z)$ is a polynomial of degree $n-1$ at most

$$P_{n-1}(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}. \tag{15}$$

The coefficients a_s $(s = 0, \ldots, n-1)$ have to be determined from the conditions of the problem.

Multiply (14) by z and let $z \to \infty$. Then from (10) and 4.21 (1) we have

$$a_{n-1} = -\frac{X + iY}{\pi}, \tag{16}$$

where $X + iY$ is the vector resultant of the external force applied to the boundary. It remains to determine $a_0, a_1, \ldots, a_{n-2}$.

Let $X_s + iY_s$ be the vector resultant of the external force applied to the segment A_s. Then

$$X_s + iY_s = \int_{a_s}^{b_s}[s(t) + i\,p(t)]\,dt = i\int_{a_s}^{b_s}[p(t) - i\,s(t)]\,dt. \tag{17}$$

Since $W(z)$ is given by (14) we can use the Plemelj formulae to find $p(t) - i\,s(t)$ from (5) as shown in 4.41. Thus (17) will give a linear relation between the n coefficients $a_0, \ldots, a_{n-1}$ of (15). Thus if we are

given the external force $X_s + iY_s$ applied to each of the n segments, we shall have n linear equations to find the coefficients.

Again (14) determines $W(z)$ and therefore from (4) the derivative $\upsilon'(t)$ of the displacement at every point of the boundary, but it determines the displacement only up to an added constant c_s in the form $\upsilon(t) + c_s$. If we postulate that the constants c_s are all to be equal on each of the segments A_s, we can again determine the coefficients without the knowledge of $X_s + iY_s$. Indeed if we integrate from the end of one segment A_s to the beginning of the next we have

$$\int_{b_s}^{a_{s+1}} \upsilon'(t)\, dt = \{\upsilon(a_{s+1}) + c_{s+1}\} - \{\upsilon(b_s) + c_s\}$$

and the condition that $c_s = c_{s-1}$ leads to a linear equation between the coefficients $a_0, a_1, \ldots, a_{n-1}$.

We can write down $n - 1$ such equations and these together with (11) suffice to determine the coefficients.

4.41. Determination of pressure and shear on the boundary

Referring to 4.40 the pressure $p(t)$ and the shear $s(t)$ are given on the part A'' of the boundary and it remains to determine them on the remaining part A'. We use 4.30 (1) which gives

$$-2\{p(t_0) - is(t_0)\} = W^L(t_0) - W^R(t_0)\,, \quad (t_0 \text{ on } A') \tag{1}$$

Also from 4.40 (4)

$$4\mu\,\upsilon(t_0) = \varkappa W^L(t_0) + W^R(t_0)\,, \quad (t_0 \text{ on } A') \tag{2}$$

where $\upsilon(t_0)$ is given. By addition we get

$$-2\{p(t_0) - i\,s(t_0)\} = (\varkappa + 1)\, W^L(t_0) - 4\mu\,\upsilon(t_0)\,. \tag{3}$$

We determine $W^L(t_0)$ from 4.40 (14) and the Plemelj formula, namely

$$W^L(t_0) = \frac{1}{2} f(t_0) + \chi^L(t_0)\, P_{n-1}(t_0) + \frac{\chi^L(t_0)}{2\pi i} \int_A \frac{f(t)\, dt}{\chi^L(t)\,(t - t_0)} \tag{4}$$

and therefore from (3) and 4.40 (7)

$$p(t_0) - i\,s(t_0) = \frac{\varkappa - 1}{\varkappa}\mu\,\upsilon'(t_0) - \frac{1}{2}(\varkappa + 1)\,\chi^L(t_0)\,\{F(t_0) + P_{n-1}(t_0)\} \tag{5}$$

where

$$F(t_0) = \frac{1}{2\pi i} \int_A \frac{f(t)\, dt}{\chi^L(t)\,(t - t_0)}\,. \tag{6}$$

We proceed to consider various applications of the foregoing. The typical situation will be that in which a rigid beam acted upon by forces and couples lies in equilibrium on an elastic half-plane. Analogous

interpretations are, for example, a (two-dimensional) building on elastic foundations, a punch, die or chisel acting on an elastic half-plane, the action of one geological stratum on another.

4.42. Rigid beam with a horizontal base which adheres

A rigid beam whose base is straight and of breadth $2\,l$ is placed with the base horizontal on a half-plane of elastic material and is pushed into the material with a vertical force P the base remaining horizontal and being welded or adhering to the material with which it is in contact.

Take the origin at the centre of the base.

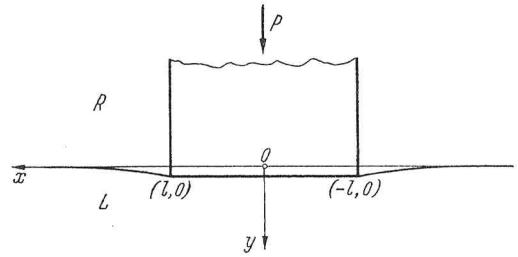

Fig. 4.42

The displacement is constant and so $\upsilon'(t) = 0$ under the base. Therefore from 4.40 (14), since $n = 1$

$$W(z) = P_0\,\chi(z)\,, \quad P_0 \text{ a constant} \tag{1}$$

$$\chi(z) = \frac{(z - l)^{\tau - 1}}{(z + l)^{\tau}}\,, \quad \tau = \frac{1}{2} + i\lambda\,, \quad \lambda = \frac{\ln \varkappa}{2\pi}\,. \tag{2}$$

Also from 4.40 (16)

$$P_0 = a_0 = -\frac{iP}{\pi}$$

so that

$$W(z) = -\frac{iP}{\pi}(z - l)^{-1/2 + i\lambda}(z + l)^{-1/2 - i\lambda}\,, \tag{3}$$

and this is the solution.

To find the pressure under the beam, since $\arg(t - l) = \pi + \arg(l - t)$, we have

$$\chi^L(t) = (l + t)^{-1/2 - i\lambda}(l - t)^{-1/2 + i\lambda}\, e^{i\pi(\tau - 1)}\,. \tag{4}$$

Therefore from 4.41-(5)

$$p(t) = \frac{(\varkappa + 1)\,P}{2\pi\sqrt{\varkappa}} \frac{1}{\sqrt{(l^2 - t^2)}} \cos\left[\left(\frac{\ln \varkappa}{2\pi}\right) \ln \frac{l + t}{l - t}\right]. \tag{5}$$

Now $p(t) \geqq 0$ so long as

$$\frac{\ln \varkappa}{2\pi} \ln \frac{l+t}{l-t} \leqq \frac{\pi}{2}$$

so that even if $\varkappa = 3$, $p(t)$ is positive when t is in the interval $(-0.9998\, l, +0.9998\, l)$ and contact will not cease owing to $p(t)$ becoming negative except very close to the edges of the beam, and indeed in the neighbourhood of such an edge we should have plastic yield anyway.

4.43. Adhering rigid beam tilted by a couple

Suppose a beam whose base AB adheres to an elastic half-plane, to be held tilted by a couple of moment M.

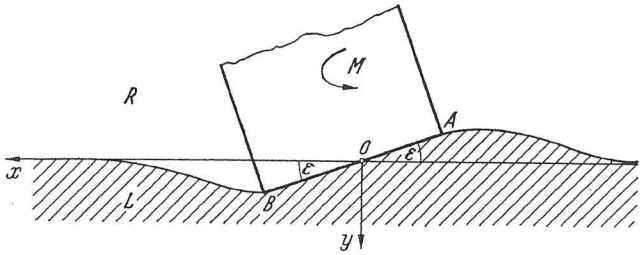

Fig. 4.43

Then if $p(t)$ is the pressure on the plane, taking moments about the origin,

$$M = \int_{AB} t\, P(t)\, dt\,. \tag{1}$$

As an example suppose the rigid beam of 4.42 is held tilted through the small angle ε by a couple of moment M.

The additional displacement due to rotation is $i\,\varepsilon\, z$ and therefore at the point t of the base

$$\upsilon(t) = i\,\varepsilon\, t + \text{constant}, \quad (-l < t < l)\,. \tag{2}$$

Referring to 4.40 (15), (16) we see that $n = 1$ and so

$$P_0(z) = a_0 = 0\,, \quad \text{since } X + iY = 0$$

Therefore from 4.40 (14)

$$W(z) = \frac{\chi(z)}{2\pi i} \int_{-l}^{l} \frac{4\,\mu\, i\, \varepsilon\, dt}{\varkappa\, \chi^L(t)\,(t-z)}\,, \quad \chi(z) = \frac{(z-l)^{\tau-1}}{(z+l)^{\tau}}\,, \tag{3}$$

$$\tau = \frac{1}{2} + \frac{i \ln \varkappa}{2\pi} = \frac{1}{2} + i\lambda\,. \tag{4}$$

Now if C is a lacet round AB, we have from 3.41

$$\frac{1}{2\pi i} \int_{-l}^{l} \frac{dt}{\chi^L(t)\,(t-z)} = \frac{\varkappa}{\varkappa+1}\, \frac{1}{2\pi i} \int_C \frac{dt}{\chi(t)\,(t-z)}\,. \tag{5}$$

Expanding in inverse powers of t we find

$$\frac{1}{\chi(t)} = t + 2i\lambda l + O\left(\frac{1}{t}\right).$$

Therefore from 3.13 (10)

$$\frac{1}{2\pi i} \int \frac{dt}{\chi(t)\,(t-z)} = \frac{1}{\chi(z)} - (z + 2i\lambda l)\,.$$

Therefore from (3)

$$W(z) = \frac{4\,\varepsilon\,\mu i}{\varkappa+1} \{1 - (z + 2i\lambda l)\, \chi(z)\} \tag{6}$$

which solves the problem in terms of ε

We now proceed to express the solution in terms of the given moment M.

The pressure under the beam can be found from 4.41 (5) or even more simply in this case from

$$-2\,(p(t) - is(t))$$
$$= W^L(t) - W^R(t) = \frac{4\,\varepsilon\,\mu i}{\varkappa+1} [-(t + 2i\lambda l)\, \{\chi^L(t) - \chi^R(t)\}]\,.$$

But $\chi^R(t) = -\varkappa \chi^L(t)$ on AB and so

$$p(t) - is(t) = 2\varepsilon\,\mu i\, \{(t + 2i\lambda l)\, \chi^L(t)\}\,.$$

Therefore, using the method of 3.41

$$\begin{aligned} M &= \int_{-l}^{l} t\, p(t)\, dt = \operatorname{Re} \int_{-l}^{l} t\, \{p(t) - is(t)\}\, dt \\ &= \operatorname{Re} \frac{2\varepsilon\,\mu i}{\varkappa+1} \oint_C t\, (t + 2i\lambda l)\, \chi(t)\, dt\,. \end{aligned} \tag{7}$$

Expanding the integrand in powers of $1/t$, which is justifiable if C is, say, enlarged to a circle of sufficiently large radius, the coefficient of $1/t$ is found to be $\frac{1}{2}(1 + 4\lambda^2) l^2$ and therefore

$$M = \frac{2\pi\mu\varepsilon}{(\varkappa+1)} (1 + 4\lambda^2)\, l^2 \tag{8}$$

so that

$$W(z) = \frac{2Mi}{\pi(1 + 4\lambda^2) l^2} \{1 - (z + 2i\lambda l)\, \chi(z)\}\,. \tag{9}$$

We get the case when the beam is both pushed down and twisted by adding (9) to 4.42 (3).

4.50. The geometrical conditions at an interface

Consider two elastic two-dimensional bodies B_1, B_2, held just in contact at the origin. Let the equations of the sections be

$$y_1 = \varphi_1(x)\,, \quad y_2 = \varphi_2(x) \tag{1}$$

and generally let suffix 1 refer to B_1 and suffix 2 to B_2.

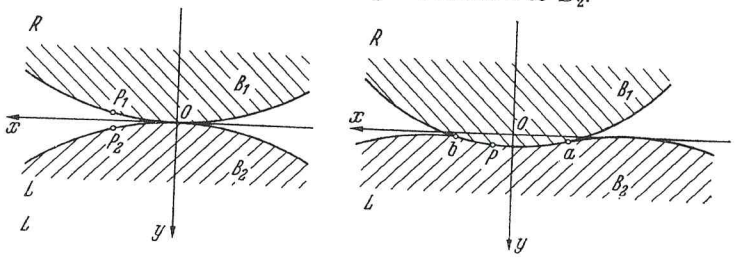

Fig. 4.50

If the body B_1 is now pressed against B_2 by external applied force an interface of contact will be formed in which an arc ab is common to both bodies.

In this process the points $P_1(x_1, y_1)$ of B_1 and $P_2(x_2, y_2)$ of B_2 will come into coincidence at a common point $P.(t, y)$ of ab, while P_1, P_2 will undergo displacements (u_1, v_1), (u_2, v_2) for which

$$x_1 + u_1 = x_2 + u_2 = t\,, \quad y_1 + v_1 = y_2 + v_2 = y \tag{2}$$

and therefore, using (1)

$$\varphi_1(x_1) + v_1 = \varphi_2(x_2) + v_2$$

or

$$\varphi_1(t - u_1) + v_1 = \varphi_2(t - u_2) + v_2\,. \tag{3}$$

These are the geometrical conditions at the interface.

We now turn to the case where the displacements are small, and the interface ab can be assimilated to a portion of the x-axis. Obvious necessary conditions are that $\varphi_1(x)$, $\varphi_1'(x)$, $\varphi_2(x)$, $\varphi_2'(x)$ should be small on the interface. It then follows that, to the first order, (3) becomes

$$\varphi_1(t) + v_1 = \varphi_2(t) + v_2 \quad \text{on} \quad ab \tag{4}$$

or in the notation of "left" and "right".

$$v_2^L - v_1^R = \varphi_1(t) - \varphi_2(t) = \Phi(t)\,, \tag{5}$$

where $\Phi(t)$ is a known function.

Differentiating this we have

$$v_2'^L - v_1'^R = \Phi'(t) \tag{6}$$

which is the geometrical displacement boundary condition for small deformation.

4.52. Conditions satisfied by $W(z)$ for limiting friction at the interface

Consider the interface ab between the bodies B_1, B_2 of 4.50. If friction is limiting, the pressure $p(t)$ and shear $s(t)$ are connected by

$$s(t) = f\,p(t) = \tan\gamma\, p(t)\,, \tag{1}$$

where f is the coefficient and γ is the angle of limiting friction. Therefore at the interface

$$p(t) - i\,s(t) = (1 - i\,f)\,p(t) = e^{-i\gamma}\sec\gamma\, p(t)\,. \tag{2}$$

We now suppose that the bodies approximate to half-planes bounded by the x-axis. For the body B_2 which occupies the region L we have from the stress boundary condition on the interface of contact ab

$$-2e^{-i\gamma}\sec\gamma\, p(t) = W_2^L(t) - W_2^R(t) \tag{3}$$

and similarly for B_1 which occupies the region R

$$2e^{-i\gamma}\sec\gamma\, p(t) = W_1^L(t) - W_1^R(t)\,. \tag{4}$$

Therefore by subtraction of (4) from (3)

$$W_1^L(t) + W_2^L(t) = W_1^R(t) + W_2^R(t) \tag{5}$$

on the interface.

But $W_1(z)$ and $W_2(z)$ are holomorphic in the whole plane cut along the interface. Therefore (5) shows that $W_1(z) + W_2(z)$ is holomorphic in the whole plane including the interface. If the stresses tend to zero at infinity it follows from Liouville's theorem that

$$W_1(z) + W_2(z) \equiv 0\,. \tag{6}$$

Returning to (3) we have by taking the complex conjugate

$$-2e^{i\gamma}\sec\gamma\, p(t) = \overline{W_2^L(t)} - \overline{W_2^R(t)} = \overline{W}_2^R(t) - \overline{W}_2^L(t)$$

from 4.30 (8) and combining this with (3)

$$e^{-i\gamma}\{\overline{W}_2^R(t) - \overline{W}_2^L(t)\} = e^{i\gamma}\{W_2^L(t) - W_2^R(t)\}$$

and so

$$\{\overline{W}_2^L(t) + e^{2i\gamma}W_2^L(t)\} = \overline{W}_2^R(t) + e^{2i\gamma}W_2^R(t)$$

on the interface.

But $W_2(z)$ is holomorphic at all points not on the interface. Therefore $\overline{W}_2(z) + e^{2i\gamma}W_2(z)$ is holomorphic in the whole plane including the interface, and vanishes at infinity. Therefore by Liouville's theorem

$$\overline{W}_2(z) + e^{2i\gamma}W_2(z) \equiv 0 \tag{7}$$

and similarly

$$\overline{W}_1(z) + e^{2i\gamma}W_1(z) \equiv 0\,. \tag{8}$$

Thus (6), (7), (8) are the conditions satisfied everywhere, when the friction at the interface is limiting. Of these condition (6) is independent of the angle of friction and therefore subsists for smooth contact.

For smooth contact (7) and (8) become

$$\overline{W}_2(z) + W_2(z) \equiv 0\,, \quad \overline{W}_1(z) + W_1(z) \equiv 0\,.$$

Combining these with (6) we have for smooth contact

$$\overline{W}_1(z) = -W_1(z) = W_2(z)\,, \tag{9}$$

$$\overline{W}_2(z) = -W_2(z) = W_1(z)\,. \tag{10}$$

4.54. The equation satisfied by $W(z)$ at an interface

Assuming limiting equilibrium, we combine 4.50 (6) with 4.52 (6), (7), (8) thereby obtaining the boundary equations to determine $W_1(z)$ and $W_2(z)$.

From the displacement boundary condition for B_2, 4.30 (6), and its complex conjugate we have

$$4\mu_2\,(u_2' + i v_2') = \varkappa_2 W_2^L(t) + W_2^R(t) \tag{1}$$

$$4\mu_2\,(u_2' - i v_2') = \varkappa_2 \overline{W_2^L(t)} + \overline{W_2^R(t)} = \overline{W}_2^L(t) + \varkappa_2\,\overline{W}_2^R(t) \quad \text{from 4.30 (8)}$$

$$= -e^{2i\gamma} W_2^L(t) - \varkappa_2\, e^{2i\gamma} W_2^R(t) \quad \text{from 4.52 (7).} \tag{2}$$

By subtraction

$$8\mu_2\, i v_2' = (\varkappa_2 + e^{2i\gamma})\, W_2^L(t) + (\varkappa_2\, e^{2i\gamma} + 1)\, W_2^R(t)\,. \tag{3}$$

By similar steps we find that

$$8\mu_1\, i v_1' = (\varkappa_1\, e^{2i\gamma} + 1)\, W_1^L(t) + (\varkappa_1 + e^{2i\gamma})\, W_1^R(t)$$

$$= -(\varkappa_1\, e^{2i\gamma} + 1)\, W_2^L(t) - (\varkappa_1 + e^{2i\gamma})\, W_2^R(t) \tag{4}$$

from 4.52. (6) Combining (3) and (4) with 4.50 (6) we have the equation satisfied by $W_2(z)$ at the boundary, namely

$$W_2^L(t) = c\, W_2^R(t) + \Phi_2'(t)\,, \quad \text{on the interface.} \tag{5}$$

$$c = -\frac{\dfrac{\varkappa_2 e^{2i\gamma} + 1}{\mu_2} + \dfrac{\varkappa_1 + e^{2i\gamma}}{\mu_1}}{\dfrac{\varkappa_2 + e^{2i\gamma}}{\mu_2} + \dfrac{\varkappa_1 e^{2i\gamma} + 1}{\mu_1}}\,, \quad \Phi_2'(t) = \frac{8\, i\, \Phi'(t)}{\dfrac{\varkappa_2 + e^{2i\gamma}}{\mu_2} + \dfrac{\varkappa_1 e^{2i\gamma} + 1}{\mu_1}}\,. \tag{6}$$

From (5) we can determine $W_2(z)$ and hence $W_1(z) = -W_2(z)$.

In fact writing

$$e^{2\pi i \tau} = c\,, \quad \chi(z) = (z-b)^{\tau-1}\,(z-a)^{-\tau} \tag{7}$$

we have, when the stress vanishes at infinity

$$W_2(z) = \frac{\chi(z)}{2\pi i} \int_{ab} \frac{\Phi_2'(t)}{\chi^L(t)\,(t-z)}\, dt + P_0 \chi(z)\,, \tag{8}$$

where P_0 is a constant.

It is often convenient, instead of $\chi(z)$, to use the function

$$\psi(z) = (b-z)^{\tau-1}\,(z-a)^{-\tau} = e^{-i\pi(\tau-1)}\,\chi(z)\,, \tag{9}$$

since

$$\arg(b-z) = -\pi + \arg(z-b)\,.$$

This function has the property that

$$\psi^L(t) = (b-t)^{\tau-1}\,(t-a)^{-\tau} = e^{2\pi i \tau}\,\psi^R(t)\,, \tag{10}$$

which is real-valued when a, b, t are real and $a < t < b$.

Also

$$\lim_{z\to\infty} z\,\psi(z) = e^{-i\pi(\tau-1)}\,, \tag{11}$$

from (9) and 3.40 (4).

Notes.

(i) If the contact is smooth we put $\gamma = 0$.

(ii) If the material composing one of the bodies is rigid, say B_1, we put $\mu_1 = \infty$.

(iii) The *form* of the boundary equation (5) is unaffected by particular values of the elastic constants or the angle of friction.

(iv) The assumption that B_1 is rigid does not detract from the generality of the method of solution.

(v) When B_1 is rigid, its general shape is unimportant, for conditions at the interface of contact alone intervene.

4.60. Smooth rigid beam with straight horizontal base

This is the problem of 4.42 with adhesion replaced by smooth contact. We use 4.54 with a proper adaptation of the constants, the beam being the body B_1. We put $\mu_1 = \infty$ for rigidity, $\gamma = 0$ for smoothness, $\Phi'(t) = 0$ since in 4.50 (1) $\varphi_1(x) =$ constant, $\varphi_2(x) =$ constant. Therefore from 4.54 (8), dropping the now unnecessary suffix 2,

$$W(z) = P_0 \chi(z) = -i P_0\, \psi(z)\,, \tag{1}$$

since in this case $c = -1$ and therefore $\tau = \frac{1}{2}$. Therefore

$$W(z) = -i P_0\,(l-z)^{-1/2}\,(l+z)^{-1/2}.$$

But from 4.21 (1)

$$\lim_{z\to\infty} z\, W(z) = -i P/\pi\,.$$

Therefore

$$W(z) = -\frac{P}{\pi}\,\frac{1}{\sqrt{(l^2 - z^2)}} \tag{2}$$

which solves the problem.

The pressure under the base is given by

$$-2\{p(t)-i\,s(t)\}=-\frac{P}{\pi}\{\psi^L(t)-\psi^R(t)\}$$

whence

$$p(t)=\frac{P}{\pi}\frac{1}{\sqrt{(l^2-t^2)}}$$

which is infinite at the ends where plastic yielding must occur.

4.61. Smooth rigid beam with straight inclined base

Here we suppose the beam to be acted upon by a vertically downwards force P as in 4.42, not applied along the axis of symmetry of the beam, so that reduced to the centre of the base, there is a force iP and a couple M, see fig. 4.43. The result is that the base tilts through a (small) angle ε. With the same notations as in 4.60, we now have $\Phi'(t)=i\,\varepsilon$, and therefore from 4.54 (8), (9), noting that $\tau=\frac{1}{2}$,

$$W(z)=\frac{\psi(z)}{2\pi i}\int_{-l}^{l}\frac{8\mu i\varepsilon}{\varkappa+1}\frac{dt}{\psi^L(t)\,(t-z)}+P_0\psi(z) \tag{1}$$

and since $\lim\limits_{z\to\infty} zW(z)=-iP/\pi$, we have $iP_0=-iP/\pi$, while

$$\psi(z)=(l^2-z^2)^{-1/2}. \tag{2}$$

Thus

$$W(z)=\frac{8\mu i\varepsilon}{\varkappa+1}\frac{1}{\sqrt{(l^2-z^2)}}\frac{1}{2\pi i}\int_{-l}^{l}\frac{\sqrt{(l^2-t^2)}}{t-z}\,dt-\frac{P}{\pi\sqrt{(l^2-z^2)}}. \tag{3}$$

Applying the method of 3.41 we have

$$\frac{1}{2\pi i}\int_{-l}^{l}\frac{\sqrt{(l^2-t^2)}\,dt}{t-z}=\frac{1}{2\pi i}\int_{-l}^{l}\frac{dt}{\psi^L(t)\,(t-z)}=\frac{1}{2}\frac{1}{2\pi i}\oint_C\frac{dt}{\psi(t)\,(t-z)}$$

where C is a lacet round the base. Expand the integrand in powers of $1/t$ and apply 3.13 (10). Then the required integral is $\frac{1}{2}\{i\,z+\sqrt{(l^2-z^2)}\}$ and therefore

$$W(z)=\frac{4\mu\varepsilon}{\varkappa+1}\left\{i-\frac{z}{\sqrt{(l^2-z^2)}}\right\}-\frac{P}{\pi}\frac{1}{\sqrt{(l^2-z^2)}}. \tag{4}$$

For the pressure under the beam we have

$$-2\{p(t)-i\,s(t)\}=W^L(t)-W^R(t)=-\left(\frac{4\mu\varepsilon t}{\varkappa+1}+\frac{P}{\pi}\right)(\psi^L(t)-\psi^R(t))$$

and since

$$\psi^R(t)=-\psi^L(t)=-\frac{1}{\sqrt{(l^2-t^2)}}$$

we have

$$p(t)=\left\{\frac{4\pi\mu\varepsilon t}{\varkappa+1}+\frac{P}{\pi}\right\}\frac{1}{\sqrt{(l^2-t^2)}} \tag{5}$$

and therefore $p(t)\geqq 0$ for all t, in particular $t=-l$, if

$$P\geqq\frac{4\pi\mu\varepsilon l}{\varkappa+1}. \tag{6}$$

The moment can be found by substituting $t=l\sin\theta$ to give

$$M=\int_{-l}^{l}t\,p(t)\,dt=\frac{2\mu\pi^2l^2}{\varkappa+1}\varepsilon. \tag{7}$$

4.63. Smooth rigid beam with a curved base

Fig. 4.63 depicts a smooth rigid beam with a curved base, which for simplicity we shall suppose capable only of vertical motion, pressed into

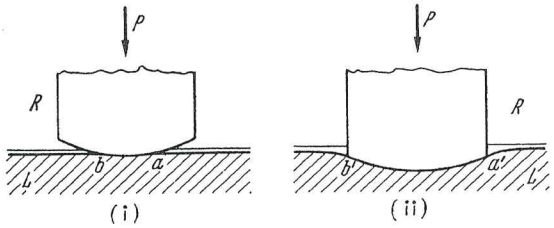

Fig. 4.63

an elastic half-plane by a vertical force P. If P is weak enough (fig. (i)), only part, say ab of the curved base will come into contact with the plane and the pressure will remain bounded and will in fact be zero at a and b. If P is increased sufficiently (fig. (ii)) the corners a', b' (if there are any such corners) will finally come into contact with the plane causing unbounded pressure at these corners. Such corners would not exist if the beam were, for example, a circular cylinder whose axis is parallel to the plane.

Let us seek to determine the points a, b when the corners do not "dig in" and the force P is given. We start with 4.54 (6), putting $\mu_1=\infty$, $\gamma=0$, and assuming that $\Phi_2'(t)$ is given by the profile $y_1=\varphi(x)$ of the beam and $y_2=$ constant for the plane. Dropping the suffix 2 we then have $c=-1$, $\tau=\frac{1}{2}$ and so

$$W^L(t)=-W^R(t)+\Phi'(t)\,,\quad \Phi'(t)=\frac{8\mu i\varphi'(t)}{\varkappa+1}. \tag{1}$$

The solution of this equation has to be bounded at a and b, which is not the case with 4.54 (8). It is, however, easy to modify the function

$$\chi(z)=(z-a)^{-1/2}(z-b)^{-1/2}. \tag{2}$$

By multiplication by $(z-a)(z-b)$ we get $(z-a)^{1/2}(z-b)^{1/2}$. It is now convenient to introduce the function

$$\omega(z) = (z-a)^{1/2}(b-z)^{1/2} = -i(z-a)^{1/2}(z-b)^{1/2} \tag{3}$$

choosing that branch for which

$$\omega(z)/z \to -i\,, \quad \text{when } z \to \infty\,, \tag{4}$$

for this agrees with 3.40 (4). Note also that

$$\omega^L(t) = (t-a)^{1/2}(b-t)^{1/2}\,,\ \omega^R(t) = -\omega^L(t) \tag{5}$$

is then real-valued when $a < t < b$.

In terms of $\omega(t)$ (1) gives

$$W(z) = \frac{\omega(z)}{2\pi i}\int_b^a \frac{\Phi'(t)\,dt}{\omega^L(t)\,(t-z)} + P_0\,\omega(z)\,, \tag{6}$$

where P_0 is an arbitrary constant. This solution is bounded at a and b, where it is in fact zero. The solution which we want must, however, also be bounded at infinity since for large values of $|z|$

$$W(z) = -\frac{iP}{\pi z} + O\left(\frac{1}{z^2}\right) \text{ from 4.21 (1).} \tag{7}$$

As a preliminary remark the integral in (6) $\to 0$ as $z \to \infty$ whereas $P_0\,\omega(z) \to -iP_0 z$ from (4). Therefore for $W(z)$ to tend to zero at infinity we must have $P_0 = 0$. Thus (6) becomes

$$W(z) = \frac{\omega(z)}{2\pi i}\int_a^b \frac{\Phi'(t)\,dt}{\omega^L(t)\,(t-z)} = \frac{1}{2\pi i}\int_a^b \frac{\Phi'(t)}{\omega^L(t)}\left(\frac{\omega(z)}{t-z}\right)dt\,. \tag{8}$$

Expanding by the binomial theorem

$$\frac{\omega(z)}{t-z} = i\left\{1 + \left[t - \frac{1}{2}(a+b)\right]\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right\}$$

which substituted in (8) leads to

$$W(z) = \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots\,, \tag{9}$$

where

$$\alpha_0 = \frac{1}{2\pi}\int_a^b \frac{\Phi'(t)\,dt}{\omega^L(t)}\,, \quad \alpha_1 = \frac{1}{2\pi}\int_a^b \frac{t\,\Phi'(t)\,dt}{\omega^L(t)} - \frac{1}{2}(a+b)\,\alpha_0\,.$$

Comparing (9) with (7) we see that we must have

$$\alpha_0 = 0\,, \quad \alpha_1 = -iP/\pi$$

and having regard to (1) this means that

$$\int_a^b \frac{\varphi'(t)\,dt}{\omega^L(t)} = 0\,, \tag{10}$$

$$\int_a^b \frac{t\,\varphi'(t)\,dt}{\omega^L(t)} = -\frac{(\varkappa+1)P}{4\mu}\,. \tag{11}$$

In (10) and (11) we have two equations to determine a and b; when a and b have been found, $W(z)$ is given by (8).

It is necessary that the pressure $p(t)$ be never negative over the range $a \leqq t \leqq b$, and this condition must therefore be examined in each individual case.

4.65. Smooth rigid beam with a circular base

We consider the case in which the profile of the base is an arc of a circle and for simplicity we assume symmetry about the y-axis so that when the beam is pressed down by a vertical force P to form the interface ab we shall have $a = -l$, $b = l$.

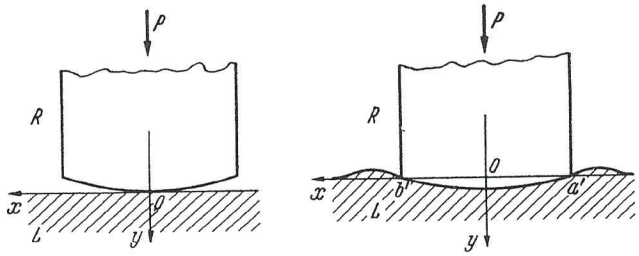

Fig. 4.65

For a circle of radius r which touches the x-axis at the origin and whose centre is at $(0, -r)$ we have $x^2 + (y+r)^2 = r^2$ and therefore if r is large, we can write, in the neighbourhood of the origin,

$$y_1 = -\frac{x^2}{2r} = \varphi(x) \tag{1}$$

in the notation of 4.63. Therefore $\varphi' = -t/r$.

Therefore in 4.63 (8) $\omega(z) = (l^2 - z^2)^{1/2}$ and

$$W(z) = -\frac{8\,i\mu}{(\varkappa+1)\,r}\,\frac{\omega(z)}{2\pi i}\int_{-l}^{l} \frac{t\,dt}{\omega^L(t)\,(t-z)}\,. \tag{2}$$

Using the method 3.41 and 3.13 (10) we find that

$$W(z) = \frac{-4\,i\,\mu}{(\varkappa+1)\,r}\left\{z - i\,(l^2 - z^2)^{1/2}\right\} \tag{3}$$

which is bounded at infinity since, from 4.63 (4), $\omega(z)/z \to -i$ when $z \to \infty$.

The value of l is determined from 4.63 (10) and (11) of which the first is satisfied identically, for $t/\omega^L(t)$ is an odd function and the second gives, putting $t = l \sin\theta$,

$$\frac{(\varkappa+1)P}{4\mu} = \int_{-l}^{l} \frac{t^2\,dt}{r\,\omega^L(t)} = \frac{1}{2r}\pi l^2 . \tag{4}$$

This equation determines l when P is given or P when l is given. For the pressure we have

$$-\{p(t) - i\,s(t)\} = W^L(t) - W^R(t) = \frac{-4i\mu}{(\pi+1)r}\{-2i\,(l^2-t^2)^{1/2}\}$$

so that $s(t) = 0$ and

$$p(t) = \frac{4\mu}{(\varkappa+1)r}\cdot(l^2-t^2)^{1/2}$$

and this is positive in the interval $(-l, l)$ vanishing at the ends. Thus all the conditions of the problem are satisfied.

4.70. Rough rigid beam with a straight horizontal base

This is the problem of 4.42 with adhesion replaced by friction. We consider the case in which the friction is limiting. Then if a vertical force P acts on the beam, a horizontal force in the x-direction $P \tan\gamma$ will just produce limiting equilibrium, where γ is the angle of friction. Thus the force applied to the half-plane by the beam is $P(\tan\gamma + i) = iPe^{-i\gamma}\sec\gamma$ In the formulae of 4.54 we have $\mu_1 = \infty$, $\Phi_2'(t) = 0$ and therefore

$$W(z) = P_0\chi(z) = P_0 e^{i\pi(\tau-1)}\,\psi(z)$$

and since

$$\lim_{z\to\infty} z\,W(z) = -iPe^{-i\gamma}\sec\gamma/\pi ,$$

we have

$$W(z) = \frac{iP\sec\gamma}{\pi}e^{i(\pi\tau-\gamma)}(l-z)^{\tau-1}(l+z)^{-\tau}, \tag{1}$$

$$e^{2\pi i\tau} = -\frac{\varkappa e^{2i\gamma}+1}{\varkappa+e^{2i\gamma}}. \tag{2}$$

Also for the pressure $p(t)$ and the shear $s(t) = \tan\gamma\,p(t)$ we have

$$-2(1-i\tan\gamma)\,p(t) = W^L(t) - W^R(t) = \frac{iP\sec\gamma}{\pi}e^{i(\pi\tau-\gamma)}(1-e^{-2\pi i\tau})\,\psi^L(t)$$

and so

$$p(t) = \frac{P\sin\pi\tau}{\pi}\,\frac{1}{(l+t)^{\tau}(l-t)^{1-\tau}}$$

which is positive and agrees with 4.60 for the case of no friction, for then $\tau = 1/2$.

4.71. Rough rigid beam with straight inclined base

We consider the same beam as in 4.70 but now the vertical force P and the horizontal force $P\tan\gamma$ for limiting equilibrium are so applied that the base of the beam remains tilted at the small angle ε as in fig. 4.43. In this case τ is still given by 4.70 (2), but now $\varphi'(t) = \varepsilon$ and therefore 4.54 (8) gives

$$W(z) = \frac{8\mu\varepsilon i}{\varkappa+e^{2i\gamma}}\,\frac{\psi(z)}{2\pi i}\int_{-l}^{l}\frac{dt}{\psi^L(t)\,(t-z)} + \frac{iP\sec\gamma}{\pi}e^{i(\pi\tau-\gamma)}\,\psi(z)$$

the last term being necessarily the same as 4.70 (1), for it follows by the same limiting process. Here

$$\psi(z) = (l+z)^{-\tau}(l-z)^{\tau-1} \tag{1}$$

$$\varkappa + e^{2i\gamma} = i(\varkappa+1)\operatorname{cosec}\pi\tau\cos\gamma\,e^{-i(\pi\tau-\gamma)} . \tag{2}$$

Now from 3.41

$$\frac{1}{2\pi i}\int_{-l}^{l}\frac{dt}{\psi^L(t)\,(t-z)} = \frac{1}{1-e^{2\pi i\tau}}\,\frac{1}{2\pi i}\oint_C\frac{dt}{\psi(t)\,(t-z)}$$

while

$$\frac{1}{\psi(t)} = t\left(1+\frac{l}{t}\right)^{\tau}\left(1-\frac{l}{t}\right)^{1-\tau}e^{i\pi(\tau-1)}$$
$$= \left[t + l\,(2\tau-1) + O\left(\frac{1}{t}\right)\right]e^{i\pi(\tau-1)} .$$

Therefore from 3.13 (10) we have

$$W(z) = \frac{4\mu\varepsilon i\,e^{-i\gamma}\sec\gamma}{\varkappa+1} + i\,e^{i(\pi\tau-\gamma)}\sec\gamma\left\{\frac{4\mu\varepsilon}{\varkappa+1}[z+l\,(2\tau-1)] + \frac{P}{\pi}\right\}\psi(z) \tag{3}$$

which agrees with 4.61 (4) when $\gamma = 0$ and $\tau = \frac{1}{2}$.

For the pressure we have

$$-2p(t)\,(1-i\tan\gamma) = W^L(t) - W^R(t)$$

which is proportional to $\psi^L(t) - \psi^R(t) = (1-e^{-2\pi i\tau})\,\psi^L(t)$. Thus

$$p(t) = \left\{\frac{4\mu\varepsilon}{\varkappa+1}[t+l\,(2\tau-1)] + \frac{P}{\pi}\right\}\sin\pi\tau\,(l+t)^{-\tau}(l-t)^{\tau-1} .$$

This will be positive for $-l < t < l$ provided that the content of the curled brackets is positive when $t = -l$, that is, if

$$P > \frac{8\mu\varepsilon l\pi(1-\tau)}{\varkappa+1} .$$

4.73. Rough rigid beam with a curved base

Looking at fig. 4.63 if we assume the friction to be limiting, we must supply a horizontal force $P \tan \gamma$ from right to left and by the same reasoning as used before we get again 4.63 (8) namely

$$W(z) = \frac{1}{2\pi i} \int_a^b \frac{\Phi'(t)}{W^L(t)} \frac{\omega(z)}{t-z} dt \tag{1}$$

but here 4.63 (3) is replaced by

$$\omega(z) = (z-a)^{1-\tau}(b-z)^{\tau}, \quad \lim_{z\to\infty} \omega(z)/z = e^{-i\pi\tau} \tag{2}$$

where

$$e^{2\pi i\tau} = -\frac{\varkappa e^{2i\gamma}+1}{\varkappa + e^{2i\gamma}}. \tag{3}$$

Expanding by the binomial theorem we have

$$\frac{\omega(z)}{t-z} = \frac{(z-a)^{1-\tau}(b-z)^{\tau}}{t-z} = e^{-i\pi(\tau-1)}\left(1-\frac{a}{z}\right)^{1-\tau}\left(1-\frac{b}{z}\right)^{\tau}\left(1-\frac{t}{z}\right)^{-1}$$
$$= e^{-i\pi(\tau-1)}\left\{1+\frac{[t+\tau(a-b)-a]}{z}+O\left(\frac{1}{z^2}\right)\right\}$$

which substituted in (1) leads to

$$W(z) = \alpha_0 + \frac{a_1}{z} + \cdots$$

where

$$\alpha_0 = \frac{e^{-i\pi(\tau-1)}}{2\pi i}\int_a^b \frac{\Phi'(t)\,dt}{\omega^L(t)},$$

$$\alpha_1 = \frac{e^{-i\pi(\tau-1)}}{2\pi i}\int_a^b \frac{t\,\Phi'(t)\,dt}{\omega^L(t)} + [\tau(a-b)-a]\,\alpha_0.$$

Comparing this with 4.21 (1) we see that

$$\alpha_0 = 0\,,\ \alpha_1 = -P(\tan\gamma + i)/\pi = -iPe^{-i\gamma}\sec\gamma/\pi$$

and therefore

$$\int_a^b \frac{\varphi'(t)\,dt}{\omega^L(t)} = 0\,,\quad \int_a^b \frac{t\,\varphi'(t)\,dt}{\omega^L(t)} = \frac{-(\varkappa+1)P}{4\mu\sin\pi\tau} \tag{4}$$

and these two equations determine a and b when P is given, or P when a and b are given compatible with the first of (4).

4.75. Rough rigid beam with a circular base

We suppose the equilibrium to be limiting. Then as in 4.65 we can take

$$\varphi'(t) = -t/r$$

so that if we take $a = -l$, $b = l$ the first condition of 4.73 (4) is satisfied identically while the second gives

$$\int_{-l}^{l} \frac{t^2\,dt}{r\,\omega^L(t)} = \frac{(\varkappa+1)P}{4\mu\sin\pi\tau}.$$

Now from 3.41, if C is a lacet round $(-l, l)$

$$\frac{1}{2\pi i}\int_{-l}^{l}\frac{t^2\,dt}{\omega^L(t)} = \frac{1}{1-e^{-2\pi i\tau}}\,\frac{1}{2\pi i}\oint_C \frac{t^2}{\omega(t)}dt$$

$$\frac{t^2}{\omega(t)} = e^{i\pi\tau}t\left\{\left(1+\frac{(\tau-1)l}{t}+\frac{(\tau-1)(\tau-2)l^2}{2t^2}+\cdots\right)\right.$$
$$\left.\times\left(1+\frac{\tau l}{t}+\frac{(-\tau)(-\tau-1)}{2t^2}l^2+\cdots\right)\right.$$

and the residue is

$$l^2 e^{i\pi\tau}\left\{\frac{\tau(\tau+1)}{2}+\tau(\tau-1)+\frac{(\tau-1)(\tau-2)}{2}\right\}$$
$$= \frac{1}{2}l^2 e^{i\pi\tau}((2\tau-1)^2+1)$$

and so

$$\int_{-l}^{l}\frac{t^2\,dt}{\omega(t)} = \frac{2\pi i e^{i\pi\tau}}{1-e^{-2\pi i\tau}}\left(-\frac{1}{2}l^2((2\tau-1)^2+1)\right)$$
$$= \frac{2\pi i}{-2i\sin\pi\tau}\cdot -\frac{1}{2}l^2((2\tau-1)^2+1) = \frac{1}{2}\pi l^2\frac{(2\tau-1)^2+1}{\sin\pi\tau}.$$

Therefore

$$P = \frac{2\mu\pi l^2}{(\varkappa+1)r}[(2\tau-1)^2+1)].$$

Examples IV

1. Use 4.11 (3), (4) to show that the moment about the origin of the standard concentrated force (X, Y) applied at z_0 is the real part of $i z_0 (X - iY)$.

2. Show that the displacement due to a single standard concentrated force is unbounded at infinity.

3. Two forces, equal in magnitude, opposite in direction have a common line of action, but are applied at different points of this line. Prove that the displacement at infinity is bounded.

Discuss whether this result also holds when the forces act in parallel but distinct lines.

4. In the case of a plate weakened by a single crack, use the solution to verify that the stresses at the edges of the crack are zero.

5. Find the complex stresses when the plane has two cracks namely from a to b and from $-b$ to $-a$ and the displacement is given on all edges.

6. In the mixed problem 4.16 calculate the displacement at the middle point of the upper edge of the slit.

7. In the notation of 4.23 prove that.

$$4\mu\,(u+iv) = \varkappa W(z) + W(\bar{z}) - (z-\bar{z})\,\overline{W}(\bar{z}) + \text{constant}.$$

Discuss the form this result would assume if the rotation does not vanish at infinity.

8. Instead of continuing $W(z)$ across the boundary from the stress formula, continue it from the displacement formula to give

$$\varkappa W(z) = \overline{\{W(\bar{z}) + \bar{z}W'(\bar{z}) + w(\bar{z})\}}, \quad (z \text{ in } R).$$

Hence prove that the displacement boundary condition can be written in the form

$$4\mu\, v'(t) = \varkappa W^L(t) - \varkappa W^R(t)$$

and find the corresponding stress boundary condition.

9. Apply the Plemelj formulae to verify that

$$\varkappa W^L(t) + W^R(t) = \frac{4\mu h i}{1+t^2}$$

in the problem of 4.35.

10. In Problem III for the half-plane show that

$$2\,[p(t_0) - is(t_0)] = \frac{2(\varkappa-1)}{\varkappa}\mu v'(t_0) - (\varkappa+1)\,\chi^L(t_0)\,P_{n-1}(t_0) - \\ - \frac{(\varkappa+1)\,\chi^L(t_0)}{2\pi i}\int_{A'} \frac{4\mu v'(t)\,dt}{\varkappa\,\chi^L(t)\,(t-t_0)}$$

11. In the solution of Problem III prove that at any point t_0 of the part A'' of the boundary

$$4\mu\, v'(t_0) = (\varkappa+1)\,W^L(t_0)$$

$$= (\varkappa+1)\,\chi^L(t_0)\,P_{n-1}(t_0) + (\varkappa+1)\,\frac{\chi^L(t_0)}{2\pi i}\int_{A'} \frac{4\mu v'(t)\,dt}{\varkappa\,\chi^L(t)\,(t-t_0)}.$$

12. For the rigid beam which adheres, prove that when the base is horizontal the shearing force under the base is

$$\frac{(\varkappa+1)\,P}{2\pi\sqrt{\varkappa}}\,\frac{1}{\sqrt{(l^2-t^2)}}\,\sin\left[\left(\frac{\ln\varkappa}{2\pi}\ln\frac{l+t}{l-t}\right)\right]$$

and deduce that the resultant horizontal force on the base is zero.

13. For a rigid beam which adheres and whose base is horizontal, show that the fraction of the base over which the pressure remains constantly non-negative is $\tanh\,[\pi^2/(2\ln\varkappa)]$.

14. Discuss the stress distribution in the foundations when a building with a rigid horizontal base rests on elastic horizontal foundations.

15. In the case of a smooth rigid beam with a straight inclined base, find at what distance from the centre of the base the line of action of the applied force must meet the base in order to cause a given angle of tilt.

16. With the notation of 4.72 prove that

$$\varkappa + e^{2i\gamma} = i\cos\gamma\, e^{-i(\pi\tau-\gamma)}\,\frac{(\varkappa+1)}{\sin\pi\tau}$$

and find a corresponding expression for $\varkappa\, e^{2i\gamma} + 1$.

17. In the case of a rough rigid beam with a circular base find the pressure under the base and show that it is everywhere positive in the region of contact.

18. Prove that for a standard concentrated force $X + iY$ at the point z_0 the stress at the circumference of a circle of radius a round z_0 is $(X+iY)/(2\pi a)$.

19. Prove that $\{v^L(t)\}' = \{v'(t)\}^L$.

20. The interval a, b of the boundary of an elastic half-plane is subjected to uniform hydrostatic pressure p. Show that

$$W(z) = \frac{ip}{\pi}\ln\left(\frac{b-z}{a-z}\right)$$

and calculate the displacement at any point of the boundary, distinguishing points of the interval a, b from points outside it. Show also that the logarithmic singularities of $W(z)$ are removable.

Chapter V

Circular boundaries

In this chapter we are concerned mainly with circular discs and holes and the concentric circular annulus.

5.1. Stress and displacement at circular boundaries

We take the origin at the centre and denote the chief stresses and shear at the point (r, θ) of the material by $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{r\theta}$. The arc measured along the circumference from the point where it crosses the x-axis is

$$s = r\theta \tag{1}$$

since

$$z = re^{i\theta} \tag{2}$$

if we differentiate along the circumference,

$$\frac{dz}{ds} = ire^{i\theta}\frac{d\theta}{ds} = \frac{iz}{r}. \tag{3}$$

Since the sum of the chief stresses i. e. Θ is invariant we have from 2.10 (3)

$$\widehat{rr} + \widehat{\theta\theta} = W(z) + \overline{W}(\bar{z}) + (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}}, \tag{4}$$

while taking the conjugate of 2.20 (7)

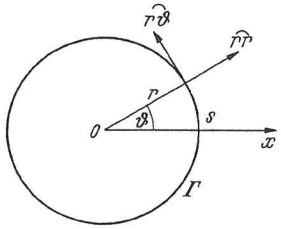

Fig. 5.1

$$2\,(\widehat{rr} + i\,\widehat{r\theta}) = W(z) + \overline{W}(\bar{z}) - \bar{z}\,\overline{W}'(\bar{z}) - \frac{\bar{z}}{z}\,\overline{w}(\bar{z}) + (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}} + 4\nu\frac{\bar{z}}{z}\frac{\partial^2 Q}{\partial \bar{z}^2}. \tag{5}$$

If u_r, u_θ are the radial and transverse components of the displacement 2.20 (9) gives

$$4\mu\,(u_r + i\,u_\theta) = \frac{\bar{z}}{r}\left\{\varkappa' W(z) - z\,\overline{W}(\bar{z}) - {}'\overline{w}(\bar{z}) + 4\nu\frac{\partial Q}{\partial \bar{z}}\right\}. \tag{6}$$

5.11. Effective stress and displacement

Referring to 1.75 we call *effective stress*, stress derived from the stress function $\chi + \nu Q$, so that denoting effective values by suffix e we have, if $\nu = 2\,(\varkappa - 1)/(\varkappa + 1)$,

$$\widehat{xx}_e + \widehat{yy}_e = \frac{4\,\partial^2(\chi + \nu Q)}{\partial z\,\partial \bar{z}} = W(z) + \overline{W}(\bar{z}) \tag{1}$$

$$\widehat{yy}_e - \widehat{xx}_e + 2i\,\widehat{xy}_e = \frac{4\,\partial^2(\chi + \nu Q)}{\partial z^2} = \bar{z}\,W'(z) + w(z)\,. \tag{2}$$

Thus effective stress includes stress derived from the body-potential, and a knowledge of effective stress leads at once to actual stress and vice versa.

In polar coordinates we have from 5.1 (4) and (5)

$$\widehat{rr}_e + \widehat{\theta\theta}_e = \widehat{rr} + \widehat{\theta\theta} - (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}} = W(z) + \overline{W}(\bar{z}) \tag{3}$$

$$\begin{aligned} 2\,(\widehat{rr}_e + i\,\widehat{r\theta}_e) &= 2\,(\widehat{rr} + i\,\widehat{r\theta}) - (8 - 4\nu)\frac{\partial^2 Q}{\partial z\,\partial \bar{z}} - 4\nu\frac{\partial^2 Q}{\partial \bar{z}^2}\cdot\frac{\bar{z}}{z} \\ &= W(z) + \overline{W}(\bar{z}) - \bar{z}\,\overline{W}'(\bar{z}) - (\bar{z}/z)\,\overline{w}(\bar{z})\,. \end{aligned} \tag{4}$$

Similarly we call *effective displacement*, that derived from the stress function $\chi + \nu Q$. Therefore from 2.12 (2)

$$4\mu\,(u_e + i\,v_e) = 4\mu\,(u + i\,v) - 4\nu\frac{\partial Q}{\partial \bar{z}} = \varkappa' W(z) - z\,\overline{W}(\bar{z}) - {}'\overline{w}(\bar{z}) \tag{5}$$

and in polar coordinates from 5.1 (6)

$$\begin{aligned} 4\mu\,(u_r + i\,u_\theta)_e &= 4\mu\,(u_r + i\,u_\theta) - (4\nu\bar{z}/r)\,\partial Q/\partial \bar{z} \\ &= (\bar{z}/r)\,\{\varkappa' W(z) - z\,\overline{W}(\bar{z}) - {}'\overline{w}(\bar{z})\}\,. \end{aligned} \tag{6}$$

The use of effective values simplifies the writing without any loss of generality.

Differentiate (5) partially with respect to θ. Since $\partial z/\partial\theta = i\,z$, $\partial\bar{z}/\partial\theta = -\,i\,\bar{z}$ we find that

$$4\mu\left(\frac{\partial u_e}{\partial\theta} + i\frac{\partial v_e}{\partial\theta}\right) = i\,z\,\{\varkappa W(z) - \overline{W}(\bar{z}) + \bar{z}\,\overline{W}'(\bar{z}) + \frac{\bar{z}}{z}\,\overline{w}(\bar{z})\}\,. \tag{7}$$

5.2. On the continuation of $W(z)$ across a boundary

To fix our ideas let the boundary A consist of a single simple closed contour, whose equation referred to an origin inside the contour is

$$F(z, \bar{z}) = 0\,. \tag{1}$$

A positive sense of description prescribed for A will distinguish a left-hand region L from a right-hand region R. We shall suppose the material to occupy the region L.

If n is a positive real number, the boundary and the curve

$$F\left(\frac{z}{n}, \frac{\bar{z}}{n}\right) = 0 \tag{2}$$

are similar and similarly situated (i. e. homothetic) with respect to the origin. The boundary is therefore that member of the homothetic family (2) which corresponds to $n = 1$. We shall suppose that $n \leqq 1$ or $n \geqq 1$ according as the region L is interior or exterior to A.

The boundary equation (1) determines z as a function of $\bar{z}$, in general, many-valued. We can however choose a branch such that

$$z = z(\bar{z}) \tag{3}$$

on the boundary.

Whether z is on the boundary or not we write

$$\overline{Z} = z(\bar{z})\,. \tag{4}$$

We shall assume a property which can always be tested, namely that if z is in L, then $\overline{Z}$ is in R and vice versa.

Referring to fig. 2.20, let C be one of the curves (2). Then from 2.20 (7)

$$2\,(\widehat{nn}_e + i\,\widehat{ns}_e) = W(z) + \overline{W}(\bar{z}) + \{z\,\overline{W}'(\bar{z}) + \overline{w}(\bar{z})\}\,\varphi(z, \bar{z})\,, \tag{5}$$

where suffix e denotes effective stress,

$$\varphi(z, \bar{z}) = (d\bar{z}/ds)/(dz/ds)\,, \tag{6}$$

and $W(z)$, $w(z)$ are holomorphic functions in the region L occupied by the material. Observe, however, that $W(z)$, $w(z)$ are undefined in the region R. We proceed to define W in the region R as follows. In (5) write zero for $\widehat{nn}_e + i\,\widehat{ns}_e$ and $\overline{Z}$ for z. Then rearranging we have

$$\begin{aligned} W(\overline{Z}) &= -\,\overline{W}(\bar{z}) - \{\overline{Z}\,\overline{W}'(\bar{z}) + \overline{w}(\bar{z})\}\,\varphi(\overline{Z}, \bar{z}) \\ &= -\,\overline{\{W(z) + \{Z\,W'(z) + w(z)\}\,\overline{\varphi}(Z, z)\}} \end{aligned} \tag{7}$$

where as usual, the bar denotes the conjugate complex. Further when

z is in L the functions $W(z)$, $w(z)$ are defined and therefore so is the right-hand side of (7). We therefore take (7) to define $W(\overline{Z})$ when $\overline{Z}$ is a point of R. This is the required continuation of W across the boundary. Mathematically it corresponds to analytic continuation through such parts, if any, of the boundary on which the effective stress is zero.

As to the function $\varphi(z, \bar{z})$ we have from (2)

$$F_z\left(\frac{z}{n}, \frac{\bar{z}}{n}\right)\frac{dz}{ds} + F_{\bar{z}}\left(\frac{z}{n}, \frac{\bar{z}}{n}\right)\frac{d\bar{z}}{ds} = 0$$

and therefore

$$\varphi(z, \bar{z}) = -F_z\left(\frac{z}{n}, \frac{\bar{z}}{n}\right) \Big/ F_{\bar{z}}\left(\frac{z}{n}, \frac{\bar{z}}{n}\right), \tag{8}$$

which is the same at all corresponding points of the homothetic curves.

If we add (5) and (7), we get

$$\begin{aligned} W(z) - W(\overline{Z}) + \{z\varphi(z, \bar{z}) - \overline{Z}\varphi(\overline{Z}, \bar{z})\}\, \overline{W}'(\bar{z}) \\ + \{\varphi(z, \bar{z}) - \varphi(\overline{Z}, \bar{z})\}\, \overline{w}(\bar{z}) = 2\,(\widehat{nn_e} + i\,\widehat{ns_e})\,. \end{aligned} \tag{9}$$

In this let z remaining in L tend to a point t of the boundary. Then $\overline{Z}$ remaining in R will also tend to t, while the terms in the curled brackets tend to zero. Therefore

$$W^L(t) - W^R(t) = 2\,(\widehat{nn_e} + i\,\widehat{ns_e})^L\,, \tag{10}$$

where L on the right denotes stress on the boundary approached from the region L.

Thus by the Plemelj theorem

$$W(z) = \frac{1}{2\pi i}\oint_A \frac{2\,(\widehat{nn_e} + i\,\widehat{ns_e})^L\,dt}{t - z} + \psi(z) \tag{11}$$

taken along the boundary A. The integral here defines a function vanishing at infinity and sectionally holomorphic in the whole plane cut along A. The function $\psi(z)$ is added to take care of any required singularities which it may be necessary to introduce.

As an illustration of (7) let L be the half-plane $y \geqq 0$ so that the boundary is $F(z, \bar{z}) = z - \bar{z} = 0$. Here $\varphi(z, \bar{z}) = 1, \overline{Z} = \bar{z}$, and so (7) gives, in agreement with 4.23 (3)

$$W(\bar{z}) = \overline{-W(z) - zW'(z) - w(z)}\,, \quad (z \text{ in } L)\,.$$

5.21. Continuation of $W(z)$ across a circular boundary

Let the boundary be the circumference Γ of radius a, whose equation is

$$z\bar{z} = a^2 \quad \text{or} \quad z = a^2/\bar{z} = \overline{Z}\,. \tag{1}$$

Then

$$(d\bar{z}/ds)/(dz/ds) = -a^2/z^2 = -\bar{z}^2/a^2$$

and therefore, from 5.2 (7), $W(z)$ is continued across the boundary from the point z of L to the point $a^2/\bar{z}$ of R by

$$W\left(\frac{a^2}{\bar{z}}\right) = -\overline{W}(\bar{z}) + \bar{z}\overline{W}'(\bar{z}) + \frac{\bar{z}^2}{a^2}\overline{w}(\bar{z})\,, \quad (z \text{ in } L) \tag{2}$$

or equivalently, writing a^2/z for $\bar{z}$,

$$W(z) = -\overline{W}\left(\frac{a^2}{z}\right) + \frac{a^2}{z}\overline{W}'\left(\frac{a^2}{z}\right) + \frac{a^2}{z^2}\overline{w}\left(\frac{a^2}{z}\right), \quad (z \text{ in } R)\,. \tag{3}$$

It should be noted that z and $a^2/\bar{z}$ are inverse points with respect to the circle.

Taking the conjugate of (2) and rearranging we get

$$w(z) = \frac{a^2}{z^2}\left\{\overline{W}\left(\frac{a^2}{z}\right) + W(z) - zW'(z)\right\}, \quad (z \text{ in } L)\,. \tag{4}$$

By hypothesis $W(z)$, $w(z)$ are holomorphic in the region L occupied by the material.

If therefore L contains the point $z = \infty$, (circular hole), it follows from (3) that $W(z)$ is holomorphic in R (the interior of the circumference) except at the point $z = 0$ where it may have a pole of the second order and therefore its principal part there will be of the form

$$A + \frac{B}{z} + \frac{C}{z^2}\,. \tag{5}$$

On the other hand if $z = \infty$ is in the region R, (circular disc), $W(z)$ is holomorphic at $z = \infty$, and its principal part there is a constant A.

Let the effective radial and shear stresses at the boundary be $\widehat{aa_e}$, $\widehat{a\theta_e}$. Then from 5.2 (11),

$$W(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{2\,(\widehat{aa_e} + i\,\widehat{a\theta_e})\,dt}{t - z} + A + \frac{B}{z} + \frac{C}{z^2}\,, \tag{6}$$

where $B = C = 0$ when the material occupies a circular disc.

The sense of description of Γ will be clockwise for a circular hole and counterclockwise for a circular disc.

5.215. A general continuation theorem for the circle

Theorem. Let Γ be a circumference of radius a which by a prescribed sense of description determines regions L and R. Let

$$f_s(z, \bar{z}), \overline{F}_s(z, \bar{z})\,, \quad s = 1, 2, \ldots, n, \quad \text{and} \quad k(z, \bar{z})$$

be given analytic functions.

Then if $\varphi_s(z)$, $s = 1, 2, \ldots, n$ are functions holomorphic in L, which satisfy the functional equation

$$\sum_{s=1}^{n} f_s(z, \bar{z})\,\varphi_s(z) + \sum_{s=1}^{n} \overline{F}_s(z, \bar{z})\,\overline{\varphi}_s(\bar{z}) = k(z, \bar{z})\,, \quad (z \text{ in } L)\,, \tag{1}$$

and if we define

$$\sum_{s=1}^{n} f_s\left(z, \frac{a^2}{z}\right) \varphi_s(z) = -\sum_{s=1}^{n} \overline{F}_s\left(z, \frac{a^2}{z}\right) \overline{\varphi}_s\left(\frac{a^2}{z}\right), \quad (z \text{ in } R), \tag{2}$$

we have

$$\sum_{s=1}^{n} f_s\left(z, \frac{a^2}{z}\right) \varphi_s(z) = \frac{1}{2\pi i} \oint \frac{k\left(t, \frac{a^2}{t}\right)}{t-z} dt + \psi(z), \tag{3}$$

(z in L or R), where $\psi(z)$ is holomorphic in L and on Γ.

Proof. In (1) write zero for $k(z, \bar{z})$, $a^2/\bar{z}$ for z and leave $\bar{z}$ unaltered. Then

$$\sum_{s=1}^{n} f_s\left(\frac{a^2}{\bar{z}}, \bar{z}\right) \varphi_s\left(\frac{a^2}{\bar{z}}\right) + \sum_{s=1}^{n} \overline{F}_s\left(\frac{a^2}{\bar{z}}, \bar{z}\right) \overline{\varphi}_s(\bar{z}) = 0, \tag{4}$$

which is equivalent to (2) if we write z for $a^2/\bar{z}$ and a^2/z for $\bar{z}$.

Subtract (4) from (1) and let z, remaining in L, tend to t, a point of the boundary. Then $a^2/\bar{z}$ remains in R and also tends to t, therefore

$$\left[\sum_{s=1}^{n} f_s\left(t, \frac{a^2}{t}\right) \varphi_s(t)\right]^L - \left[\sum_{s=1}^{n} f_s\left(t, \frac{a^2}{t}\right) \varphi_s(t)\right]^R = k\left(t, \frac{a^2}{t}\right), \tag{5}$$

from which (3) follows by the Plemelj theorem.

Q. E. D.

The method 5.21 is a particular case of this theorem.

We note that (2) defines its left-hand member when z is in R, in terms of the right-hand member which is defined when z is in R, for then a^2/z is in L.

5.22. Problem I for a circular hole in an infinite plate

In the case of an infinite plate with a circular hole bounded by the circumference $|z| = a$ or Γ we suppose the body-field vector to be absent for the reasons explained in 2.74. The edge of the hole will be supposed loaded with given normal stress $\widehat{aa}$ and shear $\widehat{a\theta}$.

From 5.21 (6) we have

$$W(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{2(\widehat{aa} + i\,\widehat{a\theta})}{t-z} dt + A + \frac{B}{z} + \frac{C}{z^2} \quad (z \text{ in } L \text{ or } R) \tag{1}$$

and from 2.74, in terms of the principal stresses and rotation at infinity,

$$A = \frac{1}{2}(N_\infty + T_\infty) + \frac{4\mu i \varepsilon_\infty}{\varkappa + 1}. \tag{2}$$

We shall assume that there is no dislocation.

To determine B and C we note that at $z = 0$, $W(z)$ as given by (1) has a pole of the second order whose principal part is $B/z + C/z^2$. But

for sufficiently large values of $|z|$, we have expansions of the form

$$\begin{aligned} W(z) &= \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots \quad (z \text{ in } L) \\ w(z) &= \beta_0 + \frac{\beta_1}{z} + \frac{\beta_2}{z^2} + \cdots \quad (z \text{ in } L) \end{aligned} \tag{3}$$

Substitute these in 5.21 (3) and we find that

$$W(z) = \frac{a^2 \bar{\beta}_0}{z^2} + \frac{\bar{\beta}_1}{z} + \cdots, \quad (z \text{ in } R)$$

where the dots indicate terms not involving negative powers of z. Therefore

$$B = \bar{\beta}_1, \quad C = a^2 \bar{\beta}_0.$$

Comparing (3) with 2.74 (27) we see that

$$\bar{\beta}_0 = (N_\infty - T_\infty)\, e^{2i\varphi},$$
$$\bar{\beta}_1 = \varkappa (X + iY)/\{\pi(\varkappa + 1)\},$$

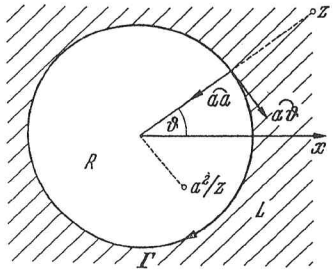

Fig. 5.22

where the resultant applied load on the boundary of the hole is

$$X + iY = -\int_0^{2\pi} (\widehat{aa} + i\,\widehat{a\theta})\, a\, e^{i\theta}\, d\theta. \tag{4}$$

Therefore we have for the complex stresses

$$\begin{aligned} W(z) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{2(\widehat{aa} + i\,\widehat{a\theta})\, dt}{t-z} + \frac{1}{2}(N_\infty + T_\infty) + \frac{4\mu i \varepsilon_\infty}{\varkappa + 1} \\ &\quad + \frac{\varkappa (X + iY)}{\pi(\varkappa + 1)} \frac{1}{z} + a^2 (N_\infty - T_\infty)\, e^{2i\varphi} \frac{1}{z^2} \quad (z \text{ in } L \text{ or } R) \end{aligned} \tag{5}$$

$$w(z) = \frac{a^2}{z^2}\left\{W(z) + \overline{W}\left(\frac{a^2}{z}\right) - zW'(z)\right\} \quad (z \text{ in } L). \tag{6}$$

The above formulae deal with all cases of a stretched plate weakened by a circular hole loaded in any prescribed manner.

Consider, for example, the case of a plate, under simple tension T at angle φ to the real axis, weakened by a circular hole of radius a. Taking the origin at the centre of the hole and supposing the hole to be unloaded we have

$$\widehat{aa} + i\widehat{a\theta} = 0, \; X + iY = 0, \; \varepsilon_\infty = 0,$$

so that

$$W(z) = \frac{1}{2} T - \frac{a^2 e^{2i\varphi} T}{z^2},$$

If $\widehat{\theta\theta}_B$ is the hoop stress at the boundary of the hole, since $\widehat{aa} = 0$ there, we have

$$\widehat{\theta\theta}_B = \Theta_B = T\{1 - 2\cos 2(\varphi - \theta)\}.$$

Therefore the hoop stress at the boundary is greatest when $\theta = \varphi \pm \pi/2$, i. e. at the extremities of the diameter perpendicular to the direction of the tension at infinity, and the maximum value is $3T$.

Thus a circular hole seriously weakens the plate.

5.23. Uniformly loaded circular hole

Let the constant stress $(X + iY)/(2\pi a)$ act on the boundary of a circular hole of radius a in an infinite plate and let no other forces be operative. Then

$$(\widehat{aa} + i\,\widehat{a\theta})\, e^{i\theta} = -(X + iY)/2\pi a\,. \tag{1}$$

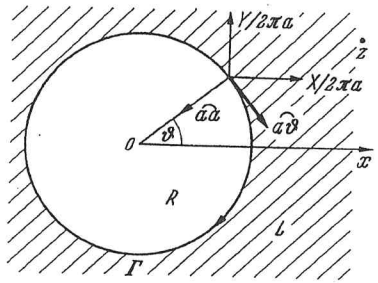

Fig. 5.23

The resultant force on the boundary is

$$\int_0^{2\pi} \frac{(X + iY)}{2\pi a}\, a\, d\theta = X + iY\,.$$

Therefore from 5.22 (5), since $t = a\, e^{i\theta}$,

$$W(z) = -\frac{1}{2\pi i}\oint \frac{2\,(X + iY)}{2\pi t\,(t - z)}\, dt + \frac{\varkappa\,(X + iY)}{\pi\,(\varkappa + 1)}\,\frac{1}{z} \tag{2}$$

where we have assumed zero rotation at infinity, and no dislocation.

Now

$$\frac{1}{t\,(t - z)} = -\frac{1}{zt} + \frac{1}{z\,(t - z)}\,.$$

Therefore when z is outside the circle, the integral reduces to $-(X + iY)/\pi z$ and therefore

$$W(z) = -\frac{X + iY}{\pi\,(\varkappa + 1)}\,\frac{1}{z}\,, \quad (z \text{ in } L) \tag{3}$$

which is independent of the radius of the hole.

For points in R the integral in (2) vanishes and

$$W(z) = \frac{\varkappa\,(X + iY)}{\pi\,(\varkappa + 1)}\,\frac{1}{z} \quad (z \text{ in } R)\,. \tag{4}$$

Therefore from 5.22 (6)

$$w(z) = \frac{a^2}{z^2}\left\{-2\,\frac{X + iY}{\pi\,(\varkappa + 1)}\cdot\frac{1}{z} + \frac{\varkappa\,(X - iY)}{\pi\,(\varkappa + 1)}\cdot\frac{z}{a^2}\right\}$$
$$= \frac{\varkappa\,(X - iY)}{\pi\,(\varkappa + 1)}\cdot\frac{1}{z} - \frac{2\,(X + iY)\,a^2}{\pi\,(\varkappa + 1)}\,\frac{1}{z^3} \quad (z \text{ in } L) \tag{5}$$

5.24. Standard concentrated force

If in 5.23 we let $a \to 0$, we get the case of a concentrated force $(X + iY)$ acting at the origin. This is the standard concentrated force, first discussed in 4.11 resulting from the limit of a uniform distribution. The complex stresses are

$$W(z) = -\frac{X + iY}{\pi\,(\varkappa + 1)}\cdot\frac{1}{z}\,, \quad w(z) = \frac{\varkappa\,(X - iY)}{\pi\,(\varkappa + 1)}\cdot\frac{1}{z}\,, \tag{1}$$

while if the force is applied at the point z_0, from 2.21,

$$W(z) = -\frac{X + iY}{\pi\,(\varkappa + 1)}\cdot\frac{1}{z - z_0}\,,$$
$$w(z) = \frac{\varkappa\,(X - iY)}{\pi\,(\varkappa + 1)}\cdot\frac{1}{z - z_0} - \frac{(X + iY)\,\bar{z}_0}{\pi\,(\varkappa + 1)}\cdot\frac{1}{(z - z_0)^2}\,. \tag{2}$$

See also 5.32.

5.25. Problem I for a circular disc

Here we are given the radial and shear stresses $\widehat{aa}$, $\widehat{a\theta}$ on the boundary.

Write

$$k(t) = 2\,(\widehat{aa}_e + i\,\widehat{a\theta}_e)\,, \quad t = a e^{i\theta} \tag{1}$$

where suffix e denotes effective stress. Then $k(t)$ is a given function and from 5.21 (6)

$$W(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{k(t)}{t - z}\, dt + A\,, \tag{2}$$

since $z = \infty$ is in the region R.

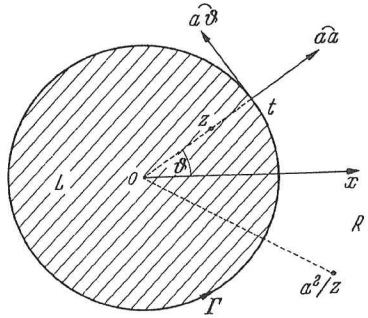

Fig. 5.25

Since $W(z)$ is holomorphic in L and R we have expansions

$$\left.\begin{aligned} W(z) &= a_0 + a_1 z + a_2 z^2 + \cdots \quad (z \text{ in } L)\\ W(z) &= A + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \quad (z \text{ in } R).\end{aligned}\right\} \tag{3}$$

To find $w(z)$ we substitute (3) in 5.21 (4) getting

$$w(z) = \frac{a^2}{z^2}\left\{(a_0 + \bar{A}) + \frac{\bar{b}_1}{a^2}\, z + \left(\frac{\bar{b}_2}{a^4} - a_2\right) z^2 + \cdots\right\} \quad (z \text{ in } L)\,.$$

But $w(z)$ is holomorphic in L. Therefore we must have

$$a_0 + \bar{A} = 0\,, \quad b_1 = 0\,. \tag{4}$$

But from (3)

$$a_0 = W(0) = \frac{1}{2\pi i}\oint_\Gamma \frac{k(t)}{t}\,dt + A$$

$$b_1 = \lim_{z\to\infty}\{z\,[W(z) - A]\}$$

$$= \lim_{z\to\infty}\frac{1}{2\pi i}\oint_\Gamma \frac{z\,k(t)\,dt}{t-z} = -\frac{1}{2\pi i}\oint_\Gamma k(t)\,dt\,.$$

Using the relations (4) we get

$$A + \bar{A} = -\frac{1}{2\pi i}\oint_\Gamma \frac{k(t)}{t}\,dt\,, \tag{5}$$

$$\oint_\Gamma k(t)\,dt = 0\,. \tag{6}$$

Thus (5) determines the real part of A. The imaginary part does not influence the stress system (cf. 2.13 (7)) but does affect the displacement by a rigid body movement which remains arbitrary. Thus so far as the stress problem is concerned we may take A to be real and therefore (2) and (5) give

$$W(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{2\,(\widehat{aa_e} + i\,\widehat{a\theta_e})\,dt}{t-z} - \frac{1}{2\pi i}\oint \frac{(\widehat{aa_e} + i\,\widehat{a\theta_e})\,dt}{t} \quad (z \text{ in } L \text{ or } R). \tag{7}$$

Observe that the second integral is half the value which arises from putting $z = 0$ in the first. For $w(z)$ we have

$$w(z) = \frac{a^2}{z^2}\left\{W(z) + \overline{W}\left(\frac{a^2}{z}\right) - z\,W'(z)\right\}, \quad (z \text{ in } L) \tag{8}$$

and the problem is solved.

Since $t = a\,e^{i\theta}$ and since $A + \bar{A}$ is real (5) and (6) give

$$\int_0^{2\pi} \widehat{a\theta_e}\,d\theta = 0\,,\ \int_0^{2\pi} (\widehat{aa_e} + i\,\widehat{a\theta_e})\,e^{i\theta}\,d\theta = 0\,. \tag{9}$$

The first of (9) states that the moment about the centre of the effective stress applied to the boundary is zero, while the second states that the resultant vector of the effective stress force applied to the boundary vanishes. The conditions are necessary in any case for the problem to have a solution at all (cf. 2.73).

5.26. Rotating circular disc

A uniform disc of radius a rotates in its plane about the centre with uniform angular speed n. From 2.11 (3) we have $Q = -n^2\,\varrho\,z^2\bar{z}^2/32$, where ϱ is the density. The effective stress at the boundary is given by 5.11 (4) in the form

$$2\,(\widehat{aa_e} + i\,\widehat{a\theta_e}) = \frac{\varkappa+3}{2\,(\varkappa+1)}\,a^2\varrho\,n^2. \tag{1}$$

Therefore from 5.25 (7)

$$W(z) = \frac{\varkappa+3}{4\,(\varkappa+1)}\,a^2\varrho\,n^2\,,\ w(z) = 0\,,\quad (z \text{ in } L) \tag{2}$$

therefore from 5.1 (5)

$$\widehat{rr} + i\,\widehat{r\theta} = \frac{\varkappa+3}{4\,(\varkappa+1)}\,\varrho\,n^2\,(a^2 - r^2)\,,\quad \text{i. e. } \widehat{r\theta} = 0\,. \tag{3}$$

Also from 5.1 (4)

$$\widehat{rr} + \widehat{\theta\theta} = \frac{(\varkappa+3)\,a^2\varrho\,n^2}{2\,(\varkappa+1)} - \frac{2\,r^2\varrho\,n^2}{\varkappa+1}\,.$$

Therefore the *hoop stress* $\widehat{\theta\theta}$ is given by

$$\widehat{\theta\theta} = \frac{\varrho\,n^2}{4\,(\varkappa+1)}\,\{a^2\,(\varkappa+3) + r^2\,(\varkappa-5)\}$$

since $\varkappa - 5$ is negative the hoop stress is greatest at the centre where $r = 0$ and so

$$\widehat{\theta\theta}_{\max} = \frac{\varkappa+3}{4\,(\varkappa+1)}\,a^2\varrho\,n^2\,. \tag{4}$$

5.27. Circular disc in equilibrium under forces at the rim

The principle being the same whatever the number of forces, for simplicity we consider the case of two forces, F at A, $z_A = a\,e^{i\alpha}$ and $-F$ at B, $z_B = -a\,e^{-i\alpha}$.

We suppose these forces to be standard concentrated forces, the force at A being the limit when $\varepsilon \to 0$ of a uniform distribution $F/(2a\varepsilon)$ on the arc A_1A_2 where $z_{A_1} = a\,e^{i(\alpha-\varepsilon)}$, and $z_{A_2} = a\,e^{i(\alpha+\varepsilon)}$. Similarly for the force at B, we have $-F/(2a\,\varepsilon)$ on the arc $B_1\,B_2$.

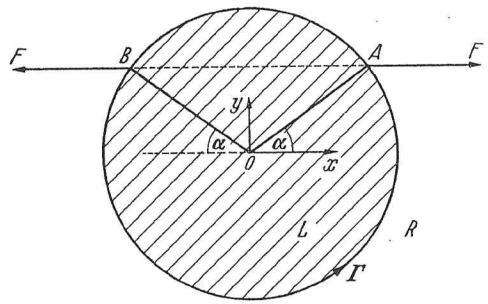

Fig. 5.27

Thus on the arc A_1A_2 we have $\widehat{aa} + i\,\widehat{a\theta} = F\,e^{-i\alpha}/(2a\,\varepsilon)$, while on the arc B_1B_2, $\widehat{aa} + i\,\widehat{a\theta} = F\,e^{i\alpha}/(2a\,\varepsilon)$. Therefore

$$\oint_\Gamma \frac{(\widehat{aa} + i\,\widehat{a\theta})\,dt}{t-z} = \frac{1}{2a\varepsilon}\int_{A_1A_2} \frac{F\,e^{-i\alpha}\,dt}{t-z} + \frac{1}{2a\varepsilon}\int_{B_1B_2} \frac{F\,e^{i\alpha}\,dt}{t-z}$$

$$= \frac{F}{2a\varepsilon}\left\{e^{-i\alpha}\ln\frac{z-z_{A_2}}{z-z_{A_1}} + e^{i\alpha}\ln\frac{z-z_{B_2}}{z-z_{B_1}}\right\}. \tag{1}$$

Now

$$z - z_{A_2} = z - a\,e^{i\alpha + i\varepsilon} = z - a\,e^{i\alpha}\,(1 + i\varepsilon + \cdots)$$
$$= (z - a\,e^{i\alpha})\left(1 - \frac{a\,e^{i\alpha}\,i\,\varepsilon}{z - a\,e^{i\alpha}} + \cdots\right)$$

Therefore the right-hand side of (1) is equal to

$$iF\left\{-\frac{1}{z - z_A} + \frac{1}{z - z_B} + O(\varepsilon)\right\}. \tag{2}$$

Similarly we show that

$$\int_0^{2\pi} \widehat{a\bar{a}}\,d\theta = \frac{2\,F\cos\alpha}{a} + O(\varepsilon)\,. \tag{3}$$

Therefore in the limit 5.25 (7) gives

$$W(z) = \frac{F}{\pi}\left\{-\frac{1}{z - a\,e^{i\alpha}} + \frac{1}{z + a\,e^{-i\alpha}} - \frac{\cos\alpha}{a}\right\}. \tag{4}$$

Observe that the boundary being unloaded, this same expression continues $W(z)$ into the region R. We find $w(z)$ from 5.21 (4) in the form

$$w(z) = \frac{F}{\pi}\left\{\frac{1}{z - a\,e^{i\alpha}} - \frac{1}{z + a\,e^{-i\alpha}} - \frac{a\,e^{-i\alpha}}{(z - a\,e^{i\alpha})^2} - \frac{a\,e^{i\alpha}}{(z + a\,e^{-i\alpha})^2}\right\}. \tag{5}$$

5.28. Circular disc in equilibrium under forces applied internally

Here again the method is sufficiently illustrated by the case of two forces, say F applied at A ($z = c$ on the real axis) and $-F$ at O.

The complex stresses $W(z)$, $w(z)$ now have singularities at O and A. For standard concentrated forces applied as described, but to material occupying the whole plane, we have from 5.24 the complex stresses

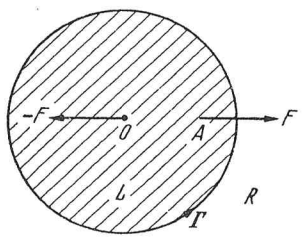

Fig. 5.28

$$W_0(z) = -\frac{G}{z_1} + \frac{G}{z}\,,$$
$$w_0(z) = \frac{\varkappa G}{z_1} - \frac{\varkappa G}{z} - \frac{c\,G}{z_1^2}\,, \tag{1}$$

where for brevity we have written

$$G = \frac{F}{\pi\,(\varkappa + 1)}\,, \quad z_1 = z - c\,. \tag{2}$$

Near the points $z = 0$, $z = c$ the complex stresses must behave like $W_0(z)$, $w_0(z)$. Therefore we can write

$$W(z) = W_0(z) + W^*(z)\,, \quad w(z) = w_0(z) + w^*(z)\,, \tag{3}$$

where $W^*(z)$, $w^*(z)$ are holomorphic in L.

Assuming body force to be absent and the boundary to be unstressed

i. e. $\widehat{a\bar{a}} + i\,\widehat{a\theta} = 0$, we have from 5.1 (5) at the boundary where $\bar{t} = a^2/t$,

$$\begin{aligned} &W^*(t) + \overline{W}^*\left(\frac{a^2}{t}\right) - \frac{a^2}{t}\,\overline{W}^{*\prime}\left(\frac{a^2}{t}\right) - \frac{a^2}{t^2}\,\overline{w}^*\left(\frac{a^2}{t}\right) \\ &= 2\,(\widehat{a\bar{a}} + i\,\widehat{a\theta}) - W_0(t) - \overline{W}_0\left(\frac{a^2}{t}\right) + \frac{a^2}{t}\,\overline{W}_0'\left(\frac{a^2}{t}\right) \\ &\quad + \frac{a^2}{t^2}\,\overline{w}_0\left(\frac{a^2}{t}\right) \\ &= 2\,(\widehat{a\bar{a}} + i\,\widehat{a\theta})^*, \quad \text{say.} \end{aligned} \tag{4}$$

Therefore from 5.25 (7)

$$W^*(z) = \frac{1}{2\,\pi i}\oint_\Gamma \frac{2\,(\widehat{a\bar{a}} + i\,\widehat{a\theta})^*\,dt}{t - z} - \frac{1}{2\,\pi i}\oint_\Gamma \frac{(\widehat{a\bar{a}} + i\,\widehat{a\theta})^*\,dt}{t}\,. \tag{5}$$

Evaluating the integrals we find that for z in L,

$$W^*(z) = \frac{F}{\pi\,(\varkappa + 1)}\left\{-\frac{2\,z}{a^2} - \frac{1}{c} - \frac{\varkappa - 1}{2\,c_1} - \frac{\varkappa}{z - c_1} + \frac{c_1^2 - a^2}{c\,(z - c_1)^2}\right\}, \; c_1 = \frac{a^2}{c}\,. \tag{6}$$

The complex stress $w(z)$ can now be obtained from 5.21 (4). We note that c, c_1 are inverse points and therefore c_1 is in the region R.

The above investigation illustrates the general method of dealing with singularities in the region L by isolating their principal parts by a substitution of the form (3).

5.32. Problem II for a circular boundary

Here we are given the effective displacement at the boundary $|z| = a$ namely $(U_a + i\,U_\theta)_e$ in polar or $U_e + i\,V_e$ in cartesian coordinates.

It is more convenient to deal not with the effective displacement $u_e + i\,v_e$ but with its partial derivative with respect to θ. This avoids the possible logarithmic singularities in $'W(z)$ and $'w(z)$.

From 5.11 (7) we have

$$4\,\mu\,(u_e' + i\,v_e') = i\,z\,\{\varkappa\,W(z) - \overline{W}(\bar{z}) + \bar{z}\,\overline{W}'(\bar{z}) + \frac{\bar{z}}{z}\,\overline{w}(\bar{z})\}\,, \tag{1}$$

where

$$u_e' = \frac{\partial u_e}{\partial \theta}\,, \quad v_e' = \frac{\partial v_e}{\partial \theta}\,. \tag{2}$$

Combine this with 5.21 (2). Then

$$4\,\mu\,(u_e' + i\,v_e') = i\,z\left\{\varkappa\,W(z) + W\left(\frac{a^2}{\bar{z}}\right) + \frac{\bar{z}}{a^2\,z}\,(a^2 - z\bar{z})\,\overline{w}(\bar{z})\right\}. \tag{3}$$

In this result let z remaining in L tend to the point $t = a\,e^{i\theta}$ of the boundary. Then $a^2/\bar{z}$ remaining in R, also tends to t, while $a^2 - z\bar{z} \to 0$. Therefore

$$i\,t\,\{\varkappa\,W^L(t) + W^R(t)\} = 4\,\mu\left(\frac{\partial U_e}{\partial \theta} + i\,\frac{\partial V_e}{\partial \theta}\right).$$

Now on the boundary $U_e + iV_e = \upsilon_e(t)$, where $\upsilon_e(t)$ is a known function of t. Therefore

$$\frac{\partial U_e}{\partial \theta} + i\frac{\partial V_e}{\partial \theta} = \frac{d\upsilon_e(t)}{dt}\frac{dt}{d\theta} = it\,\upsilon_e'(t)\,,$$

where the dash refers to differentiation with respect to t. Therefore the boundary condition is

$$W^L(t) + \frac{1}{\varkappa}W^R(t) = \frac{4\mu}{\varkappa}\upsilon_e'(t)\,. \tag{4}$$

Introduce the function $F(z)$ defined by

$$W(z) = F(z)\,, \quad (z \text{ in } L); \quad W(z) = -\varkappa F(z)\,, \quad (z \text{ in } R)\,. \tag{5}$$

Then (4) becomes

$$F^L(t) - F^R(t) = \frac{4\mu}{\varkappa}\upsilon_e'(t)$$

and so, allowing for a possible pole of the second order at $z = 0$ for the reasons indicated in 5.21,

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{4\mu\upsilon_e'(t)\,dt}{\varkappa(t-z)} + A + \frac{B}{z} + \frac{C}{z^2}\,, \tag{6}$$

where the constants B, C can be different from zero only in the case of a hole in an infinite plate.

From 5.21 (4) we have

$$w(z) = \frac{a^2}{z^2}\left\{F(z) - \varkappa\overline{F}\left(\frac{a^2}{z}\right) - zF'(z)\right\}, \quad (z \text{ in } L)\,. \tag{7}$$

Consider, for example, a rigid circular disc of radius a welded into a corresponding hole in an unloaded infinite plate and let a force $X + iY$ be applied to the disc at its centre. Since the disc is rigid $\upsilon(t) =$ constant at the boundary (for a force applied at the centre cannot cause rotation). Therefore from (6)

$$W(z) = A + \frac{B}{z} + \frac{C}{z^2}\,, \quad (z \text{ in } L); \quad W(z) = -\varkappa\left(A + \frac{B}{z} + \frac{C}{z^2}\right), \quad (z \text{ in } R)\,.$$

Substitution in (7) shows that the term in $w(z)$ independent of z is $-\varkappa\overline{C}/a^2$. Since there is no stress at infinity comparison with 2.74 (26), (27) shows that

$$A = 0\,,\ \overline{C} = 0\,,\ B = -(X + iY)/[\pi(\varkappa + 1)]\,.$$

Therefore for z in L, the expressions for $W(z)$, $w(z)$ are exactly the same as those of 5.23 (3), (5).

Thus the two problems are equivalent. It follows therefore, as in 5.24, that a standard concentrated force can be regarded as the limit when $a \to 0$, of a force $X + iY$ applied at the centre of a rigid circular disc welded on to the plate.

5.34. Problem III for a circular boundary

Let a_s, $s = 1, 2, \ldots, 2n$ be $2n$ distinct points on the circular circumference Γ or $|z| = a$.

We suppose that on the alternate arcs $a_1a_2, a_3a_4, \ldots, a_{2n-1}a_{2n}$, denoted in their totality by A_S, the applied load or effective stress is given, whereas on the complementary arcs $a_2a_3, a_4a_5, \ldots, a_{2n}a_1$, denoted in their totality by A_D, the displacement is given.

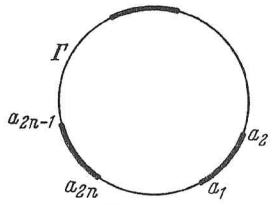

Fig. 5.34

The boundary conditions then are

$$\widehat{rr}_e + i\widehat{r\theta}_e = (\widehat{aa} + i\widehat{a\theta})_e \text{ on } A_S, \tag{1}$$

$$u_e + iv_e = U + iV = \upsilon_e(t) \quad \text{on } A_D\,, \tag{2}$$

where suffix e denotes effective values.

Therefore from 5.2 (10) and 5.32 (4) we have

$$W^L(t) - W^R(t) = 2\,(\widehat{aa} + i\widehat{a\theta})_e \quad \text{on } A_S \tag{3}$$

$$\varkappa W^L(t) + W^R(t) = 4\mu\,\upsilon_e'(t) \quad \text{on } A_D \tag{4}$$

where dashes denote partial differentiation with respect to t. These are equivalent to the single equation

$$W^L(t) = c\,W^R(t) + f(t)\,, \tag{5}$$

where

$$c = 1, f(t) = 2\,(\widehat{aa} + i\widehat{a\theta})_e \quad \text{on } A_S$$

$$c = -\frac{1}{\varkappa}, f(t) = \frac{4\mu\,\upsilon_e'(t)}{\varkappa} \text{ on } A_D$$

and this is precisely the problem resolved in 3.44. In fact we write

$$e^{2\pi i\tau} = -\frac{1}{\varkappa}\,,\ \tau = \frac{1}{2} + \frac{i\ln\varkappa}{2\pi} = \frac{1}{2} + i\lambda$$

$$\chi(z) = \frac{\{(z-a_1)(z-a_3)\ldots(z-a_{2n-1})\}^{\tau-1}}{\{(z-a_2)(z-a_4)\ldots(z-a_{2n})\}^{\tau}}\,. \tag{6}$$

In the case of a circular disc

$$W(z) = \frac{\chi(z)}{2\pi i}\oint_\Gamma \frac{f(t)\,dt}{\chi^L(t)(t-z)} + P_n(z)\,\chi(z)\,, \tag{7}$$

where $P_n(z)$ is a polynomial of degree n at most.

In the case of a circular hole

$$W(z) = \frac{\chi(z)}{2\pi i}\oint_\Gamma \frac{f(t)\,dt}{\chi^L(t)(t-z)} + \left\{\frac{B}{z} + \frac{C}{z^2} + P_n(z)\right\}\chi(z) \tag{8}$$

the additional constants B and C appearing on account of the pole of the second order which may occur at $z = 0$. They are determined from

5.22 (5) which shows that near $z = 0$

$$\chi(z)\left\{\frac{B}{z}+\frac{C}{z^2}\right\}=\frac{\varkappa(X+iY)}{\pi(\varkappa+1)}\cdot\frac{1}{z}+a^2(N_\infty-T_\infty)\,e^{2i\varphi}\cdot\frac{1}{z^2}+\cdots$$

The coefficient of z^n in $P_n(z)$ may be obtained by observing that for large values of $|z|$

$$W(z)=\frac{1}{2}(N_\infty+T_\infty)+\frac{4\mu i\varepsilon_\infty}{\varkappa+1}-\frac{X+iY}{\pi(\varkappa+1)}\frac{1}{z}+\cdots$$

The considerations which govern the determination of the coefficients in $P_n(z)$ are the same as those already explained in connection with the half-plane in 4.40.

5.36. Problem IV for a circular boundary

Let the boundary be the circumference Γ, $|z| = a$. We are given the normal displacement U_a and the tangential or shear stress $\widehat{a\theta}$ on the boundary.

If $z = re^{i\theta}$, in the absence of body force, we have from 5.11 (4), (6)

$$\left\{zW'(z)+\frac{z}{\bar z}w(z)\right\}-\left\{\bar z\overline{W}'(\bar z)+\frac{\bar z}{z}\bar w(\bar z)\right\}=4i\widehat{r\theta}\,, \tag{1}$$

$$\{\varkappa\bar z\,'W(z)-z\bar z\,\overline{W}(\bar z)-\bar z\,'\bar w(\bar z)\}$$
$$+\{\varkappa z\,'\overline{W}(\bar z)-z\bar z\,W(z)-z\,'w(z)\}=8\mu r u_r. \tag{2}$$

To these we apply the general continuation theorem of 5.215 which gives for (1)

$$zW'(z)+\frac{z^2}{a^2}w(z)=\frac{1}{2\pi i}\oint_\Gamma\frac{4i\widehat{a\theta}}{t-z}dt+\psi_1(z) \tag{3}$$

with the continuation definition

$$zW'(z)+\frac{z^2}{a^2}w(z)=\frac{a^2}{z}\overline{W}'\left(\frac{a^2}{z}\right)+\frac{a^2}{z^2}\bar w\left(\frac{a^2}{z}\right),\ (z\text{ in }R)\,. \tag{4}$$

For (2) we obtain

$$\frac{\varkappa a^2}{z}\,'W(z)-a^2W(z)-z\,'w(z)=\frac{1}{2\pi i}\oint_\Gamma\frac{8\mu a U_a}{t-z}dt+\psi_2(z)\,, \tag{5}$$

with

$$\frac{\varkappa a^2}{z}\,'W(z)-a^2W(z)-z\,'w(z)=-\varkappa z\,'\overline{W}\left(\frac{a^2}{z}\right)$$
$$+a^2\overline{W}\left(\frac{a^2}{z}\right)+\frac{a^2}{z}\,'\bar w\left(\frac{a^2}{z}\right),\ (z\text{ in }R). \tag{6}$$

Typical problems of this type are furnished by the *smooth* contact between a plate containing a circular hole and an inserted elastic disc of the same or different material. The interface condition may be treated as in 4.50.

5.37. Elastic circular disc inserted in a rigid plate

Suppose that we insert an elastic circular disc of radius $a+\varepsilon$, ε small, into a circular hole of radius a in a rigid plate, the contact being *smooth*. This is an example of Problem IV with $U_a=\varepsilon$, $\widehat{a\theta}=0$. Therefore from 5.36 (3)

$$zW'(z)+\frac{z^2}{a^2}w(z)=\psi_1(z) \tag{1}$$

with

$$zW'(z)+\frac{z^2}{a^2}w(z)=\frac{a^2}{z}\overline{W}'\left(\frac{a^2}{z}\right)+\frac{a^2}{z^2}\bar w\left(\frac{a^2}{z}\right),\quad (z\text{ in }R) \tag{2}$$

and from 5.36 (5)

$$\frac{\varkappa a^2}{z}\,'W(z)-a^2W(z)-z\,'w(z)=8\mu a\varepsilon+\psi_2(z)\text{ or }\psi_2(z), \tag{3}$$

according as z is in L or R, with

$$\frac{\varkappa a^2}{z}\,'W(z)-a^2W(z)-z\,'w(z)=-\varkappa z\,'\overline{W}\left(\frac{a^2}{z}\right)+a^2\overline{W}\left(\frac{a^2}{z}\right)$$
$$+\frac{a^2}{z}\,'\bar w\left(\frac{a^2}{z}\right),\quad (z\text{ in }R)\,. \tag{4}$$

The region occupied by the material being simply connected, dislocations are impossible and $'W(z)$, $'w(z)$ are holomorphic in L.

In (3) write $a^2/\bar z$ for z (z in R) and take conjugates. Then using (4) and (3) again we get

$$8\mu a\varepsilon+\overline{\psi}_2\left(\frac{a^2}{z}\right)=-\psi_2(z)\,. \tag{5}$$

Since the function of the right of (2) is holomorphic for z in R, it vanishes at infinity and therefore

$$zW'(z)+\frac{z^2}{a^2}w(z)=0\quad (z\text{ in }L) \tag{6}$$

and therefore from (1), $\psi_1(z)=0$.

Again if we assume, without loss of generality, that $'W(0)=0$, the function on the right of (4) is holomorphic at $z=\infty$ and therefore the function on the left of (4) is holomorphic when z is in L.

Therefore, when z is in L, $\psi_2(z)$ is a constant, say c, which we can take to be real, and so from (5) $c=-4\mu a\varepsilon$. Therefore, from (3),

$$\frac{\varkappa a^2}{z}\,'W(z)-a^2W(z)-z\,'w(z)=4\mu a\varepsilon. \tag{7}$$

Divide this equation by z, differentiate and then eliminate $w(z)$ by using (6). We then get the differential equation satisfied by $'W(z)$ namely

$$('W(z))'-\frac{1}{z}\frac{2\varkappa}{\varkappa+1}\,'W(z)=-\frac{4\mu\varepsilon}{a(\varkappa+1)}\,,$$

which gives, on integration,

$$'W(z) = \frac{4\mu\varepsilon}{\varkappa - 1}\frac{z}{a} + Dz^{\frac{2\varkappa}{\varkappa+1}}.$$

Since $2\varkappa/(\varkappa+1)$ is not an integer the last term is many valued. But $'W(z)$ is holomorphic in L. Therefore $D = 0$ and

$$'W(z) = \frac{4\mu\varepsilon}{\varkappa - 1}\frac{z}{a}. \tag{8}$$

and therefore from (6)

$$w(z) = 0. \tag{9}$$

5.4. Plate cut along arcs of a circle

The circumference Γ, $|z| = a$, separates the plane into two regions L and R of which L is the interior if we take the positive sense of description of Γ to be counterclockwise.

The material is cut along n arcs $A_s = a_{2s-1}\,a_{2s}$, $(s = 1, 2, \ldots, n)$ of Γ, and we write

$$A = A_1 + A_2 + \cdots + A_n. \tag{1}$$

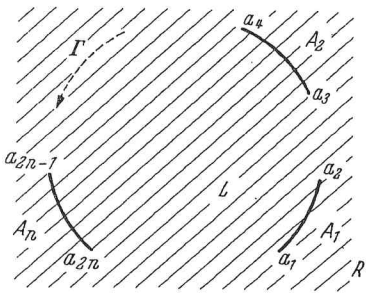

Fig. 5.4

Each cut has an inner edge L and an outer edge R. In Problem I we are given the effective stresses

$$(\widehat{aa_e} + i\,\widehat{a\theta_e})^L, \quad (\widehat{aa_e} + i\,\widehat{a\theta_e})^R$$

on the inner and outer edges. In Problem II we are given the effective displacements, and therefore their partial derivatives with respect to θ at the edges of the cuts.

Mathematically Problems I and II differ in no essential details. We shall therefore consider Problem I only.

The complex stresses $W(z)$, $w(z)$ are holomorphic in the whole plane cut along A.

We have

$$2\,(\widehat{rr_e} + i\,\widehat{r\theta_e}) = W(z) + \overline{W}(\bar{z}) - \bar{z}\,\overline{W}'(\bar{z}) - \frac{\bar{z}}{z}\,\overline{w}(\bar{z}). \tag{2}$$

Write

$$\Omega(z) = -\overline{W}\left(\frac{a^2}{z}\right) + \frac{a^2}{z}\,\overline{W}'\left(\frac{a^2}{z}\right) + \frac{a^2}{z^2}\,\overline{w}\left(\frac{a^2}{z}\right), \tag{3}$$

from which it appears that $\Omega(z)$ is holomorphic in the whole cut plane except perhaps at $z = 0$ where it may have a pole of the second order.

In (3) write $a^2/\bar{z}$ for z, and add to (2). Then

$$2\,(\widehat{rr_e} + i\,\widehat{r\theta_e}) = W(z) - \Omega\left(\frac{a^2}{\bar{z}}\right) + \left(\frac{\bar{z}^2}{a^2} - \frac{\bar{z}}{z}\right)\overline{w}(\bar{z}). \tag{4}$$

In this let $z \to t$, a point of the boundary Γ, first from L, secondly from R. Then using the boundary values we have

$$2\,(\widehat{aa_e} + i\,\widehat{a\theta_e})^L = W^L(t) - \Omega^R(t)$$

$$2\,(\widehat{aa_e} + i\,\widehat{a\theta_e})^R = W^R(t) - \Omega^L(t).$$

Thus by subtraction and addition

$$[W(t) + \Omega(t)]^L - [W(t) + \Omega(t)]^R = 2p_1(t), \tag{5}$$

$$[W(t) - \Omega(t)]^L + [W(t) - \Omega(t)]^R = 2p_2(t), \tag{6}$$

where $p_1(t)$, $p_2(t)$ are known functions of t, and we solve these equations by the method of 3.43 to give $W(z) + \Omega(z)$ and $W(z) - \Omega(z)$ whence $W(z)$, $\Omega(z)$ are obtained by addition and subtraction. To examine the form of the solution we must look into the behaviour of $W(z)$ and $\Omega(z)$ at $z = 0$ where $W(z)$ is holomorphic but $\Omega(z)$ may have a pole of the second order, and at $z = \infty$ where both functions are holomorphic.

We assume, for simplicity, there is no dislocation, and no rotation at infinity.

Near $z = \infty$ we have from 2.74 (26) (27)

$$W(z) = A_0 - \frac{B_0}{z} + O\left(\frac{1}{z^2}\right), \quad A_0 = \frac{1}{2}(N_\infty + T_\infty), \quad B_0 = \frac{X + iY}{\pi(\varkappa + 1)} \tag{7}$$

$$\begin{aligned} w(z) &= A_1 + \frac{B_1}{z} + O\left(\frac{1}{z^2}\right), \\ A_1 &= (N_\infty - T_\infty)\,e^{-2i\varphi}, \quad B_1 = \frac{\varkappa(X - iY)}{\pi(\varkappa + 1)}. \end{aligned} \tag{8}$$

Substitution in (3) then shows that

$$\Omega(z) = \frac{\overline{A}_1 a^2}{z^2} + \frac{\overline{B}_1}{z} + \text{holomorphic function, near } z = 0. \tag{9}$$

If in (3) we write $a^2/\bar{z}$ for z and then take conjugates, we get

$$w(z) = \frac{a^2}{z^2}\,\overline{\Omega}\left(\frac{a^2}{z}\right) + \frac{a^2}{z^2}\,W(z) - \frac{a^2}{z}\,W'(z), \tag{10}$$

and this has to be holomorphic near $z = 0$.

$$W(z) = \alpha_0 + \alpha_1 z + \cdots, \text{ near } z = 0, \tag{11}$$

$$\Omega(z) = \beta_0 + \frac{\beta_1}{z} + \cdots, \text{ near } z = \infty. \tag{12}$$

Substitution in (10) gives

$$w(z) = (\alpha_0 + \bar{\beta}_0)\frac{a^2}{z^2} + \frac{\beta_1}{z} + \bar{\beta}_2 - \alpha_2 a^2 + \cdots, \text{ near } z = 0 \tag{13}$$

and therefore $w(z)$ is holomorphic only if

$$\alpha_0 + \bar{\beta}_0 = 0\,, \quad \beta_1 = 0\,. \tag{14}$$

Thus

$$\Omega(z) = -\bar{\alpha}_0 + O\left(\frac{1}{z^2}\right), \quad \text{near } z = \infty\,. \tag{15}$$

We now see that the behaviour of $W(z)$, $\Omega(z)$ near $z = \infty$ is given by (7), (15), while their behaviour near $z = 0$ is given by (11) and (9). Returning to (5) and noting that $W(z)$, $\Omega(z)$ are bounded at infinity we have on the model of 3.31 and 3.43

$$W(z) + \Omega(z) = \frac{1}{2\pi i}\oint_A \frac{2\,p_1(t)\,dt}{t-z} + \gamma_0 + \frac{\bar{B}_1}{z} + \frac{\bar{A}_1 a^2}{z^2}\,, \tag{16}$$

$$W(z) - \Omega(z) = \frac{\chi(z)}{2\pi i}\oint_A \frac{2\,p_2(t)\,dt}{\chi^L(t)\,(t-z)} + \chi(z)\left\{P_n(z) + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2}\right\}, \tag{17}$$

where $\gamma_0, \gamma_1, \gamma_2$ are constants to be determined,

$$P_n(z) = c_0 + c_1 z + \cdots + c_n z^n \tag{18}$$

$$e^{2\pi i \tau} = -1\,, \quad \tau = \frac{1}{2} \tag{19}$$

$$\chi(z) = \prod_{s=1}^{2n} (z - a_s)^{-1/2}\,. \tag{20}$$

The determination of the coefficients $c_0, \ldots, c_n$ proceeds on the same lines as in the case of the cut-plane, 4.13. This problem was first proposed and solved by MUSKHELISHVILI.

5.41. Plate under tension cut along a circular arc

Let a cut be made along the arc $a_1 a_2$ where $a_1 = a\,e^{i\alpha}$, $a_2 = a\,e^{-i\alpha}$, and let the plate be subject to tension at infinity. Then in the notation of 5.4

$$n = 1,\ X + iY = 0\,, \quad p_1(t) = p_2(t) = 0\,,$$

$$B_0 = B_1 = 0,\ \chi(z) = (z - a\,e^{i\alpha})^{-1/2}\,(z - a\,e^{-i\alpha})^{-1/2}.$$

Therefore from 5.4 (16), (17) by addition and subtraction

$$W(z) = \frac{1}{2}\gamma_0 + \frac{a^2\bar{A}_1}{2z^2} + \frac{1}{2}\chi(z)\left\{c_1 z + c_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2}\right\} \tag{1}$$

$$\Omega(z) = \frac{1}{2}\gamma_0 + \frac{a^2\bar{A}_1}{2z^2} - \frac{1}{2}\chi(z)\left\{c_1 z + c_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2}\right\}. \tag{2}$$

Since $z - a\,e^{i\alpha} = -a\,e^{i\alpha}(1 - z\,e^{-i\alpha}/a) = e^{i\pi}\,a\,e^{i\alpha}(1 - z\,e^{-i\alpha}/a)$ near $z = 0$

$$\begin{aligned}\chi(z) &= -\frac{1}{a}\left(1 - \frac{z\,e^{i\alpha}}{a}\right)^{-1/2}\left(1 - \frac{z\,e^{-i\alpha}}{a}\right)^{-1/2} \\ &= -\frac{1}{a} - \frac{z\cos\alpha}{a^2} - \frac{z^2(1 + 3\cos 2\alpha)}{4a^3} + \cdots\end{aligned} \tag{3}$$

Therefore near $z = 0$

$$\begin{aligned}W(z) = \frac{1}{2}\gamma_0 + \frac{a^2\bar{A}_1}{2z^2} - \frac{1}{2}\frac{\gamma_2}{az^2} - \frac{1}{2}(a\,\gamma_1 + \gamma_2\cos\alpha)\frac{1}{a^2 z} \\ - \frac{1}{2}\left\{\frac{c_0}{a} + \frac{\gamma_1\cos\alpha}{a^2} + \frac{\gamma_2(1 + 3\cos 2\alpha)}{4a^3}\right\} + \cdots\end{aligned} \tag{4}$$

and since $W(z)$ must be holomorphic at $z = 0$

$$\gamma_2 = a^3\bar{A}_1, \quad \gamma_1 = -a^2\cos\alpha\,\bar{A}_1\,. \tag{5}$$

Now from (4), (5) and 5.4 (11)

$$\alpha_0 = W(0) = \frac{1}{2}\gamma_0 - \frac{c_0}{2a} + \frac{1}{4}\bar{A}_1\sin^2\alpha\,,$$

while from (2) and 5.4 (15)

$$-\bar{\alpha}_0 = \Omega(\infty) = \frac{1}{2}\gamma_0 - \frac{1}{2}c_1\,.$$

Eliminating α_0 we get

$$c_1 = \gamma_0 + \bar{\gamma}_0 - \frac{\bar{c}_0}{a} + \frac{1}{2}A_1\sin^2\alpha\,. \tag{6}$$

Again near $z = \infty$, we have $\chi(z) = \frac{1}{z} + \frac{a\cos\alpha}{z^2} + \cdots$ and therefore

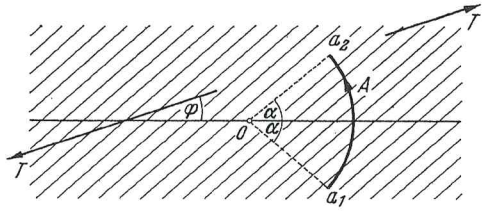

Fig. 5.41

from (1) and 5.4 (7), by equating coefficients

$$c_1 + \gamma_0 = 2A_0,\ c_0 + c_1\,a\cos\alpha = 0\,. \tag{7}$$

From (5), (6), (7) we can determine all the unknown coefficients $\gamma_0, \gamma_1, \gamma_2, c_0, c_1$. In particular for all round tension T and a semicircular cut

$$A_0 = T,\ A_1 = 0,\ \alpha = \frac{\pi}{2}$$

and therefore

$$\gamma_1 = \gamma_2 = 0,\ c_0 = 0,\ \chi(z) = (z^2 + a^2)^{-1/2}.$$

Thus from (6), (7)

$$c_1 + \gamma_0 = 2T,\ c_1 = \gamma_0 + \bar{\gamma}_0 = 4T - 2c_1$$

so that $c_1 = 4T/3$, $\gamma_0 = 2T/3$,

$$W(z) = \frac{1}{3}T\left\{1 + \frac{2z}{\sqrt{(a^2+z^2)}}\right\},\quad \Omega(z) = \frac{1}{3}T\left\{1 - \frac{2z}{\sqrt{(a^2+z^2)}}\right\}.$$

5.5. The compatibility identity for the annulus

The term *annulus* will refer to the region between two concentric circumferences. When the circumferences are not concentric, we have an *eccentric annulus*, which will be discussed in the next chapter.

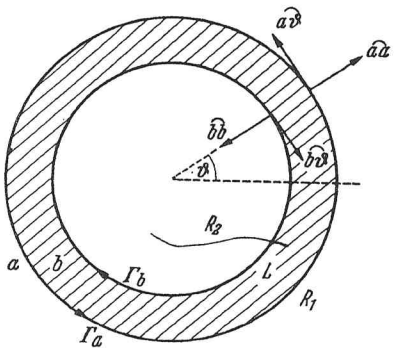

Fig. 5.5

We consider material occupying the region between an outer circumference Γ_a of radius a and an inner concentric circumference Γ_b of radius b.

Counterclockwise description of Γ_a and clockwise description of Γ_b determine the region L occupied by the material and two right-hand regions R_1 exterior to Γ_a and R_2 interior to Γ_b.

In the situation here described we may regard the annulus as representing a ring-shaped region or washer cut from a plate, or the crosssection of a tube or pipe, according as we are dealing with generalized plane stress or plane deformation.

The complex stresses $W(z)$, $w(z)$ are holomorphic at every point between Γ_a and Γ_b. The points of L thus correspond by inversion with respect to Γ_a to points of R_1 between Γ_a and a circumference of radius a^2/b. Similarly the points of L correspond by inversion with respect to Γ_b to points of R_2 between Γ_b and a circumference of radius b^2/a.

To each of Γ_a, Γ_b we can apply the continuation method of 5.21 to give

$$W(z) = -\overline{W}\left(\frac{a^2}{z}\right) + \frac{a^2}{z}\overline{W}'\left(\frac{a^2}{z}\right) + \frac{a^2}{z^2}\bar{w}\left(\frac{a^2}{z}\right),\quad (a^2/z \text{ in } L) \tag{1}$$

$$w(z) = \frac{a^2}{z^2}\left\{\overline{W}\left(\frac{a^2}{z}\right) + W(z) - zW'(z)\right\},\quad (z \text{ in } L) \tag{2}$$

$$W(z) = -\overline{W}\left(\frac{b^2}{z}\right) + \frac{b^2}{z}\overline{W}'\left(\frac{b^2}{z}\right) + \frac{b^2}{z^2}\bar{w}\left(\frac{b^2}{z}\right),\quad (b^2/z \text{ in } L) \tag{3}$$

$$w(z) = \frac{b^2}{z^2}\left\{\overline{W}\left(\frac{b^2}{z}\right) + W(z) - zW'(z)\right\},\quad (z \text{ in } L). \tag{4}$$

We note that $w(z)$ is defined in two different ways by (2) and (4). These values must agree and therefore we have the *compatibility identity*

$$(a^2 - b^2)\,[W(z) - zW'(z)] \equiv b^2\overline{W}\left(\frac{b^2}{z}\right) - a^2\overline{W}\left(\frac{a^2}{z}\right),\quad (z \text{ in } L). \tag{5}$$

This identity replaces the condition, in the case of a single circular boundary, that $w(z)$ must be holomorphic in L.

5.51. Problem I for the annulus

We are given the effective stresses $\widehat{aa}_e + i\,\widehat{a\theta}_e$ on Γ_a and $\widehat{bb}_e + i\,\widehat{b\theta}_e$ on Γ_b. The boundary conditions from 5.2 (10) are, fig. 5.5,

$$W^L(t) - W^{R_1}(t) = 2\,(\widehat{aa}_e + i\,\widehat{a\theta}_e),\quad (t \text{ in } \Gamma_a), \tag{1}$$

$$W^L(t) - W^{R_2}(t) = 2\,(\widehat{bb}_e + i\,\widehat{b\theta}_e),\quad (t \text{ in } \Gamma_b), \tag{2}$$

which are satisfied by

$$W(z) = \frac{1}{2\pi i}\oint_{\Gamma_a}\frac{2\,(\widehat{aa}_e + i\,\widehat{a\theta}_e)}{t-z}\,dt + \frac{1}{2\pi i}\cdot\oint_{\Gamma_b}\frac{2\,(\widehat{bb}_e + i\,\widehat{b\theta}_e)}{t-z}\,dt + \chi(z), \tag{3}$$

for each integral yields a continuous function of z at points on the line of integration of the other integral.

Let $W_1(z)$, $W_L(z)$, $W_2(z)$ denote the sum of the integrals in (3) when z is in R_1, L, R_2 respectively (see 3.13). Then (3) yields successively

$$W(z) = W_1(z) + \chi(z),\quad (z \text{ in } R_1) \tag{4}$$

$$W(z) = W_L(z) + \chi(z),\quad (z \text{ in } L) \tag{5}$$

$$W(z) = W_2(z) + \chi(z),\quad (z \text{ in } R_2) \tag{6}$$

The function $\chi(z)$ must be determined so that the compatibility identity

$$(a^2 - b^2)\,[W(z) - zW'(z)] = b^2\overline{W}\left(\frac{b^2}{z}\right) - a^2\overline{W}\left(\frac{a^2}{z}\right),\quad (z \text{ in } L) \tag{7}$$

is satisfied, for example by assuming a Laurent expansion

$$\chi(z) = \sum_{n=-\infty}^{\infty}\beta_n z^n \tag{8}$$

and equating coefficients.

When $W_1(z)$, $W_L(z)$, $W_2(z)$ are all constants it is sufficient to take $\chi(z) = A$, a constant determined by

$$2(a^2 - b^2)\,A = -(a^2 - b^2)\,W_L - a^2\overline{W}_1 + b^2\overline{W}_2. \tag{9}$$

In Problem I, as proved in 2.13, the constant A is determinate save for an imaginary added constant, whose value does not affect the stresses. Therefore without loss of generality we may take A to be real, which we shall do without further comment.

5.52. Annulus under all round tension

The boundary conditions (fig. 5.5) are

$$\widehat{aa} + i\,\widehat{a\theta} = T \quad \text{on} \quad \Gamma_a\,, \quad \widehat{bb} + i\,\widehat{b\theta} = T \quad \text{on} \quad \Gamma_b$$

which yield from 5.51 (4), (5), (6),

$$W(z) = A \qquad (z \text{ in } R_1)$$

$$W(z) = 2T + A \qquad (z \text{ in } L)$$

$$W(z) = A \qquad (z \text{ in } R_2)\,.$$

Together with the compatibility identity 5.51 (7)

$$(a^2 - b^2)\,(2T + A) \equiv b^2\,\bar{A} - a^2\,\bar{A}\,.$$

Thus $A = -T$, since we take A to be real, and therefore

$$W(z) = T\,, \quad w(z) = 0 \quad (z \text{ in } L)\,.$$

5.53. Annulus in equilibrium under couples uniformly applied

In fig. 5.5 the boundary condition

$$\widehat{aa} + i\,\widehat{a\theta} = \frac{iM}{2\pi a^2} \text{ on } \Gamma_a\,, \quad \widehat{bb} + i\,\widehat{b\theta} = \frac{iM}{2\pi b^2} \text{ on } \Gamma_b\,,$$

lead to equilibrium under couples M, $-M$ applied uniformly at the boundaries.

Then from 5.51 (4)—(7)

$$W(z) = A \qquad (z \text{ in } R_1)$$

$$W(z) = \frac{iM}{\pi a^2} + A \qquad (z \text{ in } L)$$

$$W(z) = \frac{iM}{\pi a^2} - \frac{iM}{\pi b^2} + A\,, \qquad (z \text{ in } R_2)\,.$$

The compatibility identity gives $A + \bar{A} = 0$ and since A is real, $A = 0$,

$$W(z) = 0,\ \frac{iM}{\pi a^2}\,,\ \frac{iM}{\pi a^2} - \frac{iM}{\pi b^2} \text{ in } R_1,\ L,\ R_2 \text{ respectively.}$$

Substituting in 5.5 (2) we get $w(z) = iM/(\pi z^2)$. Therefore we can take

$$W(z) = 0\,, \quad w(z) = \frac{iM}{\pi z^2}\,, \quad (z \text{ in } L),$$

since the addition of an imaginary constant to $W(z)$ does not affect the stress distribution.

5.54. Lamé's problem of the tube under pressure

A tube whose cross-section is shown in fig. 5.5 is subjected to hydrostatic pressure, p_a at the external and p_b at the internal surface. From 5.51 (4)—(7) we have

$$W(z) = A \qquad (z \text{ in } R_1)$$

$$W(z) = -2p_a + A\,, \qquad (z \text{ in } L)$$

$$W(z) = -2p_a + 2p_b + A \qquad (z \text{ in } R_2)$$

and the compatibility identity gives

$$A = p_a + \frac{b^2}{a^2 - b^2}\,(p_b - p_a)\,,$$

whence

$$W(z) = \frac{-a^2 p_a + b^2 p_b}{a^2 - b^2} = P, \tag{1}$$

$$w(z) = \frac{2\,a^2 b^2\,(p_b - p_a)}{a^2 - b^2} \cdot \frac{1}{z^2} = \frac{2\,Q}{z^2}\,. \tag{2}$$

Using 5.11 (3), (4) we get for the radial and hoop stresses at distance r from the centre

$$\widehat{rr} = P - \frac{Q}{r^2}\,, \quad \widehat{\theta\theta} = P + \frac{Q}{r^2} \tag{3}$$

If we make the common hypothesis of elastic failure in a tube namely that the maximum stress difference may not exceed a certain value m, we have

$$(\widehat{\theta\theta} - \widehat{rr})_{\max} = \left(\frac{2\,Q}{r^2}\right)_{\max} = \frac{2\,a^2\,(p_b - p_a)}{a^2 - b^2} < m\,.$$

Therefore to avoid elastic failure we must have

$$p_b - p_a < \left(\frac{1}{2} - \frac{b^2}{2\,a^2}\right) m < \frac{1}{2}\,m\,.$$

Thus however thick the tube may be, it will fail if the pressure difference $p_b - p_a$ exceeds $\frac{1}{2}\,m$. Elastic failure does not necessarily

imply rupture, which will, in general, occur at a higher pressure difference. To find the displacement we have from 5.11

$$4\mu\,(u_r + i\,u_\theta) = \frac{\bar z}{r}\left\{\varkappa P z - P z + \frac{2\,Q}{\bar z}\right\}$$

ignoring a rigid body displacement of the material as a whole. Thus $u_\theta = 0$ and at distance r from the centre

$$4\mu\,u_r = (\varkappa - 1)\,P r + \frac{2\,Q}{r}. \tag{4}$$

In particular if $p_a = 0$, i. e. there is only internal pressure,

$$4\mu\,u_r = \frac{b^2}{r\,(a^2 - b^2)}\,\{(\varkappa - 1)\,r^2 + 2\,a^2\}\,p_b\,. \tag{5}$$

Again if $p_b = 0$, i. e. there is only external pressure,

$$4\mu\,u_r = \frac{a^2}{r\,(a^2 - b^2)}\,\{(\varkappa - 1)\,r^2 + 2\,b^2\}\,(-p_a)\,. \tag{6}$$

From this it appears that under internal pressure all elements move outwards, while under external pressure all elements move inwards.

5.55. Shrinkage allowance

To increase strength or to promote solidity the process of forcibly fitting or shrinking on is frequently adopted. Thus a cylindrical plug can be forced into a hole of smaller diameter; a metal tyre can be heated to expand its diameter before placing it on the wheel; gun tubes are made by shrinking several tubes one on the other successively.

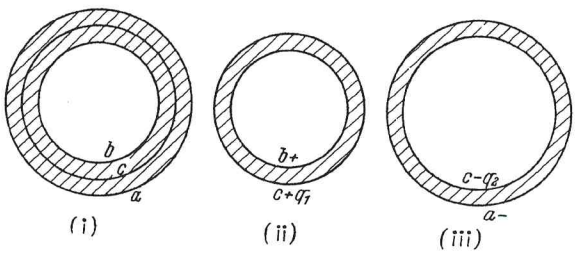

Fig. 5.55

Consider the tube consisting of two cylinders one shrunk on the other, of radii a, c, b, depicted in fig. 5.55 (i). The process of shrinking on creates a pressure p at the interface of radius c. We could, in imagination at least, separate the tubes by applying a negative pressure $-p$ i. e. a tension of this amount at the interface. The inner tube would then expand to outer radius $c + q_1$, fig. (ii), the outer would contract to inner radius $c - q_2$, fig. (iii). The difference of these $(c + q_1) - (c - q_2) = q_1 + q_2$

is called the *shrinkage allowance* necessary to create the pressure p at the interface of the assembled tube. The radii a and b also differ slightly from their original values, but this difference is unimportant so far as the fitting is concerned.

Using 5.54 (5), (6) we find that

$$4\mu\,(q_1 + q_2) = \frac{p\,c^3(a^2 - b^2)\,(\varkappa + 1)}{(a^2 - c^2)\,(c^2 - b^2)}\,.$$

5.56. Rotating annulus

When an annulus of radii a, b and density ϱ rotates about its centre with uniform angular speed n, we have as in 5.26 (1)

$$2\,(\widehat{a\,a_e} + i\,\widehat{a\,\theta_e}) = \frac{\varkappa + 3}{2\,(\varkappa + 1)}\,a^2\varrho\,n^2,\; 2\,(\widehat{b\,b_e} + i\,\widehat{b\,\theta_e}) = \frac{\varkappa + 3}{2\,(\varkappa + 1)}\,b^2\varrho\,n^2 \tag{1}$$

and therefore from 5.51

$$W(z) = A \qquad (z \text{ in } R_1)$$

$$W(z) = \frac{\varkappa + 3}{2\,(\varkappa + 1)}\,a^2\varrho n^2 + A \qquad (z \text{ in } L)$$

$$W(z) = \frac{\varkappa + 3}{2\,(\varkappa + 1)}\,(a^2 - b^2)\,\varrho n^2 + A \qquad (z \text{ in } R_2)\,.$$

The compatibility identity gives $B = C = 0$ and

$$A = -\frac{\varkappa + 3}{4\,(\varkappa + 1)}\,(a^2 - b^2)\,\varrho\,n^2$$

whence

$$W(z) = \frac{\varkappa + 3}{4\,(\varkappa + 1)}\,(a^2 + b^2)\,\varrho\,n^2\,, \quad w(z) = \frac{\varkappa + 3}{2\,(\varkappa + 1)}\,\frac{a^2 b^2\,\varrho\,n^2}{z^2}. \tag{2}$$

Since from 5.26, $Q = -\,\varrho\,n^2 z^2 \bar z^2/32$, 5.11 (4) gives

$$\widehat{r\,r} + i\,\widehat{r\,\theta} = \frac{\varkappa + 3}{4\,(\varkappa + 1)}\left(a^2 + b^2 - r^2 - \frac{a^2 b^2}{r^2}\right)\varrho\,n^2. \tag{3}$$

Also from 5.11 (3)

$$\widehat{r\,r} + \widehat{\theta\,\theta} = \frac{\varkappa + 3}{2\,(\varkappa + 1)}\,(a^2 + b^2)\,\varrho\,n^2 - \frac{2}{\varkappa + 1}\,r^2\varrho\,n^2. \tag{4}$$

Therefore

$$\widehat{\theta\,\theta} = \frac{\varrho\,n^2}{4\,(\varkappa + 1)}\left\{(\varkappa + 3)\left(a^2 + b^2 + \frac{a^2 b^2}{r^2}\right) + (\varkappa - 5)\,r^2\right\} \tag{5}$$

The maximum of this occurs at the inner boundary where $r = b$

$$\widehat{\theta\,\theta}_{\max} = \frac{\varrho\,n^2}{2\,(\varkappa + 1)}\,\{a^2\,(\varkappa + 3) + b^2\,(\varkappa - 1)\}\,. \tag{6}$$

The limit of this when $b \to 0$ is $(\varkappa + 3)\,a^2\varrho\,n^2/[2\,(\varkappa + 1)]$ which is twice $\widehat{\theta\,\theta}_{\max}$ for a rotating disc (cf. 5.26 (4)). This apparent paradox

is explained by the fact that a simply-connected region (the disc) cannot be regarded as the limit of a doubly-connected region (the annulus) for the centre remains a point not belonging to the region.

5.6. Problem II for the annulus

Here we are given the effective displacement $\upsilon_e^a(t)$ at the outer boundary Γ_a and $\upsilon_e^b(t)$ at the inner boundary Γ_b. We use the function $F(z)$ introduced in 5.32 such that

$$W(z) = -\varkappa F(z) \quad (z \text{ in } R_1)\,, \tag{1}$$

$$W(z) = F(z) \quad (z \text{ in } L)\,, \tag{2}$$

$$W(z) = -\varkappa F(z) \quad (z \text{ in } R_2)\,, \tag{3}$$

where $F(z)$ satisfies the boundary conditions

$$F^L(t) - F^{R_1}(t) = \frac{4\mu}{\varkappa}\,[\upsilon_e^a(t)]'\,,$$

$$F^L(t) - F^{R_2}(t) = \frac{4\mu}{\varkappa}\,[\upsilon_e^b(t)]'\,.$$

These are satisfied by

$$F(z) = \frac{1}{2\pi i}\oint_{\Gamma_a}\frac{4\mu\,[v_e^a(t)]'\,dt}{\varkappa\,(t-z)} + \frac{1}{2\pi i}\oint_{\Gamma_b}\frac{4\mu\,[v_e^b(t)]'\,dt}{\varkappa\,(t-z)} + \Phi(z) \tag{4}$$

since each integral is holomorphic on the path of integration of the other and where $\Phi(z)$ is continuous across both boundaries.

Let $F_1(z), F_L(z), F_2(z)$ denote the sum of the integrals in (4) when z is in R_1, L, R_2 respectively. Then (4) yields successively

$$W(z) = -\varkappa\,[F_1(z) + \Phi(z)]\,, \quad (z \text{ in } R_1) \tag{5}$$

$$W(z) = F_L(z) + \Phi(z) \quad (z \text{ in } L) \tag{6}$$

$$W(z) = -\varkappa\,[F_2(z) + \Phi(z)] \quad (z \text{ in } R_2)\,. \tag{7}$$

The function $\Phi(z)$ must be determined so that the compatibility equation

$$(a^2 - b^2)\,[W(z) - zW'(z)] = b^2\overline{W}\left(\frac{b^2}{z}\right) - a^2\overline{W}\left(\frac{a^2}{z}\right), \quad (z \text{ in } L) \tag{8}$$

is satisfied, for example by the method of 5.51 by assuming a Laurent series.

5.61. Rotational dislocation of an unloaded annulus

We consider the annulus of fig. 5.5. In the notation of 2.73 taking $n = 1$, and dropping the suffix, we have $P = 0$ since the annulus is unloaded and $\alpha = 0$ since the dislocation is to be rotational only. Therefore the simplest form of $W(z)$, obtained by differentiating 2.73 (13) is

$$W(z) = \frac{2\mu\gamma}{\pi(\varkappa+1)}\ln z + A, \tag{1}$$

where γ is the angle of rotation and A is a constant which we can take to be real.

It follows from 5.2 (10) that the stress $\widehat{rr} + i\,\widehat{r\theta}$ given by (1) vanishes on Γ_a and Γ_b so that (1) is suited to an unloaded annulus. The compatibility identity then gives

$$(a^2 - b^2)\left\{\frac{2\mu\gamma}{\pi(\varkappa+1)}\ln z + A - \frac{2\mu\gamma}{\pi(\varkappa+1)}\right\}$$
$$\equiv b^2\left\{\frac{2\mu\gamma}{\pi(\varkappa+1)}\ln\frac{b^2}{z} + A\right\}$$
$$- a^2\left\{\frac{2\mu\gamma}{\pi(\varkappa+1)}\ln\frac{a^2}{z} + A\right\}$$

and therefore

$$A = \frac{\mu\gamma}{\pi(\varkappa+1)}\left\{1 - \frac{a^2\ln a^2 - b^2\ln b^2}{a^2 - b^2}\right\} \tag{2}$$

whence

$$W(z) = \frac{\mu\gamma}{\pi(\varkappa+1)}\left\{2\ln z + 1 - \frac{a^2\ln a^2 - b^2\ln b^2}{a^2 - b^2}\right\}, \tag{3}$$

$$w(z) = -\frac{2\mu\gamma}{\pi(\varkappa+1)}\left\{\frac{\ln a^2 - \ln b^2}{a^2 - b^2}\cdot\frac{a^2b^2}{z^2}\right\} \tag{4}$$

so that

$$'W(z) = \frac{\mu\gamma}{\pi(\varkappa+1)}\left\{2z\ln z - z\left(1 + \frac{a^2\ln a^2 - b^2\ln b^2}{a^2 - b^2}\right)\right\} \tag{5}$$

$$''w(z) = +\frac{2\mu\gamma a^2b^2}{\pi(\varkappa+1)}\,\frac{\ln a^2 - \ln b^2}{a^2 - b^2}\ln z\,. \tag{6}$$

We use these results in the next section to discuss the bending of a curved bar.

5.62. Curved bar bent by couples

In the solution for rotational dislocation of an annulus, let us isolate for consideration the sector between two radii AB, CD.

If $X - iY$, $-M$ are the force and moment exerted across CD by the material on the right we have from 2.40 (1) and (2) $X - iY = 0$ and

$$2M = \frac{\mu\gamma}{\pi(\varkappa+1)}\left[a^2 - b^2 - \frac{4a^2b^2}{(a^2-b^2)}\left(\ln\frac{a}{b}\right)^2\right].$$

This gives the couple required to hold the bar bent as shown in fig. 5.62. Alternatively it gives the constant γ when the couple M is prescribed.

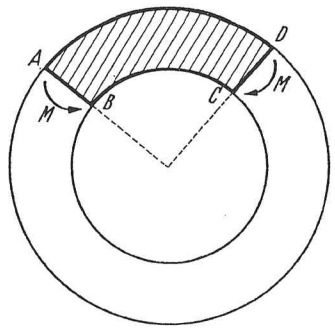

Fig. 5.62

The above solution is exact provided that the couple is applied as a hoop stress at AB and CD compatible with the complex stresses. To find this hoop stress we have

$$2\,(\widehat{\theta\theta} - i\,\widehat{r\theta}) = W(z) + \overline{W}(\bar{z}) + \bar{z}\,\overline{W}'(\bar{z}) + \frac{\bar{z}}{z}\,\overline{w}(\bar{z})$$

so that

$$\widehat{\theta\theta} = \frac{2\,\mu\,\gamma}{\pi\,(\varkappa+1)}\left\{\ln r + 1 - \frac{a^2 \ln a - b^2 \ln b}{a^2 - b^2} - \frac{a^2 b^2}{(a^2-b^2)\,r^2}\ln\frac{a}{b}\right\}.$$

5.63. Translational dislocation of an unloaded annulus

If α is the translation, and the rotation is zero, (cf. fig. 2.61 (iii)), we have from 2.73 (13)

$$W(z) = \frac{-2\,\mu\,\alpha\,i}{\pi\,(\varkappa+1)}\,\frac{1}{z} + h'(z)\,, \tag{1}$$

where $h'(z)$ is holomorphic for z in L. If we assume a Laurent expansion $h'(z) = p_0 + \left(p_1 z + \frac{q_1}{z}\right) + \left(p_2 z^2 + \frac{q_2}{z^2}\right) + \cdots$, wherein, on account of (1), $q_1 = 0$, and if we assume without loss of generality that p_0 is real, the insertion of (1) into the compatibility identity shows that

$$p_0 = 0,\; p_1 = \frac{-4\,\mu\,i\,\bar{\alpha}}{\pi\,(\varkappa+1)}\Big/(a^2+b^2)$$

and that all the remaining coefficients vanish.

Thus we find, for z in L,

$$W(z) = \frac{-2\,\mu\,i}{\pi\,(\varkappa+1)}\left\{\frac{\alpha}{z} + \frac{2\,\bar{\alpha}\,z}{a^2+b^2}\right\} \tag{1}$$

$$w(z) = \frac{2\,\mu\,i}{\pi\,(\varkappa+1)}\left\{\frac{\bar{\alpha}}{z} - \frac{2\,a^2 b^2 \alpha}{a^2+b^2}\,\frac{1}{z^3}\right\}. \tag{2}$$

For the stress across a concentric arc we have from 5.11 (4)

$$2\,(\widehat{rr} + i\,\widehat{r\theta}) = \frac{4\,\mu\,i\,\bar{\alpha}}{\pi\,(\varkappa+1)\,\bar{z}}\,\frac{(a^2-r^2)\,(r^2-b^2)}{(a^2+b^2)\,r^2} \tag{3}$$

which vanishes, as it should, for $r = a$ and $r = b$. Also if $\bar{\alpha}/\bar{z} = \alpha/z$ is real, we see that $\widehat{rr} = 0$.

Again we find that

$$\widehat{rr} + \widehat{\theta\theta} = \frac{2\,\mu\,i}{\pi\,(\varkappa+1)}\left(1 - \frac{2\,r^2}{a^2+b^2}\right)\left(\frac{\bar{\alpha}}{\bar{z}} - \frac{\alpha}{z}\right) \tag{4}$$

which also vanishes when $\bar{\alpha}/\bar{z} = \alpha/z$, is real.

Thus if α/z is real, that is to say along the diameter $\theta = \arg\alpha$, the stress reduces to shear only.

We apply this idea to the semicircular crane hook.

5.64. The semicircular crane hook

Fig. 5.64 shows schematically the hook of a crane assimilated to a bent bar consisting of an annulus halved by a vertical diameter $ABCD$. Referring to 5.63 we take translational dislocation $\alpha = ih$ where h is real. Then on AB, CD α/z is real and therefore if our half ring is considered part of the complete annulus, it is loaded at AB, CD with shear only, where from 5.63 (3)

$$\widehat{r\theta} = \frac{2\,\mu\,h}{\pi\,(\varkappa+1)}\,\frac{(a^2-r^2)\,(r^2-b^2)}{(a^2+b^2)\,r^3} \quad \text{on } AB$$

and the negative of this on CD. If the shear is applied in this way, we have for the load W,

$$W = \int_b^a \widehat{r\theta}\,dr = \frac{2\,\mu\,h}{\pi\,(\varkappa+1)}\left\{\ln\frac{a}{b} - \frac{a^2-b^2}{a^2+b^2}\right\}. \tag{1}$$

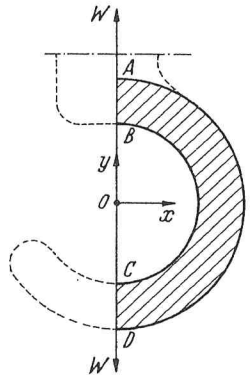

Fig. 5.64

Examples V

1. In the case of a uniformly loaded circular hole in an infinite plate, apply the Plemelj formula to obtain the expressions for $W^L(t_0)$, $W^R(t_0)$ where t_0 is a point on the boundary of the hole.

2. Concentrated forces $X + iY$, $-(X + iY)$ are applied at the points z_0, $z_0 + re^{i\beta}$ respectively. Find the form assumed by the complex stresses when $r \to 0$ assuming that $r\,(X + iY) = X_0 + iY_0$ throughout, X_0, Y_0 being finite.

3. Find the stress distribution in a plate, stretched to tension T at infinity, which has a circular hole of radius a in whose boundary the displacement is purely radial and of amount $h \sin n\theta$ at the point $a\,e^{i\theta}$, n being an integer.

4. Discuss the distribution of stress in an infinite plate, due solely to hydrostatic pressure p at the boundary of a circular hole of radius a.

Show that the displacement at distance r from the centre is

$$\frac{p}{4\mu r}\{(\varkappa-1)\,r^2+2a^2\}\,.$$

5. Find $W(z)$ in the case of a plane, under simple tension at infinity, cut along a circular arc.

Consider in particular the case of a semicircular cut.

6. Discuss the stress distribution in a circular disc subject to uniform hydrostatic pressure at the boundary.

7. At the boundary of a circular hole of radius a in an infinite plate a couple of moment M is uniformly applied by a shear $-M/(2\pi a^2)$ at the boundary. Prove that $W=0$, $w=-Mi/(\pi z^2)$.

8. Find the complex stresses for a standard concentrated force and a concentrated couple applied at the point $z=z_0$.

9. In the case of a rotating disc show that at any point of the disc $W(z)$ is proportional to the maximum value of the hoop stress.

10. For an annulus under all round tension show that the displacement is a linear function of the distance from the centre.

11. Express the radial and hoop stresses in a tube in terms of the internal pressure and the excess of this pressure over the external pressure.

12. An annulus has radii $a, b, a>b$. A rigid cylindrical plug of radius $b+\varepsilon$ is forced into the hole. Calculate the new value of the outer radius.

13. In the solution for the rotational dislocation of an unloaded annulus, prove that the stress $\widehat{rr}+i\,\widehat{r\theta}$ vanishes at each boundary.

14. An annulus is cut along a radius and opened out through a small angle ε. Calculate the couple necessary to close the gap again.

15. Show, by taking moments, that the hoop stress specified when a bar is bent by couples does in fact agree with the moment required to bend the bar.

16. In the case of the semicircular hook, prove that the hoop stress across the horizontal section is

$$\frac{2\mu h}{\pi(\varkappa+1)}\,\frac{(a^2-r^2)(r^2-b^2)-2r^4}{(a^2+b^2)\,r^3}$$

and verify that the resultant is equal to the load.

17. A circular disc of radius a is acted on by uniform hydrostatic pressure p at the rim. Show that the displacement at distance r from the centre is $-p(\varkappa-1)\,r/4\mu$.

18. A plate, weakened by a circular hole of radius a, is under simple tension T at infinity parallel to the x-axis. Show that

$$W(z)=\frac{1}{2}\,T\,(1-2a^2/z^2)\ \text{and that}$$

$$w(z)=-T\left(1-\frac{a^2}{z^2}+\frac{3a^4}{z^4}\right).$$

Find the maximum hoop stress, and prove that the transverse displacement at $z=r\,e^{i\theta}$ is

$$u_\theta=-\frac{T}{4\mu r}\left\{a^2(\varkappa-1)+r^2-\frac{a^4}{r^2}\right\}\sin 2\theta\,.$$

19. An infinite plate is weakened by a circular hole of radius a. Into the hole a smooth elastic disc of radius $a+\varepsilon$, (ε small) is forced. Assuming complete circular symmetry prove that at the rim of the hole the radial displacements are related by $u_r-u_r'=\varepsilon$, where u_r refers to the disc. If p is the pressure at the interface prove that

$$p=\frac{4\varepsilon\,\mu\mu'}{a\,[2\mu'+\mu\,(\varkappa'-1)]}\,,$$

where $\mu', \varkappa'$ are elastic constants for the disc.

20. In the preceding example show that in the case of a rigid disc

$$\widehat{rr}=\frac{-2\mu a\varepsilon}{r^2}\,,\quad \widehat{\theta\theta}=\frac{2\mu a\varepsilon}{r^2}\,,\quad \widehat{r\theta}=0$$

$$u_r=\frac{\varepsilon a}{r}\,,\quad u_\theta=0\,.$$

21. The complex stresses

$$W(z)=\frac{1}{2}\,T\left(1-\frac{\alpha a^2}{z^2}\right),\quad w(z)=-T\left(1-\frac{\beta a^2}{z^2}+\frac{\gamma a^4}{z^4}\right)$$

solve the problem of Ex. 18 above by taking $\alpha=2$, $\beta=1$, $\gamma=3$. Show that by proper choice of α, β, γ the problem of a rigid disc attached to the plate along the edge of the disc, the plate being then subject to uniform streching can be solved. [$\alpha=-2/\varkappa$, $\beta=-(\varkappa-1)/2$, $\gamma=3/\varkappa$].

22. Show how the same complex stresses as in 21 can be used to solve the problem in which a smooth rigid disc fills a circular hole in a plate which is then stretched.

Prove that in this case the normal stress can be tension rather than pressure at some parts of the boundary, but that this can be remedied by inserting a disc of radius $a+\varepsilon$, where ε is properly chosen (see ex. 20 above).

23. Describe how the method of ex. 21, 22 can be applied to the case where the attached or inserted disc is elastic.

24. In the case of a circular disc in equilibrium under two forces at the rim show that

$$\widehat{xx} = \frac{F}{\pi}\left[\frac{2\cos^3\theta_1}{r_1} + \frac{2\cos^3\theta_2}{r_2} - \frac{\cos\alpha}{a}\right]$$

where $z - a\,e^{i\alpha} = r_1 e^{-i\theta_1}$, $z + a\,e^{-i\alpha} = r_2 e^{i\theta_2}$.

Calculate also the radial and hoop stresses.

25. A disc of radius a rotates with angular speed n about its centre and carries a particle of mass m fixed to the disc at a point distance c from the centre. Evaluate the complex stresses.

26. A disc of radius a rotates about a point distance c from the centre ($c < a$) with angular speed n. Calculate the complex stresses.

27. A circular disc of radius a and of mass m hangs in equilibrium under gravity, with its plane vertical, suspended from a point distant ca from its centre. Prove that, with the y-axis vertically upwards

$$W(z) = \frac{mg}{\pi(\varkappa+1)}\left\{\frac{i}{z_1} + \frac{i\varkappa}{z_2} - \frac{2iz}{a^2} - \frac{c}{a}\left[\frac{1}{2}(\varkappa-1) + \frac{1}{c^2}\right] + \right.$$
$$\left. + \left(\frac{1}{c} - \frac{1}{c^3}\right)\frac{ia}{z_2^2}\right\},$$

where $z_1 = z + i\,ca$, $z_2 = z + \dfrac{ia}{c}$, and calculate $w(z)$.

28. An infinite plate under simple tension T at an angle φ with the x-axis is weakened by a circular hole. Show that the hoop stress is $\widehat{\theta\theta} = T\,[1 - 2\cos(2\theta - 2\varphi)]$ at the edge of the hole. Find where the maximum of this occurs.

29. An infinite plate is weakened by a circular hole of radius a. Forces $-iP$, iP are applied to the rim of the hole at $ae^{i\alpha}$, $ae^{-i\alpha}$ respectively. Prove that

$$W(z) = \frac{iP}{\pi}\left\{\frac{1}{z - ae^{i\alpha}} - \frac{1}{z - ae^{-i\alpha}}\right\}$$

and find $w(z)$.

30. A circular disc is deformed into an ellipse by the displacements $u_0 = a\varepsilon\cos\theta$, $v_0 = -a\varepsilon\sin\theta$ at the boundary, where ε is infinitesimal. Show that the state of stress is constant tension parallel to the major axis and equal constant compression parallel to the minor axis.

31. An infinite plate has a circular hole of radius a into which is inserted a rigid circular core also of radius a. Prove that the hoop stress at the point $re^{i\theta}$, when the plate is under simple tension T, at infinity, along the initial line is

$$\widehat{\theta\theta} = \frac{1}{2}T\left\{1 - \frac{1}{2}(\varkappa-1)\frac{a^2}{r^2} - \left(1 - \frac{3\varkappa a^4}{r^4}\right)\cos 2\theta\right\}.$$

32. If hydrostatic pressure p is substituted for simple tension in the preceding example, show that

$$\widehat{rr} = -p\left\{1 + \frac{(\varkappa-1)a^2}{r^2}\right\}, \quad \widehat{\theta\theta} = -p\left\{1 - \frac{(\varkappa-1)a^2}{r^2}\right\}.$$

33. A rigid core of radius a is inserted into a circular hole in an infinite elastic plate; no forces act on the plate except those due to the core. A force $P = X + iY$ is applied to the core at its centre, taken as origin, together with a couple of moment M. Assuming the material to adhere to the core, and the couple to produce a rotation ε, show that

$$W(z) = -\frac{P}{\pi(\varkappa+1)}\cdot\frac{1}{z}, \quad \frac{M}{\pi} = 4\mu\varepsilon a^2,$$
$$w(z) = \frac{\varkappa\bar{P}}{\pi(\varkappa+1)}\frac{1}{z} - \frac{4\mu i\varepsilon a^2}{z^2} - \frac{2P}{\pi(\varkappa+1)}\frac{a^2}{z^3}$$

34. A rigid core of radius $a + \varepsilon$, ε small, is forced into a hole of radius a in an elastic plate. Find the stress distribution in the plate, with the assumption that the contact between core and plate is smooth, and a force $X + iY$ is applied to the core at its centre.

Discuss the effect of simple tension, and all round tension in the plate at infinity.

Chapter VI

Curvilinear boundaries

In this chapter we consider discs and holes in which the boundary is, in general, other than circular. In particular a method is developed for dealing with doubly-connected regions by conformal mapping on an annulus.

6.1. Curvilinear coordinates

Let $f(t)$ be a holomorphic function of the complex variable t and write

$$z = f(t), \quad t = \xi + i\eta. \tag{1}$$

The ascription of particular values ξ_0 to ξ and η_0 to η determines a point z_0 given by

$$z_0 = f(\xi_0 + i\eta_0) \tag{2}$$

as the intersection of the curves

$$\xi = \xi_0, \quad \eta = \eta_0. \tag{3}$$

We call the system (ξ, η) *curvilinear coordinates*. Conventionally we suppose the curve $\xi = \xi_0$ to be described in the sense in which η increases

and the curve $\eta = \eta_0$ to be described in the sense in which ξ increases. It then follows that

$$(\arg dt)_{\eta=\eta_0} = \arg d\xi = 0\,, \quad (\arg dt)_{\xi=\xi_0} = \arg(i\,d\eta) = \frac{\pi}{2}\,. \tag{4}$$

Let

$$\frac{dz}{dt} = f'(t) = J e^{i\alpha}\,, \quad J^2 = f'(t)\,\bar{f}'(\bar{t})\,. \tag{5}$$

Then

$$(\arg dz) - (\arg dt) = \alpha$$

and therefore

$$(\arg dz)_{\eta=\eta_0} = \alpha\,, \quad (\arg dz)_{\xi=\xi_0} = \alpha + \frac{\pi}{2}\,. \tag{6}$$

Thus the curves $\xi = \xi_0$, $\eta = \eta_0$ cut orthogonally, the normal to one being the tangent to the other, and therefore, from (6), α is the inclination to the x-axis of the normal to $\xi = \xi_0$.

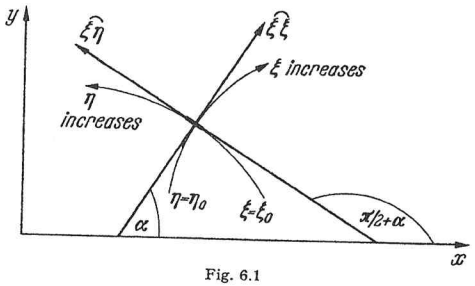

Fig. 6.1

If ds_ξ, ds_η are elements of arc of $\xi = \xi_0$, $\eta = \eta_0$, by taking the modulus of (5) we have

$$ds_\xi = J\,d\eta\,,\; ds_\eta = J\,d\xi\,,\; J^2 = f'(t)\,\bar{f}'(\bar{t})\,. \tag{7}$$

Again from (5)

$$e^{2i\alpha} = \frac{dz}{dt}\Big/\frac{d\bar{z}}{d\bar{t}} = f'(t)/\bar{f}'(\bar{t}) = z'/\bar{z}'\,. \tag{8}$$

6.11. The stresses in curvilinear coordinates

An element of arc of the curve $\xi = \xi_0$ is perpendicular to the direction in which ξ increases. We therefore denote the normal stress across this arc by $\widehat{\xi\xi}$ and the tangential or shear stress by $\widehat{\xi\eta}$ (fig. 6.1). Similarly across an element of arc of $\eta = \eta_0$ we have the components $\widehat{\eta\eta}$, $\widehat{\eta\xi}$.

In the notation of fundamental stress combinations we write

$$\Theta_t = \widehat{\xi\xi} + \widehat{\eta\eta} = \widehat{xx} + \widehat{yy} = \Theta \tag{1}$$

$$\Phi_t = \widehat{\eta\eta} - \widehat{\xi\xi} + 2i\,\widehat{\xi\eta} = \Phi e^{2i\alpha} = (\widehat{yy} - \widehat{xx} + 2i\,\widehat{xy})\,f'(t)/\bar{f}'(\bar{t}) \tag{2}$$

from 1.24 and 6.1 (8).

For the displacement, if u_ξ, u_η are the components in the directions of ξ, η increasing respectively, we have

$$u + iv = (u_\xi + iu_\eta)\,e^{i\alpha} = (u_\xi + iu_\eta)\,\frac{f'(t)}{|f'(t)|}\,. \tag{3}$$

6.12. Mapping on a circle

Let the closed curve C in the z-plane be $\xi = \xi_0$ in the orthogonal net given by

$$z = f(t)\,, \quad t = \xi + i\eta\,. \tag{1}$$

If we write

$$\zeta = e^t = e^\xi(\cos\eta + i\sin\eta)\,, \tag{2}$$

substitution into (1) leads to

$$z = m(\zeta)\,, \tag{3}$$

where the *mapping function* $m(\zeta)$ will be assumed to be also holomorphic.

On the curve C

$$z = m\,[e^{\xi_0}(\cos\eta + i\sin\eta)]\,,$$

so that C corresponds point by point to the circumference Γ, $|\zeta| = e^{\xi_0}$ in the ζ-plane, and η is the polar angle

In like manner the curve C_1 defined by $\xi = \xi_1$ in the net (1) will map into a concentric circumference Γ_1, $|\zeta| = e^{\xi_1}$.

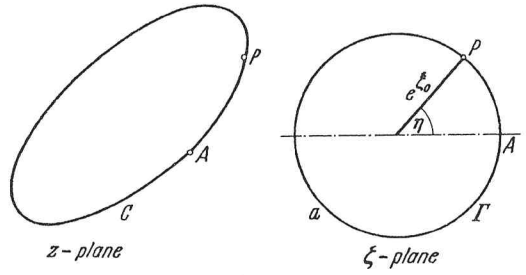

Fig. 6.12

Thus the region between C and C_1 in the z-plane will map into the concentric annulus bounded by Γ and Γ_1.

The mapping is *conformal*, that is to say a small region round z maps into a geometrically similar small region round ζ.

Proof. Let d_1z, d_2z be two differently directed increments of z issuing from z and let $d_1\zeta$, $d_2\zeta$ be the corresponding increments of ζ issuing from ζ. From (3) $dz/d\zeta = m'(\zeta)$ and therefore $\arg dz = \arg d\zeta + \arg m'(\zeta)$. Therefore

$$\arg d_1z - \arg d_2z = \arg d_1\zeta - \arg d_2\zeta$$

so that the angle between corresponding pairs of elements of arc on the two planes is the same. Therefore the angles of a small triangle at z are the same as the angles of its map at ζ, so that the triangles are similar. Therefore any small area at z is similar to its map. Q. E. D.

The linear scale of the mapping is given by

$$|dz| = |m'(\zeta)|\,|d\zeta|\,. \tag{4}$$

From (2) $d\zeta/dt = \zeta$ and therefore from 6.1 (5)

$$J e^{i\alpha} = \zeta m'(\zeta)\,, \quad J^2 = \zeta\bar{\zeta}\, m'(\zeta)\,\bar{m}'(\bar{\zeta})\,, \tag{5}$$

$$e^{2i\alpha} = \frac{\zeta m'(\zeta)}{\bar{\zeta}\,\bar{m}'(\bar{\zeta})}\,, \tag{6}$$

while from (4) the linear scale of the mapping is

$$[m'(\zeta)\,\bar{m}'(\bar{\zeta})]^{1/2} = J/(\zeta\bar{\zeta})^{1/2}\,. \tag{7}$$

Notes. (i) The above reasoning breaks down at any point at which $m'(\zeta) = 0$ or $m'(\zeta) = \infty$. Such points, always finite in number, must be excluded or taken into account by a limiting process.

(ii) If the map consists of the whole region interior to the circumference Γ, we must have an expansion about $\zeta = 0$ of the form

$$m(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \ldots, \quad a_1 \neq 0\,, \tag{8}$$

for $a_1 = 0$ would lead to $m'(0) = 0$. We can arrange that $a_0 = 0$, by translation of the origin in the z-plane. Further we can arrange for a_1 to be real, by rotation of the axes in the ζ-plane.

(iii) If the map consists of the whole plane exterior to Γ, we must have an expansion about $\zeta = \infty$ of the form

$$m(\zeta) = b_1\zeta + b_0 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots, \quad b_1 \neq 0\,, \tag{9}$$

for no power of ζ higher than the first can occur, otherwise we should have $m'(\zeta) = \infty$ at $\zeta = \infty$. Nor can b_1 be zero if for it were, $m'(\zeta)$ would be zero at infinity. As in (ii) we can arrange that $b_0 = 0$ and b_1 is real.

We also see from (9) that $z/\zeta \to b_1$ as $|\zeta| \to \infty$, and therefore that the mappings of the distant parts of the planes tend to geometrical similarity in the large.

(iv) Instead of (2) we could, if convenient, write

$$\zeta = c e^t\,,$$

where c is a constant.

(v) Mapping on the unit circle is frequently useful.

6.13. Stresses under conformal mapping

Let the contour C be mapped on the circumference Γ, $|\zeta| = a$ by

$$z = m(\zeta)\,, \quad \zeta = e^{\xi + i\eta}\,, \quad a = e^{\xi_0}\,. \tag{1}$$

Then from 6.11 (1) and the complex conjugate of 6.11 (2)

$$2\,(\widehat{\xi\xi} + i\,\widehat{\xi\eta}) = \Theta - \bar{\Phi}\, e^{-2i\alpha}, \; 2\,(\widehat{\eta\eta} - i\,\widehat{\eta\xi}) = \Theta + \bar{\Phi}\, e^{-2i\alpha},$$

and therefore from 6.12 (6)

$$2\,(\widehat{\xi\xi} + i\,\widehat{\xi\eta}) = \Theta - \bar{\Phi}\,\frac{\bar{\zeta}\,\bar{m}'(\bar{\zeta})}{\zeta\, m'(\zeta)}\,, \tag{2}$$

$$2\,(\widehat{\eta\eta} - i\,\widehat{\eta\xi}) = \Theta + \bar{\Phi}\,\frac{\bar{\zeta}\,\bar{m}'(\bar{\zeta})}{\zeta\, m'(\zeta)}\,, \tag{3}$$

where (2) is appropriate to members of the family $\xi =$ constant, and (3) to those of $\eta =$ constant.

For the displacement, from 6.11 (3) and 6.12 (5)

$$u_\xi + i u_\eta = e^{-i\alpha}(u + iv) = \frac{\bar{\zeta}\,\bar{m}'(\bar{\zeta})}{|\bar{\zeta}\,\bar{m}'(\bar{\zeta})|}\,(u + iv)\,. \tag{4}$$

Let $\zeta = \sigma$ be a point of the boundary. Then $\bar{\zeta} = a^2/\sigma$ and therefore

$$\frac{\bar{\zeta}\,\bar{m}'(\bar{\zeta})}{\zeta\, m'(\zeta)} = \frac{a^2}{\sigma^2}\,\frac{\bar{m}'\left(\dfrac{a^2}{\sigma}\right)}{m'(\sigma)}\,. \tag{5}$$

6.2. Isotropic material undergoing elastic deformation

The preceding sections of this chapter apply to any kind of continuous deformable material and any type of response. We now consider the case of isotropic material undergoing infinitesimal elastic deformation.

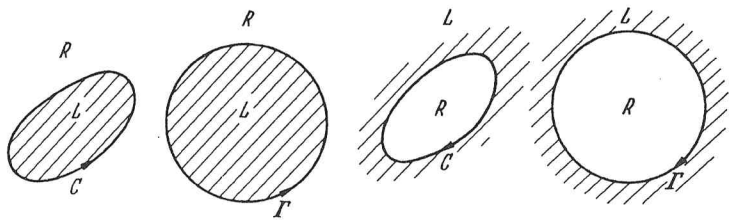

Fig. 6.2

For simplicity of exposition we shall ignore the body-vector field or, what amounts to the same thing, we shall treat all stresses and displacements as "effective" (see 5.11). The subsequent inclusion of the body-vector field introduces no new mathematical technique.

We recall that the distinction between plane deformation and generalized plane stress is simply effected by an alteration in the value of the elastic constant $\varkappa$, 1.72 (10).

Let C be the boundary of the material. We choose the sense of description of C so that the material lies in the left-hand region L. We

map C and L on the ζ-plane by

$$z = m(\zeta)\,, \quad \zeta = e^{\xi + i\eta}\,, \tag{1}$$

in such a way that C corresponds to $\xi = \xi_0$, and maps into the circumference Γ, $|\zeta| = e^{\xi_0} = a$, while L maps into the interior or exterior of Γ according as the material is inside C (a disc) or outside C (a hole).

We could map the outside of C on the inside of Γ, thus reversing the sense of description of Γ but the above procedure seems more physically intuitive.

If we denote the complex stresses by $W_0(z)$, $w_0(z)$, we have

$$\Theta = W_0(z) + \overline{W}_0(\bar{z})\,, \quad \overline{\Phi} = z\,\overline{W}_0'(\bar{z}) + \overline{w}_0(\bar{z})\,. \tag{2}$$

The results of eliminating z in the complex stresses will be written

$$W_0(z) = W_0\,[m(\zeta)] = W(\zeta)\,, \quad w_0(z) = w_0\,[m(\zeta)] = w(\zeta)\,. \tag{3}$$

Noting that

$$W_0'(z) = \frac{d\,W_0(z)}{dz} = \frac{d\,W(\zeta)}{dz} = \frac{d\,W(\zeta)}{d\zeta}\,\frac{d\zeta}{dz} = \frac{W'(\zeta)}{m'(\zeta)} \tag{4}$$

we have

$$\Theta = W(\zeta) + \overline{W}(\bar{\zeta})\,, \quad \overline{\Phi} = \frac{m(\zeta)}{\overline{m}'(\bar{\zeta})}\,\overline{W}'(\bar{\zeta}) + \overline{w}(\bar{\zeta})\,. \tag{5}$$

Therefore from 6.13 (2), (3)

$$2\,(\widehat{\xi\xi} + i\,\widehat{\xi\eta}) = W(\zeta) + \overline{W}(\bar{\zeta}) - \frac{\bar{\zeta}\,m(\zeta)}{\zeta\,m'(\zeta)}\,\overline{W}'(\bar{\zeta}) - \frac{\bar{\zeta}\,\overline{m}'(\bar{\zeta})}{\zeta\,m'(\zeta)}\,\overline{w}(\bar{\zeta})\,, \tag{6}$$

$$2\,(\widehat{\eta\eta} - i\,\widehat{\eta\xi}) = W(\zeta) + \overline{W}(\bar{\zeta}) + \frac{\bar{\zeta}\,m(\zeta)}{\zeta\,m'(\zeta)}\,\overline{W}'(\bar{\zeta}) + \frac{\bar{\zeta}\,\overline{m}'(\bar{\zeta})}{\zeta\,m'(\zeta)}\,\overline{w}(\bar{\zeta})\,. \tag{7}$$

Differentiating 2.12 (2) partially with respect to η we have

$$4\,\mu\,\frac{\partial}{\partial\eta}\,(u + iv) = \{\varkappa\,W_0(z) - \overline{W}_0(\bar{z})\}\,\frac{\partial z}{\partial\eta} - \{z\,\overline{W}_0'(\bar{z}) + \overline{w}_0(\bar{z})\}\,\frac{\partial\bar{z}}{\partial\eta}\,.$$

But $\partial z/\partial\eta = m'(\zeta)\,i\zeta$. Therefore

$$\begin{aligned} &4\mu\,\frac{\partial}{\partial\eta}\,(u + iv) \\ &= [\varkappa\,W(\zeta) - \overline{W}(\bar{\zeta})]\,i\,\zeta\,m'(\zeta) + [m(\zeta)\,\overline{W}'(\bar{\zeta}) + \overline{m}'(\bar{\zeta})\,\overline{w}(\bar{\zeta})]\,i\,\bar{\zeta}\,. \end{aligned} \tag{8}$$

6.21. Solution of Problem I for a given curvilinear boundary

Let the boundary C of the material be the curve $\xi = \xi_0$ in the net

$$z = m(\zeta)\,, \quad \zeta = e^{\xi + i\eta} \tag{1}$$

and let σ be a point on the circle Γ of radius $a = e^{\xi_0}$ into which C maps.

We are given the boundary value of the stress $\widehat{\xi\xi} + i\,\widehat{\xi\eta}$ say

$$\widehat{\xi_0\xi_0} + i\,\widehat{\xi_0\eta} = -\,p(\sigma) + i\,s(\sigma) \text{ on } C \tag{2}$$

where $p(\sigma)$ is the pressure and $s(\sigma)$ is the shear.

From 6.2 (6) we have

$$\begin{aligned} &m'(\zeta)\,W(\zeta) + m'(\zeta)\,\overline{W}(\bar{\zeta}) - \frac{\bar{\zeta}}{\zeta}\,m(\zeta)\,\overline{W}'(\bar{\zeta}) - \frac{\bar{\zeta}}{\zeta}\,\overline{m}'(\bar{\zeta})\,\overline{w}(\bar{\zeta}) \\ &\quad = 2\,(\widehat{\xi\xi} + i\,\widehat{\xi\eta})\,m'(\zeta)\,. \end{aligned} \tag{3}$$

To this we apply the general continuation theorem for the circle 5.215 to give

$$m'(\zeta)\,W(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{2\,[-p(\sigma) + i\,s(\sigma)]\,m'(\sigma)\,d\sigma}{\sigma - \zeta} + \psi(\zeta)\,, \tag{4}$$

with the continuation

$$\begin{aligned} m'(\zeta)\,W(\zeta) &= -\,m'(\zeta)\,\overline{W}\left(\frac{a^2}{\zeta}\right) \\ &+ \frac{a^2}{\zeta^2}\,m(\zeta)\,\overline{W}'\left(\frac{a^2}{\zeta}\right) + \frac{a^2}{\zeta^2}\,\overline{m}'\left(\frac{a^2}{\zeta}\right)\overline{w}\left(\frac{a^2}{\zeta}\right), \quad (\zeta \text{ in } R)\,. \end{aligned} \tag{5}$$

From (5) by writing $\bar{\zeta}$ for a^2/ζ and $a^2/\bar{\zeta}$ for ζ and taking the complex conjugate, we get

$$\begin{aligned} m'(\zeta)\,w(\zeta) &= \frac{a^2}{\zeta^2}\,\overline{m}'\left(\frac{a^2}{\zeta}\right)\left[W(\zeta) + \overline{W}\left(\frac{a^2}{\zeta}\right)\right] - \overline{m}\left(\frac{a^2}{\zeta}\right)W'(\zeta) \\ &= -\,\frac{d}{d\zeta}\left[\overline{m}\left(\frac{a^2}{\zeta}\right)W(\zeta)\right] + \frac{a^2}{\zeta^2}\,\overline{m}'\left(\frac{a^2}{\zeta}\right)\overline{W}\left(\frac{a^2}{\zeta}\right)(\zeta \text{ in } L)\,. \end{aligned} \tag{6}$$

The function $\psi(\zeta)$ in (4) must be continuous across Γ and adapted to the conditions of the problem which will be taken to be

(i) The function $w(\zeta)$ must be holomorphic in L.

(ii) If L contains the point ∞, the stresses at infinity must have given values

(iii) The solution must be non-dislocational.

This last condition is not mandatory. We are entitled to consider solutions with given dislocations.

The continuation (5) is derived from (3) in the manner explained in 5.215 by writing zero for $\widehat{\xi\xi} + i\,\widehat{\xi\eta}$ and $a^2/\bar{\zeta}$ for ζ.

If we operate in this manner and then eliminate $\overline{w}(\bar{\zeta})$, we get

$$\begin{aligned} &2\zeta\,m'(\zeta)\,[\widehat{\xi\xi} + i\,\widehat{\xi\eta}] \\ &= \zeta\,m'(\zeta)\,W(\zeta) - \frac{a^2}{\bar{\zeta}}\,m'\left(\frac{a^2}{\bar{\zeta}}\right)W\left(\frac{a^2}{\bar{\zeta}}\right) \\ &\quad + \left[\zeta\,m'(\zeta) - \frac{a^2}{\bar{\zeta}}\,m'\left(\frac{a^2}{\bar{\zeta}}\right)\right]\overline{W}(\bar{\zeta}) - \left[m(\zeta) - m\left(\frac{a^2}{\bar{\zeta}}\right)\right]\bar{\zeta}\,\overline{W}'(\bar{\zeta})\,, \end{aligned} \tag{7}$$

which expresses the stress in terms of $W(\zeta)$ and its continuation into R.

Thus, as in the case of the circle, we can dispense with the function $w(\zeta)$, remembering however that (6) must be used to express the *holomorphy condition* (i). It will be noticed that by dividing (3) by $m'(\zeta)$ we could have solved for $W(\zeta)$ rather than for $m'(\zeta)\,W(\zeta)$. It is, however simpler to discuss the singularities of this latter function.

6.22. The singularities of $m'(\zeta)\,W(\zeta)$ in the region R

We suppose the mapping of the z-plane on the ζ-plane is that indicated in fig. 6.2 i. e. inside on inside or outside on outside. From notes (ii) and (iii) of 6.12 we can write

$$z = m(\zeta) = l\zeta + f(\zeta)\,, \quad l \neq 0\,, \tag{1}$$

where $f(\zeta)$ is holomorphic in L, and l is real.

The complex stresses $W(\zeta)$, $w(\zeta)$ are holomorphic in L. We continue $W(\zeta)$ from L into R by 6.21 (5).

$$\begin{aligned} m'(\zeta)\,W(\zeta) = -\,m'(\zeta)\,\overline{W}\left(\frac{a^2}{\zeta}\right) + \frac{a^2}{\zeta^2}\,m(\zeta)\,\overline{W}'\left(\frac{a^2}{\zeta}\right) \\ + \frac{a^2}{\zeta^2}\,\overline{m}'\left(\frac{a^2}{\zeta}\right)\overline{w}\left(\frac{a^2}{\zeta}\right),\ (\zeta \text{ in } R)\,. \end{aligned} \tag{2}$$

When ζ is in R, a^2/ζ is in L and therefore $\overline{w}\left(\frac{a^2}{\zeta}\right)$, $\overline{W}\left(\frac{a^2}{\zeta}\right)$ and its derivative $-\frac{a^2}{\zeta^2}\,\overline{W}'\left(\frac{a^2}{\zeta}\right)$ are holomorphic.

Again from (1)

$$\overline{m}\left(\frac{a^2}{\zeta}\right) - \frac{la^2}{\zeta} = \overline{f}\left(\frac{a^2}{\zeta}\right), \quad \overline{m}'\left(\frac{a^2}{\zeta}\right)\frac{a^2}{\zeta^2} - \frac{la^2}{\zeta^2} = \frac{a^2}{\zeta^2}\,\overline{f}'\left(\frac{a^2}{\zeta}\right).$$

By hypothesis $\overline{f}\left(\frac{a^2}{\zeta}\right)$ and therefore its derivative $-\frac{a^2}{\zeta^2}\,\overline{f}'\left(\frac{a^2}{\zeta}\right)$ are holomorphic when ζ is in R and therefore a^2/ζ is in L. It follows that $\frac{a^2}{\zeta^2}\,\overline{m}'\left(\frac{a^2}{\zeta}\right)\overline{w}\left(\frac{a^2}{\zeta}\right)$ has the same singularities as $\frac{la^2}{\zeta^2}\,\overline{w}\left(\frac{a^2}{\zeta}\right)$ when ζ is in R. Therefore (2) shows that the singularities of $m'(\zeta)\,W(\zeta)$ when ζ is in R are at most those of $m'(\zeta)$, $m(\zeta)$ together with a pole of order 2 at most at $\zeta = 0$, if L contains the point ∞.

If R contains the point ∞ and if $m(\zeta)$ has a pole there of order n, $m'(\zeta)\,W(\zeta)$ will have a pole of order $n-1$ at most at $\zeta = \infty$.

6.3. Hypotrochoidal holes

When a circle rolls upon the interior of a fixed circle, the locus of a point fixed relatively to the moving circle is a *hypotrochoid*.

It is easy to show geometrically that, for suitable axes of reference, the points on the curve are given by equations of the form

$$x = c\left(\cos\eta + \frac{k}{n}\cos n\eta\right), \quad y = c\left(\sin\eta - \frac{k}{n}\sin n\eta\right) \tag{1}$$

and the curve will therefore be closed if n is a positive integer, for then x, y return to their original values when η increases by 2π.

Therefore the hypotrochoid is the curve $\xi = 0$ in the orthogonal net

$$z = c\left(e^t + \frac{k}{ne^{nt}}\right),\ t = \xi + i\eta \tag{2}$$

and this can therefore be mapped on the unit circumference γ, $|\zeta| = 1$, by

$$z = m(\zeta) = c\left(\zeta + \frac{k}{n\zeta^n}\right),\ \zeta = e^{\xi + i\eta},\ \ \xi = 0\,. \tag{3}$$

Similarly we can map the hypotrochoid for which $\xi = \xi_0$ on the circumference Γ, $|\zeta| = e^{\xi_0}$.

From note (iii) of 6.12 it appears that $m(\zeta)$ is suited to mapping the distant parts of the plane, but not the parts which contain the origin. We can therefore consider an infinite plate with a hypotrochoidal hole whose equation is

$$z = m(\sigma) = c\left(\sigma + \frac{k}{n\sigma^n}\right),\ \sigma = e^{i\eta}\,. \tag{4}$$

These holes form a very general class. If $k = 0$, we have a circle of radius c.

If $n = 1$, we have an ellipse whose semi-axes are $c\,(1+k)$, $c\,(1-k)$.

If in addition to $n = 1$, we have also $k = 1$, the curve is a straight slit or crack of length $2c$.

For greater values of n we have *curvilinear polygons*, a curvilinear triangle for $n = 2$, a curvilinear square for $n = 3$, and so on.

If $0 < k \leqq 1/n$ the hypotrochoids will have no point of inflexion, and if $k = 1/n$ they will be flattest at the points nearest the centre.

6.31. Theorem of SCHWARZ and CHRISTOFFEL

Consider a simple closed polygon of n-sides whose exterior angles (fig. 6.31) are $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2\pi\,. \tag{1}$$

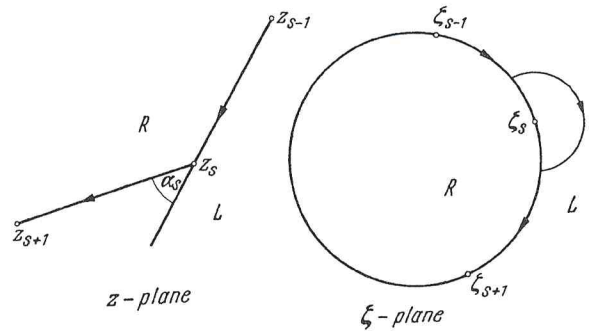

Fig. 6.31

Let the vertices of the polygon be the points z_s, $(s = 1, \ldots, n)$. We map the region outside the polygon on the region outside the circumference Γ, $|\zeta| = a$, in a ζ-plane in such a way that the vertices map respectively into the points of Γ.

$$\zeta_s = a e^{i\eta_s}, \quad s = 1, \ldots, n. \tag{2}$$

The mapping is effected by the theorem of SCHWARZ and CHRISTOFFEL [MILNE-THOMSON (5)] namely by

$$z = m(\zeta), \tag{3}$$

$$m'(\zeta) = \frac{dz}{d\zeta} = A \prod_{s=1}^{n} \left(1 - \frac{\zeta_s}{\zeta}\right)^{\alpha_s/\pi}, \tag{4}$$

where A is a constant.

Proof. On Γ let $\zeta = a e^{i\eta}$. Using (1) we then find from (4) that

$$\frac{dz}{d\eta} = B \prod_{s=1}^{n} \left[\sin \frac{1}{2}(\eta - \eta_s)\right]^{\alpha_s/\pi} \tag{5}$$

where B is a constant.

Thus as z moves from z_{s-1} to z_s, that is as η decreases from η_{s-1} to η_s, $\arg dz/d\eta$ remains constant.

On passing round z_s, $\arg(\zeta - \zeta_s)$ decreases by π and therefore from (4) $\arg dz/d\eta$ decreases by α_s and so z moves along the side $z_s z_{s+1}$.

Q. E. D.

The development of (4) in powers of $1/\zeta$, will contain a term in $1/\zeta$, and therefore z will contain a term in $\ln \zeta$ and so be many-valued unless $\sum \alpha_s \zeta_s = 0$.

To map the interior of the polygon on the interior of the circle we have the corresponding transformation given by

$$m'(\zeta) = \frac{dz}{d\zeta} = A \prod_{s=1}^{n} \left(1 - \frac{\zeta}{\zeta_s}\right)^{-\alpha_s/\pi}$$

which leads to a power series for $m(\zeta)$ convergent within the circumference. The proof is on the same lines as before.

6.32. Polygonal holes

For mapping the region exterior to a regular polygon of n sides we have $\alpha_s = 2\pi/n$ and we take the ζ_s to be roots of the equation $\zeta^n = a^n$. Then from 6.31 (4)

$$m'(\zeta) = A \prod_{s=1}^{n} \frac{(\zeta - \zeta_s)^{2/n}}{\zeta^2} = A \left(\frac{\zeta^n - a^n}{\zeta^n}\right)^{2/n} = A \left(1 - \frac{a^n}{\zeta^n}\right)^{2/n}.$$

Therefore

$$m(\zeta) = A \int^{\zeta} \left(1 - \frac{a^n}{\zeta^n}\right)^{2/n} d\zeta + D$$

where A and D are constants which affect only the scale and orientation. By the binomial theorem

$$\left(1 - \frac{a^n}{\zeta^n}\right)^{2/n} = 1 - \frac{2}{n}\frac{a^n}{\zeta^n} + \frac{2(2-n)}{n^2 2!}\frac{a^{2n}}{\zeta^{2n}} + \frac{2(2-n)(2-2n)}{n^3 3!}\frac{a^{3n}}{\zeta^{3n}} + \cdots.$$

Therefore we can write

$$m(\zeta) = A a \left[\frac{\zeta}{a} + \frac{2}{n(n-1)}\left(\frac{a}{\zeta}\right)^{n-1} + \frac{2(n-2)}{n^2(2n-1)\,2!}\left(\frac{a}{\zeta}\right)^{2n-1} + \cdots\right] + D.$$

Thus for an equilateral triangle $n = 3$

$$m(\zeta) = A a \left[\left(\frac{\zeta}{a}\right) + \frac{1}{3}\left(\frac{a}{\zeta}\right)^2 + \frac{1}{45}\left(\frac{a}{\zeta}\right)^5 + \cdots\right] + D.$$

For a square $n = 4$

$$m(\zeta) = A a \left[\left(\frac{\zeta}{a}\right) + \frac{1}{6}\left(\frac{a}{\zeta}\right)^3 + \frac{1}{56}\left(\frac{a}{\zeta}\right)^7 + \cdots\right] + D.$$

Thus we can approximate to any required accuracy the mapping functions for a regular polygonal hole, and we note that the two-term approximation gives a hypotrochoidal hole with $k = 2/m$ where m is the number of sides of the polygon.

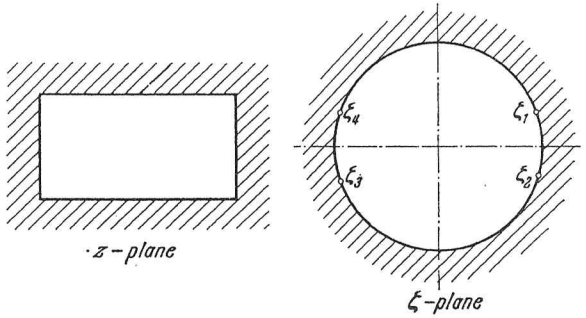

Fig. 6.32

To map a rectangular hole take the points $\zeta_1, \zeta_2, \zeta_3$ to be $\zeta_1, \bar{\zeta}_1, -\zeta_1$. Then ζ_4 by the symmetry will be $-\bar{\zeta}_1$ and so for mapping on a *unit* circle

$$m(\zeta) = A \int^{\zeta} [(\zeta - \zeta_1)(\zeta - \bar{\zeta}_1)(\zeta + \zeta_1)(\zeta + \bar{\zeta}_1)]^{1/2} \frac{d\zeta}{\zeta^2} + D.$$

Write $\zeta_1^2 = \beta$, $\bar{\zeta}_1^2 = \bar{\beta}$. Then

$$m(\zeta) = A \int^{\zeta} \left[1 - \frac{1}{2}(\beta + \bar{\beta})\frac{1}{\zeta^2} - \frac{1}{8}(\beta - \bar{\beta})^2 \frac{1}{\zeta^4} - \cdots\right] d\zeta + D$$

$$= A \left[\zeta + \frac{1}{2}(\beta + \bar{\beta})\frac{1}{\zeta} + \frac{1}{24}(\beta - \bar{\beta})^2 \frac{1}{\zeta^3} + \cdots\right] + D.$$

Thus with the two term approximation we get an ellipse.

If $\beta = i$, $\bar{\beta} = -i$ the rectangle becomes a square.

6.33. Unloaded hypotrochoidal hole in an infinite plate

Let an infinite plate under tension at infinity be weakened by the hypotrochoidal hole

$$z = m(\sigma) = c\left(\sigma + \frac{k}{n\sigma^n}\right), \quad \sigma = e^{i\eta} \tag{1}$$

which is therefore mapped on the unit circumference γ by

$$z = m(\sigma)\,. \tag{2}$$

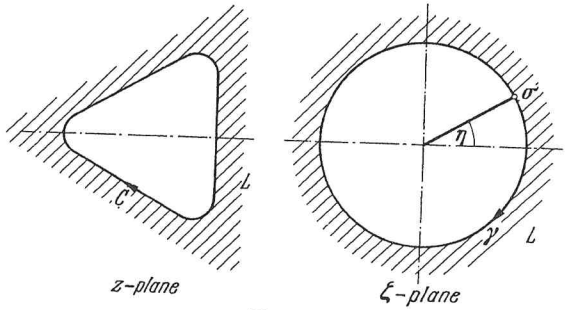

Fig. 6.33

Mapping exterior on exterior we have

$$z = m(\zeta) = c\left(\zeta + \frac{k}{n\zeta^n}\right) \tag{3}$$

We suppose the plate to be under no applied forces, the rotation at ∞ to be zero, and the situation to be non-dislocational. In the neighbourhood of infinity (3) becomes $z = c\zeta$ and therefore from 2.74 (26) and (27)

$$W(\zeta) = \frac{1}{2}(N_\infty + T_\infty) + \frac{\alpha_2}{c^2\zeta^2} + \cdots, \tag{4}$$

$$w(\zeta) = (N_\infty - T_\infty)\, e^{-2i\varphi} + \frac{\beta^2}{c^2\zeta^2} + \cdots \tag{5}$$

From (3)

$$m'(\zeta) = c\left(1 - \frac{k}{\zeta^{n+1}}\right)$$

and so the only singularity of $m'(\zeta)\,W(\zeta)$ in the region R consists of a pole of order $n+1$ at most at $\zeta = 0$.

Thus from 6.21 (4) we have

$$m'(\zeta)\,W(\zeta) = A_0 + \frac{A_1}{\zeta} + \frac{A_2}{\zeta^2} + \cdots + \frac{A_{n+1}}{\zeta^{n+1}}, \quad (\zeta \text{ in } L \text{ or } R) \tag{6}$$

and comparing this with (4) we see that

$$A_0 = \frac{1}{2}c\,(N_\infty + T_\infty), \quad A_1 = 0. \tag{7}$$

To obtain the remaining coefficients A_s we use the holomorphy condition for $w(\zeta)$ as given by 6.21 (6).

Now

$$\overline{m}\left(\frac{1}{\zeta}\right) W(\zeta) = \left(\frac{k\zeta^n}{n} + \frac{1}{\zeta}\right)\left(A_0 + \frac{A_2}{\zeta^2} + \frac{A_3}{\zeta^3} + \cdots + \frac{A_{n+1}}{\zeta^{n+1}}\right)\left(1 - \frac{k}{\zeta^{n+1}}\right)^{-1}$$
$$= \frac{k}{n}\left(A_0\zeta^n + A_2\zeta^{n-2} + A_3\zeta^{n-3} + \cdots A_{n-1}\zeta + A_n\right) + O\left(\frac{1}{\zeta}\right),$$

while

$$\frac{1}{\zeta^2}\,\overline{m}'\left(\frac{1}{\zeta}\right)\overline{W}\left(\frac{1}{\zeta}\right) = \frac{1}{\zeta^2}\left(\overline{A}_0 + \overline{A}_2\zeta^2 + \overline{A}_3\zeta^3 + \cdots + \overline{A}_{n+1}\zeta^{n+1}\right).$$

Therefore from 6.21 (6) we have for $m'(\zeta)\,w(\zeta)$ the expression

$$\begin{aligned}(\overline{A}_{n+1} - kA_0)\,\zeta^{n-1} + \overline{A}_n\zeta^{n-2} + \left[\overline{A}_{n-1} - (n-2)\frac{k}{n}A_2\right]\zeta^{n-3} + \cdots \\ + \left[\overline{A}_3 - 2\frac{k}{n}A_{n-2}\right]\zeta + \left[\overline{A}_2 - \frac{k}{n}A_{n-1}\right] + O\left(\frac{1}{\zeta^2}\right)\end{aligned} \tag{8}$$

and since this must be holomorphic in L, the coefficients of all the positive powers of ζ must vanish. The equations derived from the coefficients of the pairs ζ^{n-4}, ζ; ζ^{n-5}, ζ^2; and so on, show that all the coefficients from A_3 to A_{n-2} vanish. Therefore we are left with $A_n = 0$, $\overline{A}_{n+1} = kA_0$ and

$$\overline{A}_{n-1} - (n-2)\frac{k}{n}A_2 = 0, \quad \overline{A}_2 - \frac{k}{n}A_{n-1} = c\,(N_\infty - T_\infty)\,e^{-2i\varphi} = B, \tag{9}$$

say, the last equation being obtained by comparison of (8) with (5). Therefore solving we get

$$\begin{aligned}\left(1 - \frac{k}{\zeta^{n+1}}\right) cW(\zeta) = A_0\left(1 + \frac{k}{\zeta^{n+1}}\right) \\ + \frac{1}{1 - \frac{k^2}{n^2}(n-2)}\left\{\frac{\overline{B}}{\zeta^2} + \frac{(n-2)\,kB}{n\zeta^{n-1}}\right\}\end{aligned} \tag{10}$$

where

$$A_0 = \frac{1}{2}c\,(N_\infty + T_\infty), \quad B = c\,(N_\infty - T_\infty)\,e^{-2i\varphi}. \tag{11}$$

For *simple tension* $N_\infty = 0$, $T_\infty = T$

$$A_0 = \frac{1}{2}cT, \quad B = -cTe^{-2i\varphi}. \tag{12}$$

For *all round* tension T

$$A_0 = cT, \; B = 0\,. \tag{13}$$

Formula (10) applies if $n > 2$.

If $n = 1$, we have from (8)

$$c\left(1 - \frac{k}{\zeta^2}\right) w(\zeta) = (\overline{A}_2 - kA_0) + \frac{\overline{A}_1}{\zeta} + \cdots$$

and therefore $\bar{A}_2 - kA_0 = (N_\infty - T_\infty)\, e^{-2i\varphi}$ and

$$c\left(1-\frac{k}{\zeta^2}\right) W(\zeta) = A_0\left(1+\frac{k}{\zeta^2}\right) + \frac{\bar{B}}{\zeta^2}. \tag{14}$$

If $n = 2$, we have

$$c\left(1-\frac{k}{\zeta^3}\right) w(\zeta) = (\bar{A}_3 - kA_0)\,\zeta + \bar{A}_2 + O\left(\frac{1}{\zeta^2}\right).$$

Therefore $\bar{A}_3 = kA_0$, $\bar{A}_2 = B$.

$$c\left(1-\frac{k}{\zeta^3}\right) W(\zeta) = A_0\left(1+\frac{k}{\zeta^3}\right) + \frac{\bar{B}}{\zeta^2}. \tag{15}$$

6.34. Unloaded infinite plate with a hypotrochoidal hole under pressure

In this case the tension at infinity is zero and therefore 6.33 (7) is replaced by $A_0 = 0$, $A_1 = 0$ so that if p is the pressure, 6.21 (4) gives

$$c\left(1-\frac{k}{\zeta^{n+1}}\right) W(\zeta) = 2pc - 2pc\left(1-\frac{k}{\zeta^{n+1}}\right) + \sum_{s=2}^{n+1} \frac{A_s}{\zeta^s}, \quad (\zeta \text{ in } L)$$

$$c\left(1-\frac{k}{\zeta^{n+1}}\right) W(\zeta) = 2pc + \sum_{s=2}^{n+1} \frac{A_s}{\zeta_s}, \quad (\zeta \text{ in } R)$$

where the integration has been carried out with aid of 3.13 (6).

In the present case in 6.33 (9) we have $B = 0$ so that the holomorphy condition for $w(\zeta)$ shows that all the coefficients A_s vanish and so

$$W(\zeta) = -2p + \frac{2p}{1-\frac{k}{\zeta^{n+1}}} = \frac{2pk}{\zeta^{n+1}-k} \quad (\zeta \text{ in } L)$$

$$W(\zeta) = \frac{2p}{1-\frac{k}{\zeta^{n+1}}} = \frac{2p\zeta^{n+1}}{\zeta^{n+1}-k} \quad (\zeta \text{ in } R).$$

The case of a hole under pressure in a plate under tension is got by superposing by addition these results on those of 6.33.

6.4. Elliptic holes

We consider a plate weakened by an elliptic hole whose semi-axes are

$$a = c\,(1+k), \quad b = c\,(1-k), \tag{1}$$

where the boundary of the hole is

$$z = c\left(\zeta + \frac{k}{\zeta}\right) = m(\zeta),\ \zeta = e^{\xi+i\eta},\ \xi = 0,\ \sigma = e^{i\eta} = \zeta. \tag{2}$$

When the plate is under simple tension T at infinity at angle φ to the real axis and the hole is unloaded 6.33 (14) gives

$$W(\zeta) = \frac{\frac{1}{2}T(\zeta^2+k)}{\zeta^2-k} - \frac{Te^{2i\varphi}}{\zeta^2-k} \quad (\zeta \text{ in } L \text{ or } R). \tag{3}$$

We get the case of all round tension T by superposing on (3) tension T at right angles which changes φ into $\varphi + \frac{\pi}{2}$ and so

$$W(\zeta) = \frac{T(\zeta^2+k)}{\zeta^2-k} \quad (\zeta \text{ in } L \text{ or } R). \tag{4}$$

When the hole is under hydrostatic pressure p and the plate is otherwise unstressed we have from 6.34

$$W(\zeta) = \frac{2pk}{\zeta^2-k}, \quad (\zeta \text{ in } L); \quad W(\zeta) = \frac{2p\zeta^2}{\zeta^2-k}, \quad (\zeta \text{ in } R). \tag{5}$$

All the above results can be obtained directly from 6.21 (4) by putting $\psi(\zeta) = A + B/\zeta + C/\zeta^2$ and determining the constants from the holomorphy condition.

6.41. Stress concentration factor for an elliptic hole

When a plate is weakened by a hole we call *stress concentration factor* the ratio $\widehat{\eta\eta}_0/T$, where $\widehat{\eta\eta}_0$ is the hoop stress at the boundary of the hole and T is the tension at infinity.

Consider an unloaded elliptic hole with simple tension T (at angle φ to the real axis) at infinity.

To find the hoop stress we have, at the boundary of the hole, $\widehat{\xi\xi} = 0$ and therefore

$$\widehat{\eta\eta}_0 = \widehat{\eta\eta}_0 + \widehat{\xi\xi}_0 = W(\zeta) + \overline{W}(\bar{\zeta}), \quad \xi = 0.$$

Therefore from 6.4 (3), the stress concentration factor is

$$f = \frac{\widehat{\eta\eta}_0}{T} = \frac{\frac{1}{2}(e^{2i\eta}+k) - e^{2i\varphi}}{e^{2i\eta}-k} + \frac{\frac{1}{2}(e^{-2i\eta}+k) - e^{-2i\varphi}}{e^{-2i\eta}-k}$$
$$= \frac{1-k^2+2k\cos 2\varphi - 2\cos(2\eta - 2\varphi)}{1-2k\cos 2\eta + k^2} \tag{1}$$

6.42. The spreading of cracks

For an unloaded elliptic hole in a plate under simple tension perpendicular to the major axis of the hole we have, from 6.41, the stress concentration factor

$$f = \frac{1-k^2-2k+2\cos 2\eta}{1-2k\cos 2\eta + k^2}$$

and $df/d\eta = 0$ gives $\sin 2\eta = 0$, so that the points of maximum stress concentration are at the ends of the axes giving

$$f_A = 1 + \frac{2a}{b}, \quad f_B = -1 .$$

Thus maximum tension occurs at the ends of the major axis, maximum compression at the ends of the minor axis. We note that $f_A \to \infty$ when $b \to 0$. This indicates that for a slender elliptic hole in a plate under increasing tension perpendicular to the axis of the hole, elastic failure will start at the ends of the major axis. This result affords an explanation of the spreading of cracks. It further shows how such tendency to failure may be obviated by drilling circular (or elliptic) holes at the ends of the crack.

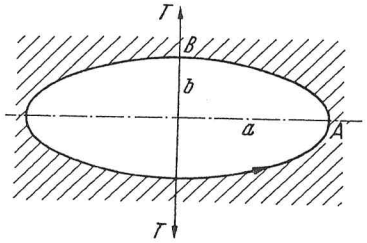

Fig. 6.42

A similar discussion for the case of all round tension indicates that failure will still start at the ends of the crack.

6.43. Crack opened by pressure

Consider an unstressed infinite plate with an elliptic crack (a, b) under pressure p. At the boundary of the crack $\widehat{\xi\xi} = -p$ and $\zeta = e^{i\eta}$, therefore from 6.4 (5) the hoop stress $\widehat{\eta\eta}_0$ is given by

$$\widehat{\eta\eta}_0 = p + W(\zeta) + \overline{W}(\zeta) = p + 4pk\frac{\cos 2\eta - k}{1 - 2k\cos 2\eta + k^2} .$$

Therefore the stress concentration factor $f = \widehat{\eta\eta}_0/p$ is

$$f = -1 + \frac{2(1-k^2)}{1 - 2k\cos 2\eta + k^2} .$$

The maximum value of f therefore occurs when the denominator is least i. e. when $\cos 2\eta = 1$ which gives the ends of the major axis

$$f_{\max} = f_A = -1 + \frac{2a}{b} .$$

Since this $\to \infty$ when $b \to 0$, failure will start at the ends of the crack.

6.5. Epitrochoidal discs

Let a circle roll on the exterior of a fixed circle. The locus of a point fixed relatively to the moving circle is an *epitrochoid*. Such an epitrochoid can be exhibited as the curve $\xi = 0$ in the orthogonal net

$$z = c\left(e^t + \frac{k e^{nt}}{n}\right), \; t = \xi + i\eta . \tag{1}$$

It is mapped on the unit circumference γ, $|\zeta| = 1$, by

$$z = m(\zeta) = c\left(\zeta + \frac{k\zeta^n}{n}\right), \; \zeta = e^{\xi + i\eta} . \tag{2}$$

From note (ii) of 6.12 it appears that (2) is adapted to mapping parts of the plane in the neighbourhood of the origin. We shall therefore consider discs of material bounded by the epitrochoid

$$z = c\left(\sigma + \frac{k\sigma^n}{n}\right), \; \sigma = e^{i\eta}, \; k \leqq 1 . \tag{3}$$

The family of epitrochoids contain many important curves. When $n = 1$ we have a circle. When $n = 2$ the curve is an elliptic limaçon, becoming a cardioid if $k = 1$. The case $n = 3$ yields an oval curve resembling an ellipse if $k \leqq \frac{1}{3}$.

If $0 < k \leqq \frac{1}{n}$, the curve has no points of inflexion, and if $k = \frac{1}{n}$ the points where the curvature vanishes coincide with those whose distance from the centre is least. The type of curve is then a *curvilinear polygon* of $n - 1$ sides (cf. 6.3).

6.51. Epitrochoidal oval in equilibrium under two forces

Consider the epitrochoidal oval

$$z = c\left(\sigma + \frac{k\sigma^3}{3}\right), \; \sigma = e^{i\eta}, \; 0 < k \leqq \frac{1}{3} . \tag{1}$$

We call semi-axes of the oval

$$a = c\left(1 + \frac{k}{3}\right), \; b = c\left(1 - \frac{k}{3}\right) . \tag{2}$$

We suppose a disc in the form of this oval to be in equilibrium under forces F at A and $-F$ at A', where A, A' are the ends of the "major" axis.

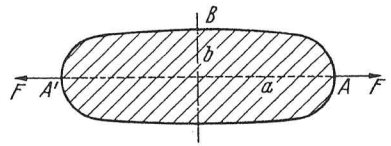

Fig. 6.51

Let ε be infinitesimal and take points A_1, A_2, A_1', A_2' on the oval given by

$$z_{A_1} = a(1 - i\varepsilon), \; z_{A_2} = a(1 + i\varepsilon), \; z_{A_1'} = -a(1 - i\varepsilon), \; z_{A_2'} = -a(1 + i\varepsilon). \tag{3}$$

To obtain standard concentrated forces F, we suppose F to be obtained as the limit, when $\varepsilon \to 0$, of a uniform stress distribution $-p(\sigma) + is(\sigma) = F/(2a\varepsilon)$ applied to arcs A_1AA_2, $A_1'A'A_2'$.

Since $m(\zeta) = c(\zeta + k\zeta^3/3)$ has a pole of order three at $\zeta = \infty$, it follows from 6.22 that $m'(\zeta)\, W(\zeta)$ will have a pole of order two at most. Therefore we see from 6.21 (4) that $c(1 + k\zeta^2)\, W(\zeta)$ will be obtained as the limit $\varepsilon \to 0$ of

$$\frac{1}{4\pi a \varepsilon i}\left\{\int\limits_{A_1 A A_2} \frac{P d\sigma}{\sigma - \zeta} + \int\limits_{A_1' A' A_2'} \frac{P d\sigma}{\sigma - \zeta}\right\} + A_0' + A_1\zeta + A_2\zeta^2\,, \qquad (4)$$

(ζ in L or R)

where $P = 2cF(1 + k\sigma^2)$, and A_0', A_1, A_2 are constants to be determined from the holomorphy condition for $w(\zeta)$.

Since

$$\frac{1 + k\sigma^2}{\sigma - \zeta} = k(\sigma + \zeta) + \frac{1 + k\zeta^2}{\sigma - \zeta}$$

the integrals in (4) are equal to

$$\begin{aligned} &2cF\left[\frac{1}{2}k(\sigma + \zeta)^2 + (1 + k\zeta^2)\ln(\sigma - \zeta)\right]_{\sigma_{A_1}}^{\sigma_{A_2}} \\ &+ 2cF\left[\frac{1}{2}k(\sigma + \zeta)^2 + (1 + k\zeta^2)\ln(\sigma - \zeta)\right]_{\sigma_{A_1'}}^{\sigma_{A_2'}} \end{aligned} \qquad (5)$$

Now $|dz| = |m'(\zeta)|\,|d\zeta|$ and therefore to the first order, since $dz = i\varepsilon a$ on going from A to A_2, $d\sigma = \dfrac{i\varepsilon a}{c(1 + k)}$ on going from $\sigma_A = 1$ to σ_{A_2}. Therefore

$$\sigma_{A_1} = 1 - \frac{i\varepsilon a}{c(1 + k)}\,,\ \sigma_{A_2} = 1 + \frac{i\varepsilon a}{c(1 + k)}\,,\ \sigma_{A_1'} = -1 + \frac{i\varepsilon a}{c(1 + k)}\,,$$

$$\sigma_{A_2} = -1 - \frac{i\varepsilon a}{c(1 + k)}\,.$$

Substitute these values in (5), and use the logarithmic series in the form $\ln(1 + q\varepsilon) = q\varepsilon$; then (4) gives

$$c(1 + k\zeta^2)\, W(\zeta) = \frac{2F}{\pi(1 - \zeta^2)} + A_0 + A_1\zeta + A_2\zeta^2\,, \qquad (6)$$

where $A_0 = A_0' - 2kF/\pi(1 + k)$.

From (6) we get

$$c\left(1 + \frac{k}{\zeta^2}\right)\overline{W}\left(\frac{1}{\zeta}\right) = -\frac{2F\zeta^2}{\pi(1 - \zeta^2)} + \overline{A}_0 + \frac{\overline{A}_1}{\zeta} + \frac{\overline{A}_2}{\zeta^2}\,. \qquad (7)$$

Substitute from (6) and (7) into 6.21 (6). We then get after some reduction

$$c\,w(\zeta) = \frac{B_2}{\zeta^4} + \frac{B_1}{\zeta^3} + \frac{B_0}{\zeta^2} + f(\zeta)\,,$$

where $f(\zeta)$ is holomorphic in L and

$$B_2 = k\left(A_0 + \frac{2F}{\pi}\right) + \overline{A}_2,\quad B_1 = \frac{2kA_1}{3} + \overline{A}_1\,,$$

$$B_0 = \frac{k}{3}\left(A_2 + \frac{2F}{\pi}\right) + \left(1 - \frac{k^2}{3}\right)\left(A_0 + \frac{2F}{\pi}\right) + \overline{A}_0\,.$$

In order that $w(\zeta)$ may be holomorphic in L we must have

$$B_2 = 0\,,\quad B_1 = 0\,,\quad B_0 = 0\,.$$

Clearly the quantities A_0, A_1, A_2 are real and solving we find that $A_1 = 0$ and

$$A_0 = -\frac{1 + \frac{k}{3} - \frac{2k^2}{3}}{1 - \frac{k^2}{3}}\,\frac{F}{\pi}\,,\quad A_2 = -\frac{k - \frac{k^2}{3}}{1 - \frac{k^2}{3}}\,\frac{F}{\pi}\,. \qquad (8)$$

Therefore

$$W(\zeta) = \frac{F}{\pi c}\,\frac{1}{1 + k\zeta^2}\left\{\frac{2}{1 - \zeta^2} - \frac{1 + \frac{k}{3} - 2\frac{k^2}{3}}{1 - \frac{k^2}{3}} - \frac{k - \frac{k^2}{3}}{1 - \frac{k^2}{3}}\,\zeta^2\right\} \qquad (9)$$

and since the boundary is unloaded this expression applies for ζ in L and for ζ in R.

As a check, putting $k = 0$, will give the case of a circle.

The foregoing solution is typical of the method to be employed for any disc with a curvilinear boundary in equilibrium under forces at the rim.

6.6. Solution of Problem II for a given curvilinear boundary

Let the displacement be given at the boundary C in the form

$$(u + iv)^{(C)} = \upsilon(\sigma)\,,\quad \sigma = a e^{i\eta}\,. \qquad (1)$$

Then at the boundary

$$\left(\frac{\partial u}{\partial \eta} + i\frac{\partial v}{\partial \eta}\right)^{(C)} = i\sigma\,\upsilon'(\sigma)\,. \qquad (2)$$

The method of solution is to obtain a Hilbert problem by means of 6.2 (8) and the continuation of $W(\zeta)$ from L into R given by 6.21 (5).

Divide 6.21 (5) by $m'(\zeta)$, next write $a^2/\bar{\zeta}$ for ζ and $\bar{\zeta}$ for a^2/ζ, and then multiply the resulting equation by $i\zeta\, m'(\zeta)$ to get

$$0 = i\zeta\, m'(\zeta)\left\{W\left(\frac{a^2}{\bar{\zeta}}\right) + \overline{W}(\bar{\zeta})\right\} - \frac{\frac{i\zeta\bar{\zeta}^2}{a^2}\, m\left(\frac{a^2}{\bar{\zeta}}\right) m'(\zeta)}{m'\left(\frac{a^2}{\bar{\zeta}}\right)}\,\overline{W}'(\bar{\zeta})$$

$$- \frac{i\zeta\bar{\zeta}^2}{a^2}\,\frac{\overline{m}'(\bar{\zeta})\, m'(\zeta)}{m'\left(\frac{a^2}{\bar{\zeta}}\right)}\,\overline{w}(\bar{\zeta})\,.$$

Add this to 6.2 (8). Then

$$4\mu\left(\frac{\partial u}{\partial \eta}+i\frac{\partial v}{\partial \eta}\right)=i\,\zeta\,m'(\zeta)\left[\varkappa W(\zeta)+W\left(\frac{a^2}{\bar\zeta}\right)\right]+$$

$$+\,i\,\bar\zeta\,\overline{W}'(\bar\zeta)\left[m(\zeta)-\frac{\zeta\bar\zeta}{a^2}\,\frac{m\left(\frac{a^2}{\bar\zeta}\right)m'(\zeta)}{m'\left(\frac{a^2}{\zeta}\right)}\right]+ \tag{3}$$

$$+\,i\,\bar\zeta\,\overline{m}'(\bar\zeta)\,\overline{w}(\bar\zeta)\left[1-\frac{\zeta\bar\zeta}{a^2}\,\frac{m'(\zeta)}{m'\left(\frac{a^2}{\bar\zeta}\right)}\right].$$

Now let ζ, remaining in L, tend to a point σ of Γ, then $a^2/\bar\zeta$ tends to the same point from R. Therefore using (2) we get the required Hilbert problem

$$[\varkappa m'(\sigma)\,W(\sigma)]^L+[m'(\sigma)\,W(\sigma)]^R=4\mu\,\upsilon'(\sigma)\,. \tag{4}$$

As in 5.32 introduce a function

$$F(\zeta)=W(\zeta)\,m'(\zeta)\,,\quad(\zeta\text{ in }L)\;;\quad F(\zeta)=-\frac{1}{\varkappa}\,W(\zeta)\,m'(\zeta)\,,\quad(\zeta\text{ in }R). \tag{5}$$

Then (4) becomes

$$F^L(\zeta)-F^R(\zeta)=\frac{4\mu}{\varkappa}\,\upsilon'(\sigma)$$

with the solution

$$F(\zeta)=\frac{1}{2\pi i}\oint_\Gamma\frac{4\mu\upsilon'(\sigma)\,d\sigma}{\varkappa(\sigma-\zeta)}+\psi(\zeta)\,, \tag{6}$$

where $\psi(\zeta)$ must be chosen on lines similar to those laid down in 6.21.

6.61. Twisting of a rigid hypotrochoidal core

Let an infinite elastic plate under tension at infinity be weakened by a hypotrochoidal hole into which a rigid core, whose rim is welded to the plate, fits exactly. We examine the state of stress when a couple M is applied to the core in such a way that it is held twisted in its own plane through a small angle ε.

Let the hole be defined by

$$z=m(\sigma)=c\left(\sigma+\frac{k}{n\sigma^n}\right),\quad\sigma=e^{i\eta}\,. \tag{1}$$

The displacement at the rim of the core and the displacement of the material (since they adhere) are the same namely

$$\upsilon(\sigma)=i\,\varepsilon\,z=i\,\varepsilon\,m(\sigma)\,. \tag{2}$$

Therefore from 6.6 (5), (6), using 3.13 (6) to perform the integrations

$$c\left(1-\frac{k}{\zeta^{n+1}}\right)W(\zeta)=-\frac{4\mu\,\varepsilon\,ick}{\varkappa\zeta^{n+1}}+\psi(\zeta)\quad(\zeta\text{ in }L)\,, \tag{3}$$

$$c\left(1-\frac{k}{\zeta^{n+1}}\right)W(\zeta)=4\mu\,\varepsilon\,ic-\varkappa\,\psi(\zeta)\qquad(\zeta\text{ in }R)\,. \tag{4}$$

Since the only possible singularity is a pole of order $n+1$ at $\zeta=0$ we see from 6.22 that

$$\psi(\zeta)=A_0+\frac{A_1}{\zeta}+\frac{A_2}{\zeta^2}+\cdots+\frac{A_{n+1}}{\zeta^{n+1}}\,. \tag{5}$$

Comparison of (5) with the form of $W(\zeta)$ at infinity given in 6.33 (4) shows that

$$A_0=\frac{1}{2}\,c\,(N_\infty+T_\infty)\,,\quad A_1=0\,. \tag{6}$$

Applying the holomorphy condition for $w(\zeta)$ using 6.21 (6) we get, as is readily seen, for $m'(\zeta)\,w(\zeta)$ an expression which differs from 6.33 (8) only in that each coefficient $\bar A_s$ is replaced by $-\varkappa\bar A_s$, $(s=2,3,\ldots,n+1)$

In the same way as in 6.33 we find that $A_n=0$ and

$$-\varkappa\bar A_{n+1}-kA_0=0 \tag{7}$$

together with

$$\begin{aligned}-\varkappa\bar A_{n-1}-(n-2)\frac{k}{n}A_2&=0\,,\\ -\varkappa\bar A_2-\frac{k}{n}A_{n-1}&=c\,(N_\infty-T_\infty)\,e^{-2i\varphi}=B\,,\end{aligned} \tag{8}$$

say, the last equation being obtained by comparison of $w(\zeta)$ at infinity with the expression 6.33 (5). Thus

$$A_2=\frac{-\varkappa\bar B}{\varkappa^2-\frac{k^2}{n^2}(n-2)}\,,\qquad A_{n-1}=\frac{(n-2)\frac{k}{n}B}{\varkappa^2-\frac{k^2}{n^2}(n-2)} \tag{9}$$

Therefore

$$c\left(1-\frac{k}{\zeta^{n+1}}\right)W(\zeta)=A_0+\frac{A_2}{\zeta^2}+\frac{A_{n-1}}{\zeta^{n-1}}-\frac{kA_0+4\mu\,\varepsilon\,ick}{\varkappa\zeta^{n+1}}\,,\quad(\zeta\text{ in }L), \tag{10}$$

$$c\left(1-\frac{k}{\zeta^{n+1}}\right)W(\zeta)=-\varkappa A_0+4\mu\,\varepsilon\,ic-\frac{\varkappa A_2}{\zeta^2}-\frac{\varkappa A_{n-1}}{\zeta^{n-1}}+\frac{kA_0}{\zeta^{n+1}}\,,\quad(\zeta\text{ in }R), \tag{11}$$

where the coefficients are given by (6), (8), (9).

These results apply when $n>1$.

When $n=1$, we find that $A_2=-(kA_0+\bar B)/\varkappa$ and so for an elliptic core

$$c\left(1-\frac{k}{\zeta^2}\right)W(\zeta)=A_0-\frac{kA_0+\bar B+4\mu\,\varepsilon\,ick}{\varkappa\zeta^2}\,,\quad(\zeta\text{ in }L), \tag{12}$$

$$c\left(1-\frac{k}{\zeta^2}\right)W(\zeta)=4\mu\,\varepsilon\,ic-\varkappa A_0+\frac{kA_0+\bar B}{\zeta^2}\,,\quad(\zeta\text{ in }R), \tag{13}$$

where $A_0 = \frac{1}{2} c\,(N_\infty + T_\infty)$, $\overline{B} = (N_\infty - T_\infty)\, e^{2i\varphi}$.

To relate the couple M and the angle of twist ε we note that as $\zeta \to \infty$, $z \to c\zeta$ and therefore from 2.74 (27)

$$w(\zeta) = (N_\infty - T_\infty)\, e^{-2i\varphi} - \frac{iM + N'}{c^2 \pi \zeta^2} + \cdots \tag{14}$$

We therefore seek the coefficient of $1/\zeta^2$ in the expansion of $w(\zeta)$ in descending powers of ζ. From 6.21 (6), for large values of $|\zeta|$, we have

$$w(\zeta) = -k\zeta^{n-1}\left[1 + O\left(\frac{1}{\zeta^{n+1}}\right)\right]\left[W(\zeta) + \overline{W}\left(\frac{1}{\zeta}\right)\right]$$
$$+ \frac{k}{n}\zeta^n\left[1 + O\left(\frac{1}{\zeta^{n+1}}\right)\right] W'(\zeta).$$

Picking out the coefficient of $1/\zeta^2$ by the use of (10) and (11) and equating this to the coefficient of $1/\zeta^2$ in (14) we get

$$M = 4\pi\mu\varepsilon c^2\left(1 + \frac{k^2}{n\varkappa}\right). \tag{15}$$

This result again holds only for $n > 1$, and it is noteworthy that the moment is independent of the stress at infinity and that moment and twist vanish together.

6.62. Rigid elliptic core in an infinite plate

We consider an elliptic core filling a hole in an infinite plate which is welded to the core along the rim. On this account the displacement at the boundary will be the same for the core as for the rim. This is the boundary condition whether the core is rigid, as here, or consists of another deformable material.

From 6.21 (6) we have for the ellipse $z = c\,(\zeta + k/\zeta)$, $\zeta = \sigma = e^{i\eta}$,

$$\left(1 - \frac{k}{\zeta^2}\right) w(\zeta) = -\frac{d}{d\zeta}\left\{\left(\frac{1}{\zeta} + k\zeta\right) W(\zeta)\right\} + \frac{1}{\zeta^2}(1 - k\zeta^2)\,\overline{W}\left(\frac{1}{\zeta}\right). \tag{1}$$

As in 6.61 we expand $w(\zeta)$ in descending powers of ζ and determine the moment which produces a given small twist ε, by comparing the coefficient of $1/\zeta^2$ with that of 6.61 (14). The necessary values of $W(\zeta)$ are given by 6.61 (12), (13) and we get

$$M = 4\mu\varepsilon\pi c^2\left(1 + \frac{k^2}{\varkappa}\right) + k\pi c^2 (N_\infty - T_\infty)\left(1 + \frac{1}{\varkappa}\right)\sin 2\varphi. \tag{2}$$

Comparing (2) with 6.61 (15) we see that in the case of an elliptic core alone of the hypotrochoidal family, the moment depends on the state of stress at infinity. The exceptions to this are $N - T = 0$ or $\sin 2\varphi = 0$, which correspond respectively to all round tension, or to the case when a principal stress is in the direction of a principal axis of the ellipse.

In the case of simple tension T_∞, we have $N_\infty = 0$ and

$$M = 4\mu\varepsilon\pi c^2\left(1 + \frac{k^2}{\varkappa}\right) - k\pi c^2 T_\infty\left(1 + \frac{1}{\varkappa}\right)\sin 2\varphi. \tag{3}$$

Let us suppose that the core is first inserted and then the plate is stressed. This result (3) can then be looked upon in several ways;

(i) it gives the couple required to produce a given twist ε,

(ii) it gives the twist produced by a given couple,

(iii) it gives the couple, namely $-kT_\infty\pi c^2\left(1 + \frac{1}{\varkappa}\right)\sin 2\varphi$, required to prevent the core twisting ($\varepsilon = 0$),

(iv) it gives the twist when no couple acts on the core ($M = 0$).

Suppose that the core is inserted but the plate is unstressed. Let a force $X + iY$ be applied to the *centre* of the core. This will produce no rotation for clearly no rotation can be produced by X alone or by Y alone. The effect of the force will be to produce a translation of the core, but as the core is rigid, this rigid body movement can be ignored, for it can be removed by a translation of the material as a whole. Comparison with 2.74 (26) then shows that instead of 6.33 (14) we must have

$$\left(1 - \frac{k}{\zeta^2}\right) W(\zeta) = -\frac{X + iY}{\pi(\varkappa + 1)}\,\frac{1}{\zeta}, \quad (\zeta \text{ in } L).$$

Since the displacement at the boundary is zero we must have

$$\left(1 - \frac{k}{\zeta^2}\right) W(\zeta) = \frac{\varkappa(X + iY)}{\pi(\varkappa + 1)}\,\frac{1}{\zeta}, \quad (\zeta \text{ in } R).$$

6.7. Rings

Let C_1, C_2 be two simple closed curves, C_2 being entirely inside C_1. The curves C_1, C_2 define a doubly-connected ring-shaped region. For such a region there exists a biuniform mapping on a circular annulus in which C_1 maps into one circumference and C_2 into the other (CARATHÉODORY).

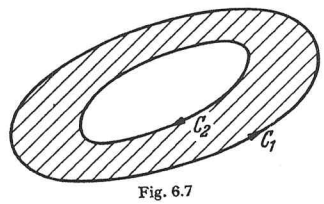

Fig. 6.7

It follows that all the methods applicable to circular boundaries and the annulus are available for the ring.

6.72. The compatibility identity for a ring

We suppose the ring to be mapped on the region between concentric circumferences of radii α, β, $\alpha > \beta$, by

$$z = m(\zeta).$$

From 6.21 (6) we can calculate $m'(\zeta)\,w(\zeta)$ using either $|\zeta| = \alpha$ or $|\zeta| = \beta$ thus obtaining

$$m'(\zeta)\,w(\zeta) = \frac{\alpha^2}{\zeta^2}\,\overline{m}'\left(\frac{\alpha^2}{\zeta}\right)\left[W(\zeta) + \overline{W}\left(\frac{\alpha^2}{\zeta}\right)\right] - \overline{m}\left(\frac{\alpha^2}{\zeta}\right)W'(\zeta),$$

$$m'(\zeta)\,w(\zeta) = \frac{\beta^2}{\zeta^2}\,\overline{m}'\left(\frac{\beta^2}{\zeta}\right)\left[W(\zeta) + \overline{W}\left(\frac{\beta^2}{\zeta}\right)\right] - \overline{m}'\left(\frac{\beta^2}{\zeta}\right)W'(\zeta).$$

By equating these two expressions we obtain the *compatibility identity*

$$\left[\frac{\alpha^2}{\zeta^2}\,\overline{m}'\left(\frac{\alpha^2}{\zeta}\right) - \frac{\beta^2}{\zeta^2}\,\overline{m}'\left(\frac{\beta^2}{\zeta}\right)\right]W(\zeta) - \left[\overline{m}\left(\frac{\alpha^2}{\zeta}\right) - \overline{m}\left(\frac{\beta^2}{\zeta}\right)\right]W'(\zeta)$$
$$= \frac{\beta^2}{\zeta^2}\,\overline{m}'\left(\frac{\beta^2}{\zeta}\right)\overline{W}\left(\frac{\beta^2}{\zeta}\right) - \frac{\alpha^2}{\zeta^2}\,\overline{m}'\left(\frac{\alpha^2}{\zeta}\right)\overline{W}\left(\frac{\alpha^2}{\zeta}\right),$$

or

$$\frac{d}{d\zeta}\left\{\left[\overline{m}\left(\frac{\alpha^2}{\zeta}\right) - \overline{m}\left(\frac{\beta^2}{\zeta}\right)\right]W(\zeta)\right\}$$
$$= \frac{\alpha^2}{\zeta^2}\,\overline{m}'\left(\frac{\alpha^2}{\zeta}\right)\overline{W}\left(\frac{\alpha^2}{\zeta}\right) - \frac{\beta^2}{\zeta^2}\,\overline{m}'\left(\frac{\beta^2}{\zeta}\right)\overline{W}\left(\frac{\beta^2}{\zeta}\right), \quad (\zeta \text{ in } L)$$

where, as in 6.21, $\overline{W}(\alpha^2/\zeta)$, $\overline{W}(\beta^2/\zeta)$ are defined by continuation.

6.8. Coaxal coordinates

The orthogonal net

$$e^t = \frac{z+c}{z-c}, \quad t = \xi + i\eta \tag{1}$$

gives

$$\left|\frac{z+c}{z-c}\right| = e^{\xi}, \ \arg(z+c) - \arg(z-c) = \eta \tag{2}$$

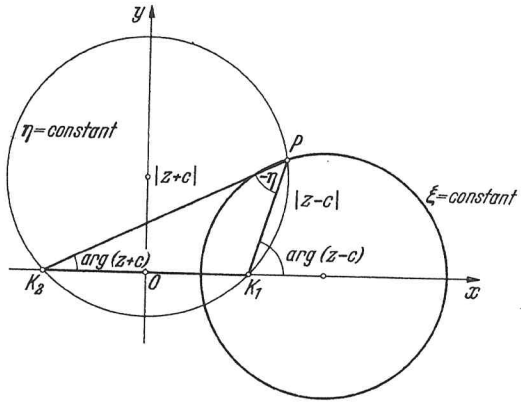

Fig. 6.8 (i)

so that the curves $\xi =$ constant are coaxal circles with limiting points $K_1(c, 0)$, $K_2(-c, 0)$ while the curves $\eta =$ constant are the orthogonal system of coaxal circles intersecting at $(c, 0)$, $(-c, 0)$.

We call (ξ, η) *coaxal coordinates*. The term *bipolar coordinates* is also used [MILNE-THOMSON (5)].

From (2) we have

$$(x - c\coth\xi)^2 + y^2 = c^2\,\mathrm{cosech}^2\xi, \quad x^2 + (y + c\cot\eta)^2 = c\,\mathrm{cosec}^2\eta. \tag{3}$$

Therefore the curve $\xi = \xi_0$ is the circle whose centre is the point $(c\coth\xi_0, 0)$ and whose radius is $c\,|\mathrm{cosech}\,\xi_0|$.

For positive values of ξ_0 these circles lie to the right of the y-axis, and as ξ_0 increases the centre moves towards $(c, 0)$ and the radius decreases while the limiting point K_1 in fig. 6.8 (ii) remains interior to all the circles for which $\xi > 0$.

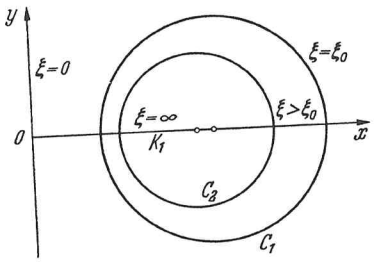

Fig. 6.8 (ii)

We now see that coaxal coordinates can be used to represent three situations.

(*a*) The material occupies an eccentric ring-shaped region between two circumferences, say $\xi = \xi_0$, $\xi = 2\xi_0$, fig. 6.8 (iii) (a).

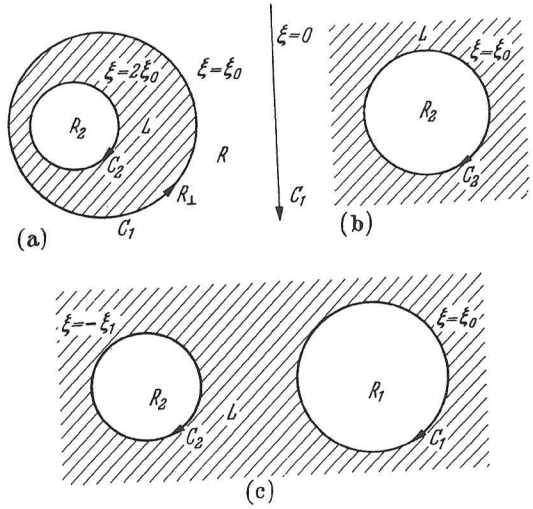

Fig. 6.8 (iii)

(*b*) The material occupies a half-plane, $\xi = 0$, within which there is a circular hole, $\xi = \xi_0$, fig. 6.8 (iii) (b).

(*c*) The material occupies the whole plane except for two circular holes, $\xi = \xi_0$, $\xi = -\xi_1$, where ξ_0, ξ_1 are both positive, fig. 6.8 (iii) (c). If $\xi_0 = \xi_1$ the holes will be of equal size.

Returning to (3) we see that the curve $\eta = \eta_0$ is a circle whose centre is the point $(0, -c\cot\eta_0)$ and whose radius is $c\,|\operatorname{cosec}\eta_0|$. Thus we can suppose the material to consist of the whole plane except for a hole bounded by arcs of circles $\eta = \eta_0$, $\eta = -\eta_1$ say. The case $\eta = \eta_0$, $\eta = -\eta_0$ is shown in fig. 6.8 (iv)

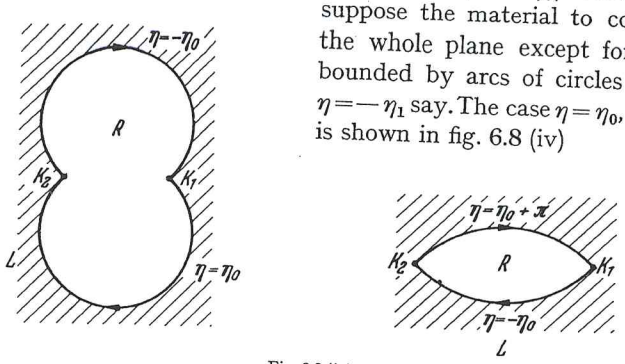

Fig. 6.8 (iv)

From (1) we have

$$\bar{z} = c\coth\frac{1}{2}t = c\,\frac{\sinh\frac{1}{2}(t+\bar{t}) - \sinh\frac{1}{2}(t-\bar{t})}{\cosh\frac{1}{2}(t+\bar{t}) - \cosh\frac{1}{2}(t-\bar{t})} \tag{4}$$

so that

$$x = \frac{c\sinh\xi}{\cosh\xi - \cos\eta}, \quad y = \frac{-\sin\eta}{\cosh\xi - \cos\eta} \tag{5}$$

Thus for a circle $\eta =$ constant, on the upper arc $\eta = -\eta_0$, and on the lower $\eta = -\eta_0 - \pi$ fig. 6.8 (v).

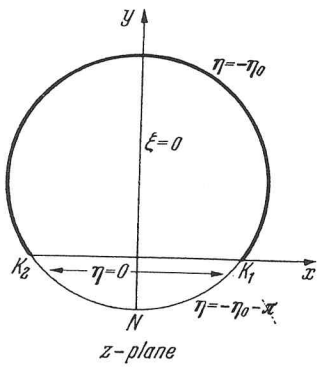

Fig. 6.8 (v)

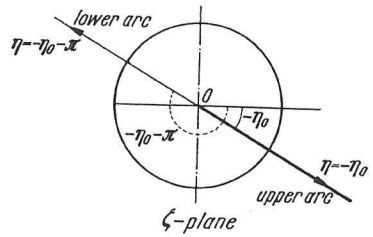

Fig. 6.8 (vi)

Also from (1) we have, if $\zeta = e^t$,

$$z = c\,\frac{e^t+1}{e^t-1} = c\,\frac{\zeta+1}{\zeta-1} = c + \frac{2c}{\zeta-1} = m(\zeta) \tag{6}$$

so that the circumference $\xi = \xi_0$ in the z-plane is mapped on the circumference $|\zeta| = e^{\xi_0}$ in the ζ-plane, and the upper arc $\eta = -\eta_0$ (cf. fig. 6.8 (vi))

maps into the ray $\arg\zeta = -\eta_0$, while the lower arc $\eta = -\eta_0 - \pi$, maps into the opposite ray $\arg\zeta = -\eta_0 - \pi$.

6.82. Mapping an eccentric on a concentric annulus

Let the eccentric annulus be defined by the circumferences C_1, C_2 $\xi = \xi_1$, $\xi = \xi_2$, $\xi_2 > \xi_1$ in the coaxal net

$$z = m(\zeta) = c + \frac{2c}{\zeta - 1}, \quad \zeta = e^{\xi + i\eta}. \tag{1}$$

C_1 maps into Γ_1, $|\zeta| = e^{\xi_1} = \beta$, and C_2 maps into Γ_2, $|\zeta| = e^{\xi_2} = \alpha > \beta$.

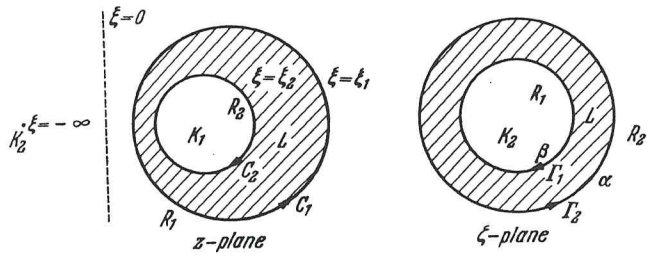

Fig. 6.82

Clearly a point in the region L between C_1 and C_2 will map into a point between Γ_1 and Γ_2. Thus the eccentric annulus maps into a concentric one but, as it were, turned inside out. The region R_1 exterior to C_1 in the z-plane maps into the region R_1 interior to Γ_1 in the ζ-plane.

6.83. Problem I for the eccentric annulus

Referring to 6.82 let the stresses be

$$2(\widehat{\xi_1\xi_1} + i\,\widehat{\xi_1\eta}) = f_1(\sigma) \quad \text{on} \quad C_1,\ \sigma = \beta e^{i\eta} \tag{1}$$

$$2(\widehat{\xi_2\xi_2} + i\,\widehat{\xi_2\eta}) = f_2(\sigma) \quad \text{on} \quad C_2,\ \sigma = \alpha e^{i\eta}\ . \tag{2}$$

Then the boundary conditions are, with $m'(\zeta) = -2c/(\zeta-1)^2$

$$[m'(\sigma)\,W(\sigma)]^L - [m'(\sigma)\,W(\sigma)]^R = m'(\sigma)\,f_1(\sigma) \quad \text{on} \quad C_1 \tag{3}$$

$$[m'(\sigma)\,W(\sigma)]^L - [m'(\sigma)\,W(\sigma)]^R = m'(\sigma)\,f_2(\sigma) \quad \text{on} \quad C_2 \tag{4}$$

and therefore

$$m'(\zeta)\,W(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{m'(\sigma)\,f_1(\sigma)\,d\sigma}{\sigma-\zeta} + \frac{1}{2\pi i}\oint_{\Gamma_2}\frac{m'(\sigma)\,f_2(\sigma)\,d\sigma}{\sigma-\zeta} + h(\zeta) \tag{5}$$

where $h(\zeta)$ is continuous across Γ_1 and Γ_2, for this satisfies (3) and (4) and applies whether ζ is in R_1, L, or R_2. The function $h(\zeta)$ must be

determined from the compatibility identity (6.72), for example by assuming a Laurent series.

6.84. Eccentric annulus under pressure

Let the annulus of fig. 6.82 be subject to hydrostatic pressure p_1 at C_1 and p_2 at C_2. It is sufficient to assume $h(\zeta) = A + \frac{B}{\zeta} + \frac{C}{\zeta^2}$. Then from 6.83

$$\frac{W(\zeta)}{(\zeta-1)^2} = \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{-2p_1 d\sigma}{(\sigma-1)^2(\sigma-\zeta)} + \frac{1}{2\pi i}\oint_{\Gamma_2}\frac{-2p_2 d\sigma}{(\sigma-1)^2(\sigma-\zeta)} + A + \frac{B}{\zeta} + \frac{C}{\zeta^2}. \tag{1}$$

Performing the integrations by means of 3.13 we get

$$\frac{W(\zeta)}{(\zeta-1)^2} = \frac{-2(p_1-p_2)}{(\zeta-1)^2} + A + \frac{B}{\zeta} + \frac{C}{\zeta^2}, \quad (\zeta \text{ in } R_2) \tag{2}$$

$$\frac{W(\zeta)}{(\zeta-1)^2} = \frac{-2p_1}{(\zeta-1)^2} + A + \frac{B}{\zeta} + \frac{C}{\zeta^2}, \quad (\zeta \text{ in } L) \tag{3}$$

$$\frac{W(\zeta)}{(\zeta-1)^2} = A + \frac{B}{\zeta} + \frac{C}{\zeta^2}, \quad (\zeta \text{ in } R_1). \tag{4}$$

Writing for brevity $Z_1 = \beta^2 - \zeta$, $Z_2 = \alpha^2 - \zeta$, the compatiblity identity reduces to

$$(\zeta^2 - \alpha^2\beta^2)\, W(\zeta) - \zeta W'(\zeta)\, Z_1 Z_2 = \frac{2\alpha^2}{\alpha^2-\beta^2}(p_1-p_2)\, Z_1^2 + \left(\frac{\bar{C}}{a^2\beta^2} - \frac{\bar{A}}{\zeta^2}\right) Z_1^2 Z_2^2, \tag{5}$$

where $W(\zeta)$ is given by (3). We note that the symbol i is explicitly absent from this equation and therefore A, B, C must be real. Equating coefficients we find that

$$C = -\alpha^2\beta^2 A. \tag{6}$$

Writing for brevity

$$\lambda = \alpha^2\beta^2,\ \mu = \alpha^2 + \beta^2,\ q = (p_2 - p_1)/(\alpha^2 - \beta^2) \tag{7}$$

we get three equations, by equating coefficients in (5),

$$A\,(2 - 2\lambda - 2\mu + \mu^2) + B\,(\mu - 2) - 2\,(p_1 - \alpha^2 q) = 0,$$

$$A\,(4\lambda - \mu - \lambda\mu) + B\,(1-\lambda) - 2\lambda q = 0,$$

$$A\,(-2\lambda + 2\lambda^2 - 2\lambda\mu + \mu^2) + B\,(2\lambda - \mu) + 2\lambda\,(p_1 + \beta^2 q) = 0.$$

These equations are consistent, for their determinant vanishes and we get

from the first two

$$\frac{A}{-\lambda q(\mu-2) + (1-\lambda)(p_1 - \alpha^2 q)} = \frac{B}{(-4\lambda + \mu + \lambda\mu)(p_1 - \alpha^2 q) + \lambda q\,(2 - 2\lambda - 2\mu + \mu^2)} = \frac{1}{(1+\lambda-\mu)^2} = \frac{1}{[(\alpha^2-1)(\beta^2-1)]^2} \tag{8}$$

and so the constants are completely determined by (6) and (8), and the problem is solved.

To express the solution in terms of a, the radius of the outer circle, b of the inner, and h the distance between their centres, we have

$$a \sinh \xi_1 = c = b \sinh \xi_2, \quad h = a \cosh \xi_1 - b \cosh \xi_2,$$

whence we obtain

$$\lambda = \left[\frac{(a^2 - b^2 + P^2)^2 - h^4}{4abh^2}\right]^2$$

$$\mu = \left(\frac{a^2 - b^2 + h^2 + P^2}{2ah}\right)^2 + \left(\frac{a^2 - b^2 - h^2 + P^2}{2bh}\right)^2,$$

where $P^4 = (a+b+h)(a+b-h)(a-b+h)(a-b-h)$.

Thus the stresses can be expressed in terms of a, b, h.

6.86. Confocal elliptic ring

In elliptic coordinates

$$z = c \cosh t, \quad t = \xi + i\eta$$

the curves $\xi =$ constant are confocal ellipses, which are mapped on concentric circumferences by $z = m(\zeta) = c\left(\zeta + \frac{1}{\zeta}\right)$, $\zeta = e^{\xi + i\eta}$.

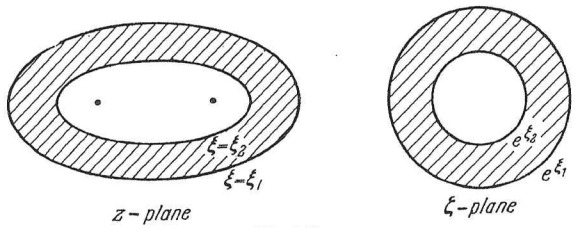

Fig. 6.86

The semi-axes of the ellipse $\xi = \xi_0$ are

$$a = c \cosh \xi_0, \quad b = c \sinh \xi_0.$$

The ring-shaped region between two such ellipses say $\xi = \xi_1$, $\xi = \xi_2$ is mapped on the concentric circular annulus between circles of radii e^{ξ_1}, e^{ξ_2}.

We can therefore set up the compatibility identity of 6.72 for the confocal elliptic ring, if we write

$$\alpha = e^{\xi_1},\ \beta = e^{\xi_2},\ m(\zeta) = c\left(\zeta + \frac{1}{\zeta}\right)$$

6.88. The hypotrochoidal ring

Such a ring is defined by attributing the values $\xi = \xi_1$, $\xi = \xi_2$ in the net (6.3)

$$z = m(\zeta) = c\left(\zeta + \frac{k}{n\zeta^n}\right),\ \zeta = e^{\xi + i\eta}. \tag{1}$$

The ring is then mapped on the annulus defined by the circumferences $|\zeta| = \alpha = e^{\xi_1}$, $|\zeta| = \beta = e^{\xi_2}$, where we assume that $\xi_1 > \xi_2$.

The confocal elliptic ring of 6.86 is a particular case.

The compatibility identity of 6.72 can then be obtained by attributing to $m(\zeta)$ the expression given in (1).

Examples VI

1. Derive the mapping of the periphery of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the circumference of a circle by starting from elliptic coordinates given by $z = c \cosh t$.

Discuss the mapping of the ring region between two confocal ellipses whose semi-axes are (a, b) and $(a^2 + \lambda)^{1/2}$, $(b^2 + \lambda)^{1/2}$.

2. Discuss the relation to polar coordinates of the curvilinear coordinates defined by $z = ce^t$.

3. Obtain the system of curves $\xi =$ constant, $\eta =$ constant defined by the cyclic coordinates given by $z = a^2/t$.

4. Show that the curvilinear coordinates defined by $z = ic \cot \frac{1}{2}\zeta$ refer to two orthogonal systems of coaxal circles.

5. A circle C_2 of radius b rolls on the inner side of a fixed circumference C_1 of radius a. A point P distant kb from the centre of C_2 remains fixed in C_2. The system starts with P on the line of centres which is taken as x-axis. If $n = (a - b)/b$ show that the coordinates of P when the line of centres has turned through the angle η are given by

$$x = (a - b)\left(\cos\eta + \frac{k}{n}\cos n\eta\right),\quad y = (a - b)\left(\sin\eta - \frac{k}{n}\sin n\eta\right).$$

Prove also that if $0 < k < 1/n$ the locus of P will have no points of inflexion.

Find the curvature.

6. In the hypotrochoid of Ex. 5 find the maximum and minimum distances of the curve from the centre when $k < 1$. If $k = 1/n$, show that the curve is flattest at the minimum distance.

7. An infinite plate under all round tension is weakened by an elliptic hole (a, b). Show that the stress concentration factors at the ends of the major and minor axes respectively are $2a/b$ and $2b/a$.

8. An otherwise unstressed infinite plate is weakened by an elliptic hole (a, b) whose boundary is subjected to uniform shear q. Show that

$$W(\zeta) = \frac{2qi(a - b)}{(a - b) - (a + b)\zeta^2}$$

and find $w(\zeta)$.

9. Let $z = c\left(\zeta + \frac{k}{\zeta}\right)$, $z_1 = c\left(\sigma_1 + \frac{k}{\sigma_1}\right)$, $z_2 = c\left(\sigma_2 + \frac{k}{\sigma_2}\right)$. Let z_1, z_2 be points on the ellipse whose semiaxes are $c(1 + k)$, $c(1 - k)$, and which forms the boundary of an elliptic hole in an infinite plate. If the arc between z_1 and z_2 is subject to hydrostatic pressure p and the rest of the boundary is unstressed, show that $W(\zeta)$ is given by

$$c\left(1 - \frac{k}{\zeta^2}\right)W(\zeta) = \frac{ip}{\pi}\left\{\frac{kc}{\zeta^2}\ln\frac{\sigma_2}{\sigma_1} + \frac{d}{d\zeta}\left[(z - z_2)\ln(\sigma_2 - \zeta) - (z - z_1)\ln(\sigma_1 - \zeta)\right] - \frac{\varkappa}{\varkappa + 1}(z_1 - z_2)\frac{1}{\zeta}\right\}.$$

Deduce the corresponding result for a circular hole.

10. In the case of a plate under all round tension weakened by a hypotrochoidal hole show that the stress concentration factor at a vertex of the hole is $2(1 + k)/(1 - k)$.

11. An infinite plate under simple tension T at angle φ to the real axis is weakened by a hypotrochoidal hole. Show that the stress concentration factor at a vertex is

$$f_s = \frac{1}{1 - k}\left\{1 + k - \frac{2n^2\cos\alpha_s\,[n + k(n - 2)]}{n^3 - k^3(n - 2)}\right\},$$

where $\alpha_s = \varphi - 2\pi s/n$, $s = 0, 1, \ldots, n - 1$.

12. Calculate directly the complex stress $W(\zeta)$ for a plate weakened by an elliptic hole

(i) when the hole is unloaded and the plate is stretched,

(ii) when the hole is loaded with hydrostatic pressure and the plate is unstretched.

13. A plate under simple tension at angle φ to the major axis is weakened by an elliptic hole. Prove that if $k = \tan\alpha$, the stress concentration factor at the rim of the hole is

$$f = \frac{\cos 2\alpha + \sin 2\alpha\cos 2\varphi - 2\cos^2\alpha\cos 2(\eta - \varphi)}{1 - \sin 2\alpha\cos 2\eta}.$$

14. Use the result of Ex. 13 to show that there are two points at which $df/d\eta = 0$ and that these correspond one with maximum tension the other with maximum compression, and that the tangents to the ellipse at these points are perpendicular.

15. Show that the radius of curvature of the epitrochoid of 6.5 is

$$\frac{1 + nk^2 + k(1+n)\cos(n-1)\eta}{c\,[1 + k^2 + 2k\cos(n-1)\eta]^{3/2}}$$

and hence show that there are real points of inflexion only if $\frac{1}{n} \leqq k \leqq 1$.

16. Trace the epitrochoidal curvilinear polygons
(i) $n = 4, k = 1/4$, (ii) $n = 5, k = 1/5$.

17. Use the theorem of SCHWARZ and CHRISTOFFEL to find epitrochoidal approximations to an equilateral triangle, a square, a regular hexagon.

18. Use the theorem of SCHWARZ and CHRISTOFFEL to obtain a three term approximation to a rectangular plate of sides a and $2a$.

19. Calculate the stress distribution when a confocal elliptic ring is in equilibrium under standard concentrated forces applied at the extremities of the major axis of the exterior bounding ellipse.

20. A hypotrochoidal equilateral triangle is in equilibrium under three standard concentrated forces of equal magnitude applied at the vertices, that is to say at the points whose distance from the centre is greatest. Calculate the complex stress $W(\zeta)$.

21. An eccentric annulus is in equilibrium under two standard concentrated forces applied to the outer boundary where the annulus has maximum and minimum thickness. Calculate the complex stress $W(\zeta)$.

22. Show that the Möbius transformation

$$z = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \quad \alpha\delta - \beta\gamma \neq 0$$

maps the z-plane on itself and circles on to circles. Hence show how how to map coaxal circles into concentric circles.

Chapter VII

The influence of anisotropy

We now turn to the study of anisotropic material. It might be thought that the difficulties would be greatly enhanced but in fact, in the mathematical sense, the anisotropic is simpler than the isotropic case, the reason being that in the anisotropic case each term in the expression of the stress function is on the same footing whereas this is not so in the case of isotropy. Nevertheless the algebraic complexity of formulae is much greater. We might put the situation this way; in the anisotropic case formulae are obtained by simpler methods but are more complicated in algebraic structure.

As explained in 1.5 we assume the xy plane to be one of elastic symmetry so that our problems are plane in the sense originally defined.

The stress components are therefore given by 1.5 (5), while the coefficients of deformation are given by 1.62 (1) in terms of $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$. The constants involved are therefore those defined by 1.62 (2).

7.01. Change of axes of reference

If we rotate the axes x, y through the angle α to become x_1, y_1, the elastic constants $l_{11}, l_{12}, l_{16}, l_{22}, l_{26}, l_{66}$ of 1.62 (2) will have to be replaced by a new set say $m_{11}, m_{12}, m_{16}, m_{22}, m_{26}, m_{66}$. Introduce the matrices

$$L = \begin{bmatrix} l_{11} & l_{12} & l_{16} \\ l_{12} & l_{22} & l_{26} \\ l_{16} & l_{26} & l_{66} \end{bmatrix}, \quad M = \begin{bmatrix} m_{11} & m_{12} & m_{16} \\ m_{12} & m_{22} & m_{26} \\ m_{16} & m_{26} & m_{66} \end{bmatrix} \tag{1}$$

$$T = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \tag{2}$$

We denote the transposed (or conjugate) of T by T_c

$$T_c = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \tag{3}$$

If U denotes the strain-energy function, 1.4 we have as a product of row and column matrices

$$2U = [\widehat{xx}\ \widehat{yy}\ \widehat{xy}] \cdot \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

But by Hooke's law in the form 1.5

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = L \cdot \begin{bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{xy} \end{bmatrix}$$

Therefore

$$2U = [\widehat{xx}\ \widehat{yy}\ \widehat{xy}] \cdot L \cdot \begin{bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{xy} \end{bmatrix} = [\widehat{x_1x_1}\ \widehat{y_1y_1}\ \widehat{x_1y_1}] \cdot M \cdot \begin{bmatrix} \widehat{x_1y_1} \\ \widehat{y_1y_1} \\ \widehat{x_1y_1} \end{bmatrix} \tag{4}$$

similarly. Also

$$[\widehat{xx}\ \widehat{yy}\ \widehat{xy}] = [\widehat{x_1x_1}\ \widehat{y_1y_1}\ \widehat{x_1y_1}] \cdot T \tag{5}$$

Therefore

$$\begin{bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{xy} \end{bmatrix} = T_c \cdot \begin{bmatrix} \widehat{x_1x_1} \\ \widehat{y_1y_1} \\ \widehat{x_1y_1} \end{bmatrix}. \tag{6}$$

Substitute these in (4). Then we see that

$$M = T \cdot L \cdot T_c \tag{7}$$

so that when L is given M is also known.

7.1. The complex variables for anisotropic material

The solution of the equation for the stress function, 1.62, in the absence of a body-field is

$$\chi = {}^1/_4 \{ \text{``}W_1(z_1) + \text{``}W_2(z_2) + \text{``}\overline{W}_1(\bar{z}_1) + \text{``}\overline{W}_2(\bar{z}_2) \} \tag{1}$$

where

$$z_1 = x + \lambda_1 y\,, \quad z_2 = x + \lambda_2 y\,, \tag{2}$$

and

$$\lambda_1 = \alpha_1 + i\beta_1,\ \lambda_2 = \alpha_2 + i\beta_2,\ \beta_1 > 0,\ \beta_2 > 0,\ \beta_1 \geqq \beta_2 \tag{3}$$

are two roots of the characteristic equation

$$l_{11}\lambda^4 - 2l_{61}\lambda^3 + (2l_{12} + l_{66})\lambda^2 - 2l_{26}\lambda + l_{22} = 0\,. \tag{4}$$

For convenience, write

$$\left.\begin{aligned} \gamma_1 &= \frac{1 - i\lambda_1}{2}, & \delta_1 &= \frac{1 + i\lambda_1}{2} \\ \gamma_2 &= \frac{1 - i\lambda_2}{2}, & \delta_2 &= \frac{1 + i\lambda_2}{2} \end{aligned}\right\} \tag{5}$$

Then

$$\gamma_1 + \delta_1 = \gamma_2 + \delta_2 = 1,\ \gamma_1 - \delta_1 = -i\lambda_1,\ \gamma_2 - \delta_2 = -i\lambda_2\,. \tag{6}$$

We can take any one of the three pairs

$$z, \bar{z};\quad z_1, \bar{z}_1;\quad z_2, \bar{z}_2$$

as independent variables, for

$$z = \frac{\bar{\gamma}_1 z_1 - \delta_1 \bar{z}_1}{\bar{\gamma}_1 - \delta_1} = \frac{\bar{\gamma}_2 z_2 - \delta_2 \bar{z}_2}{\bar{\gamma}_2 - \delta_2} \tag{7}$$

$$z_1 = \gamma_1 z + \delta_1 \bar{z} = \frac{(\lambda_1 - \bar{\lambda}_2)\, z_2 - (\lambda_1 - \lambda_2)\, \bar{z}_2}{\lambda_2 - \bar{\lambda}_2} \tag{8}$$

$$z_2 = \gamma_2 z + \delta_2 \bar{z} = \frac{(\lambda_2 - \bar{\lambda}_1)\, z_1 - (\lambda_2 - \lambda_1)\, \bar{z}_1}{\lambda_1 - \bar{\lambda}_1}\,. \tag{9}$$

The above constant coefficients can be expressed as convenient in terms of λ's or γ, δ's.

The conjugate expressions give the appropriate forms of the conjugate variables $\bar{z}$, $\bar{z}_1$, $\bar{z}_2$.

Now consider the pair

$$z_1 = x + \lambda_1 y,\quad \bar{z}_1 = x + \bar{\lambda}_1 y\,.$$

We have the operator equivalences

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_1}\,, \quad \frac{\partial}{\partial y} = \lambda_1 \frac{\partial}{\partial z_1} + \bar{\lambda}_1 \frac{\partial}{\partial \bar{z}_1}\,. \tag{10}$$

Therefore

$$\left.\begin{aligned} (\lambda_1 - \bar{\lambda}_1)\frac{\partial}{\partial z_1} &= -\bar{\lambda}_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ (\lambda_1 - \bar{\lambda}_1)\frac{\partial}{\partial \bar{z}_1} &= \lambda_1 \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \end{aligned}\right\} \tag{11}$$

Similarly

$$\left.\begin{aligned} (\lambda_2 - \bar{\lambda}_2)\frac{\partial}{\partial z_2} &= -\bar{\lambda}_2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ (\lambda_2 - \bar{\lambda}_2)\frac{\partial}{\partial \bar{z}_2} &= \lambda_2 \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \end{aligned}\right\} \tag{12}$$

7.11. The fundamental stress combinations

We have the general results

$$\begin{aligned} \Theta &= 4\frac{\partial^2 \chi}{\partial z\, \partial \bar{z}} = \widehat{xx} + \widehat{yy} \\ &= \gamma_1 \delta_1 W_1(z_1) + \bar{\gamma}_1 \bar{\delta}_1 \overline{W}_1(\bar{z}_1) + \gamma_2 \delta_2 W_2(z_2) + \bar{\gamma}_2 \bar{\delta}_2 \overline{W}_2(\bar{z}_2) \end{aligned} \tag{1}$$

$$\begin{aligned} \Phi &= 4\frac{\partial^2 \chi}{\partial z^2} = \widehat{yy} - \widehat{xx} + 2i\,\widehat{xy} \\ &= \gamma_1^2 W_1(z_1) + \gamma_2^2 W_2(z_2) + \delta_1^2 \overline{W}_1(\bar{z}_1) + \delta_2^2 \overline{W}_2(\bar{z}_2)\,, \end{aligned} \tag{2}$$

but here in addition it is useful to consider three new combinations

$$\Theta_1 = 4\frac{\partial^2 \chi}{\partial z_2\, \partial \bar{z}_2} = -\frac{4}{(\lambda_2 - \bar{\lambda}_2)^2}\{\lambda_2 \bar{\lambda}_2\, \widehat{yy} + \widehat{xx} + (\lambda_2 + \bar{\lambda}_2)\, \widehat{xy}\}\,, \tag{3}$$

$$\Theta_2 = 4\frac{\partial^2 \chi}{\partial z_1\, \partial \bar{z}_1} = -\frac{4}{(\lambda_1 - \bar{\lambda}_1)^2}\{\lambda_1 \bar{\lambda}_1\, \widehat{yy} + \widehat{xx} + (\lambda_1 + \bar{\lambda}_1)\, \widehat{xy}\}\,, \tag{4}$$

$$\Phi_{1,2} = 4\frac{\partial^2 \chi}{\partial z_1\, \partial z_2} = \frac{4}{(\lambda_1 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_2)}\{\bar{\lambda}_1 \bar{\lambda}_2\, \widehat{yy} + \widehat{xx} + (\bar{\lambda}_1 + \bar{\lambda}_2)\, \widehat{xy}\}\,, \tag{5}$$

the right-hand members being obtained by using 7.1 (11), (12).

To evaluate the above in terms of the complex stresses we use 7.1 (8) to give

$$4\frac{\partial \chi}{\partial z_2} = \text{`}W_1(z_1)\frac{\lambda_1 - \bar{\lambda}_2}{\lambda_2 - \bar{\lambda}_2} + \text{`}W_1(\bar{z}_1)\frac{\bar{\lambda}_1 - \lambda_2}{\lambda_2 - \bar{\lambda}_2} + \text{`}W_2(z_2)\,. \tag{6}$$

Operating with $\partial/\partial \bar{z}_2$ we get

$$\Theta_1 = W_1(z_1)\frac{(\lambda_1 - \bar{\lambda}_2)(\lambda_2 - \lambda_1)}{(\lambda_2 - \bar{\lambda}_2)^2} + \overline{W}_1(\bar{z}_1)\frac{(\bar{\lambda}_1 - \bar{\lambda}_2)(\lambda_2 - \bar{\lambda}_1)}{(\lambda_2 - \bar{\lambda}_2)^2} \tag{7}$$

and therefore from (3)

$$\begin{aligned} (\lambda_2 - \bar{\lambda}_2)^2 \Theta_1 &= W_1(z_1)(\lambda_1 - \bar{\lambda}_2)(\lambda_2 - \lambda_1) + \overline{W}_1(\bar{z}_1)(\bar{\lambda}_1 - \bar{\lambda}_2)(\lambda_2 - \bar{\lambda}_1) \\ &= -4\{\lambda_2 \bar{\lambda}_2\, \widehat{yy} + \widehat{xx} + (\lambda_2 + \bar{\lambda}_2)\, \widehat{xy}\}\,. \end{aligned} \tag{8}$$

Similarly we obtain

$$(\lambda_1-\bar\lambda_1)^2\Theta_2 = W_2(z_2)\,(\lambda_2-\bar\lambda_1)\,(\lambda_1-\lambda_2) + \overline{W}_2(\bar z_2)\,(\bar\lambda_2-\bar\lambda_1)\,(\lambda_1-\bar\lambda_2)$$
$$= -4\,\{\lambda_1\bar\lambda_1\,\widehat{yy} + \widehat{xx} + (\lambda_1+\bar\lambda_1)\,\widehat{xy}\}\,.$$

Operating on (6) with $\partial/\partial z_1$ and using 7.1 (9) we get

$$(\lambda_1-\bar\lambda_1)\,(\lambda_2-\bar\lambda_2)\,\Phi_{1,2} = W_1(z_1)\,(\lambda_1-\bar\lambda_2)\,(\lambda_1-\bar\lambda_1) + W_2(z_2)\,(\lambda_2-\bar\lambda_1)\,(\lambda_2-\bar\lambda_2)$$
$$= 4\,\{\bar\lambda_1\bar\lambda_2\,\widehat{yy} + \widehat{xx} + (\bar\lambda_1+\bar\lambda_2)\,\widehat{xy}\}\,. \tag{9}$$

From (1), (2) and 1.22 (9) we have

$$\widehat{xx} = \frac{1}{4}\{\lambda_1^2\,W_1(z_1) + \lambda_2^2\,W_2(z_2) + \bar\lambda_1^2\overline{W}_1(\bar z_1) + \bar\lambda_2^2\overline{W}_2(\bar z_2)\} \tag{10}$$

$$\widehat{xy} = -\frac{1}{4}\{\lambda_1 W_1(z_1) + \lambda_2 W_2(z_2) + \bar\lambda_1\overline{W}_1(\bar z_1) + \bar\lambda_2\overline{W}_2(\bar z_2)\} \tag{11}$$

$$\widehat{yy} = \frac{1}{4}\{W_1(z_1) + W_2(z_2) + \overline{W}_1(\bar z_1) + \overline{W}_2(\bar z_2)\}\,. \tag{12}$$

7.12. Elimination of body force

We use the equations 1.22 (10) in the form

$$\widehat{xx} - \widehat{xx}_0 = \frac{\partial^2\chi}{\partial y^2}\,,\quad \widehat{yy} - \widehat{yy}_0 = \frac{\partial^2\chi}{\partial x^2}\,,\quad \widehat{xy} - \widehat{xy}_0 = -\frac{\partial^2\chi}{\partial x\,\partial y} \tag{1}$$

where $\widehat{xx}_0$, $\widehat{yy}_0$, $\widehat{xy}_0$ are particular solution of the general plane equation 1.2 (1) and (2). Having found these, the general equations become

$$\frac{\partial}{\partial x}(\widehat{xx} - \widehat{xx}_0) + \frac{\partial}{\partial y}(\widehat{yx} - \widehat{yx}_0) = 0\,, \tag{2}$$

$$\frac{\partial}{\partial x}(\widehat{xy} - \widehat{xy}_0) + \frac{\partial}{\partial y}(\widehat{yy} - \widehat{yy}_0) = 0\,, \tag{3}$$

so that $\widehat{xx} - \widehat{xx}_0$, $\widehat{yy} - \widehat{yy}_0$, $\widehat{xy} - \widehat{xy}_0$ satisfy the general equations in the absence of a body-vector field.

In particular, if the y-axis is directed vertically upwards, we have $b_1 = 0$, $b_2 = g\varrho$ and therefore we can take

$$\widehat{xx}_0 = \widehat{xy}_0 = 0\,,\quad \widehat{yy}_0 = g\varrho y\,. \tag{4}$$

For the displacement we proceed as follows.

From 1.62 (1) we have, in the case when the y-axis is vertical,

$$\frac{\partial u}{\partial x} = l_{12}g\varrho y\,,\quad \frac{\partial v}{\partial y} = l_{22}g\varrho y\,,\quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = l_{26}g\varrho y \tag{5}$$

which has the particular solution

$$u_0 = l_{12}g\varrho\,xy + \frac{1}{2}l_{26}g\varrho y^2\,,\quad v_0 = \frac{1}{2}l_{22}g\varrho y^2 - \frac{1}{2}l_{12}g\varrho x^2\,. \tag{6}$$

The equation of compatibility 1.6 (2) is satisfied and therefore the displacements

$$u - u_0\,,\quad v - v_0 \tag{7}$$

correspond to the absence of gravity.

Thus we can always eliminate the effect of gravity. The procedure for other body-vector fields is on the same lines.

7.13. The displacement

To find the displacement we use

$$\frac{\partial u}{\partial x} = e_{xx}\,,\quad \frac{\partial v}{\partial y} = e_{yy}\,,\quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2e_{xy}\,. \tag{1}$$

Substituting the values of $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ given by 7.11 (10)—(12) in 1.62 (1) we have

$$4\frac{\partial u}{\partial x} = (l_{11}\lambda_1^2 + l_{12} - \lambda_1 l_{16})\,W_1(z_1) + (l_{11}\lambda_2^2 + l_{12} - \lambda_2 l_{16})\,W_2(z_2)$$
$$+ (l_{11}\bar\lambda_1^2 + l_{12} - \bar\lambda_1 l_{16})\,\overline{W}_1(\bar z_1) + (l_{11}\bar\lambda_2^2 + l_{12} - \bar\lambda_2 l_{16})\,\overline{W}_2(\bar z_2) \tag{2}$$

$$4\frac{\partial v}{\partial y} = (l_{12}\lambda_1^2 + l_{22} - \lambda_1 l_{62})\,W_1(z_1) + (l_{12}\lambda_2^2 + l_{22} - \lambda_2 l_{62})\,W_2(z_2)$$
$$+ (l_{12}\bar\lambda_1^2 + l_{22} - \bar\lambda_1 l_{62})\,\overline{W}_1(\bar z_1) + (l_{12}\bar\lambda_2 + l_{22} - \bar\lambda_2 l_{62})\,\overline{W}_2(\bar z_2) \tag{3}$$

$$4\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = (l_{61}\lambda_1^2 + l_{62} - \lambda_1 l_{66})\,W_1(z_1) + (l_{61}\lambda_2^2 + l_{62} - \lambda_2 l_{66})\,W_2(z_2)$$
$$+ (l_{61}\bar\lambda_1^2 + l_{62} - \bar\lambda_1 l_{66})\,\overline{W}_1(\bar z_1) + (l_{61}\bar\lambda_2^2 + l_{62} - \bar\lambda_2 l_{66})\,\overline{W}_2(\bar z_2)\,. \tag{4}$$

Now

$$\int W_1(z_1)\,dx = {}'W_1(z_1) + \text{constant} \tag{5}$$

$$\int W_1(z_1)\,dy = \frac{1}{\lambda_1}\,{}'W_1(z_1) + \text{constant}. \tag{6}$$

Therefore omitting a constant (rigid) displacement we have the tentative results

$$4u = (l_{11}\lambda_1^2 - l_{61}\lambda_1 + l_{21})\,{}'W(z_1) + (l_{11}\lambda_2^2 - l_{61}\lambda_2 + l_{21})\,{}'W(z_2)$$
$$+ (l_{11}\bar\lambda_1^2 - l_{61}\bar\lambda_1 + l_{21})\,{}'\overline{W}(\bar z_1) + (l_{11}\bar\lambda_2^2 - l_{61}\bar\lambda_2 + l_{21})\,{}'\overline{W}(\bar z_2) \tag{7}$$

$$4v = \left(\frac{l_{12}\lambda_1^2 - l_{62}\lambda_1 + l_{22}}{\lambda_1}\right){}'W(z_1) + \left(\frac{l_{12}\lambda_2^2 - l_{62}\lambda_2 + l_{22}}{\lambda_2}\right){}'W(z_2)$$
$$+ \left(\frac{l_{12}\bar\lambda_1^2 - l_{62}\bar\lambda_1 + l_{22}}{\bar\lambda_1}\right){}'\overline{W}(\bar z_1) + \left(\frac{l_{12}\bar\lambda_2^2 - l_{62}\bar\lambda_2 + l_{22}}{\bar\lambda_2}\right){}'\overline{W}(\bar z_2)\,. \tag{8}$$

These results will therefore be compatible with (4) if equations of the type

$$l_{61}\lambda_1^2 - l_{66}\lambda_1 + l_{62} = \frac{l_{12}\lambda_1^2 - l_{62}\lambda_1 + l_{22}}{\lambda_1} + \lambda_1\,(l_{11}\lambda_1^2 - l_{61}\lambda_1 + l_{21})$$

are satisfied. But this is indeed so, for the above is simply a rearrangement of the characteristic equation satisfied by λ_1. Therefore

$$4\,(u+iv) = p_1{}'W_1(z_1) + p_2{}'W_2(z_2) + \bar{q}_1{}'\overline{W}(\bar{z}_1) + \bar{q}_2{}'\overline{W}_2(\bar{z}_2)\,, \tag{9}$$

where

$$p_1 = l_{21} - l_{61}\lambda_1 + l_{11}\lambda_1^2 + \frac{i}{\lambda_1}\,(l_{22} - l_{62}\lambda_1 + l_{12}\lambda_1^2)\,, \tag{10}$$

$$\bar{q}_1 = l_{21} - l_{61}\bar{\lambda}_1 + l_{11}\bar{\lambda}_1^2 + \frac{i}{\bar{\lambda}_1}\,(l_{22} - l_{62}\bar{\lambda}_1 + l_{12}\bar{\lambda}_1^2)\,, \tag{11}$$

p_2, $\bar{q}_2$ are formed from λ_2 with the same coefficients.

The characteristic equation satisfied by λ_1 can be written

$$A_1\lambda_1^2 - B_1\lambda_1 + C_1 = 0\,, \tag{12}$$

where

$$\left.\begin{aligned} A_1 &= l_{11}\lambda_1^2 - l_{61}\lambda_1 + l_{21} \\ B_1 &= l_{61}\lambda_1^2 + l_{66}\lambda_1 + l_{62} \\ C_1 &= l_{12}\lambda_1^2 - l_{62}\lambda_1 + l_{22} \end{aligned}\right\}. \tag{13}$$

Thus

$$\left.\begin{aligned} p_1 &= A_1 + \frac{i}{\lambda_1}\,C_1 \\ \bar{q}_1 &= \bar{A}_1 + \frac{i}{\bar{\lambda}_1}\,\bar{C}_1 \end{aligned}\right\}. \tag{14}$$

7.2. Force and moment

With the figure and notation of 1.26 we have for the force (X, Y) and its moment M

$$\begin{aligned} (X+iY) &= -2i\left[\frac{\partial\chi}{\partial\bar{z}}\right]_{AB} \\ &= -\frac{1}{2}\,i\,[\delta_1{}'W_1(z_1) + \delta_2{}'W_2(z_2) + \bar{\gamma}_1{}'\overline{W}_1(\bar{z}_1) + \bar{\gamma}_2{}'\overline{W}_2(\bar{z}_2)]_{AB} \end{aligned} \tag{1}$$

$$\begin{aligned} M &= \mathrm{Re}\left[\chi - 2z\,\frac{\partial\chi}{\partial z}\right]_{AB} \\ &= \mathrm{Re}\,\frac{1}{4}\,[{}''W_1(z_1) + {}''W_2(z_2) + {}''\overline{W}_1(\bar{z}_1) + {}''\overline{W}_2(\bar{z}_2) \\ &\quad - 2z\,\{\gamma_1{}'W_1(z_1) + \gamma_2{}'W_2(z_2) + \delta_1{}'\overline{W}_1(\bar{z}_1) + \delta_2{}'\overline{W}_2(\bar{z}_2)\}]_{AB}\,. \end{aligned} \tag{2}$$

The same formula holds for a closed contour C merely by replacing AB by C.

In the case of a closed contour the only terms which contribute to force and moment are those containing logarithms.

7.21. Stress boundary conditions

Let AB be a boundary described in the sense AB. We consider the action upon the material in the region L to the left of AB. Let the given stress distribution be $(\widehat{nn} + i\,\widehat{ns})$ where n is the normal directed into R.

Let this normal n make the angle α with Ox. Then if $\widehat{nx}$, $\widehat{ny}$ are the stress components parallel to the axes, we have

$$\widehat{nx} + i\,\widehat{ny} = (\widehat{nn} + i\,\widehat{ns})\,e^{i\alpha}. \tag{1}$$

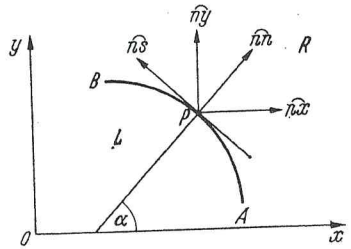

Fig. 7.21

Let $X_P + iY_P$ be the resultant force acting upon the arc AP of the boundary. Then from 7.2 (1)

$$\begin{aligned} -\frac{1}{2}\,i\,\{\delta_1{}'W_1(z_1) + \delta_2{}'W_2(z_2) + \bar{\gamma}_1{}'\overline{W}_1(\bar{z}_1) + \bar{\gamma}_2{}'\overline{W}_2(\bar{z}_2)\} + \text{constant} \\ = X_P + iY_P = \int_{AP} (\widehat{nx} + i\,\widehat{ny})\,ds = -i\int_{AP} (\widehat{nn} + i\,\widehat{ns})\,dz\,, \end{aligned}$$

since $-i\,dz = ds\,e^{i\alpha}$.

Thus the boundary condition is

$$\begin{aligned} \delta_1{}'W_1(z_1) + \delta_2{}'W_2(z_2) + \bar{\gamma}_1{}'\overline{W}_1(\bar{z}_1) + \bar{\gamma}_2{}'\overline{W}_2(\bar{z}_2) + \text{constant} \\ = \int_{AP} 2\,(\widehat{nn} + i\,\widehat{ns})\,dz\,. \end{aligned} \tag{2}$$

If we suppose z, z_1, z_2 to be functions of the parameter $\sigma = e^{i\theta}$, $\bar{\sigma} = e^{-i\theta} = 1/\sigma$, say

$$z = m(\sigma)\,, \quad z_1 = m_1(\sigma)\,, \quad z_2 = m_2(\sigma)\,, \tag{3}$$

differentiation of (2) with respect to σ gives

$$\begin{aligned} &\delta_1 W_1(z_1)\,m_1'(\sigma) + \delta_2 W_2(z_2)\,m_2'(\sigma) - \bar{\gamma}_1\overline{W}_1(\bar{z}_1)\,\overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &- \bar{\gamma}_2\overline{W}_2(\bar{z}_2)\,\overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = 2\,(\widehat{nn} + i\,\widehat{ns})\,m'(\sigma)\,. \end{aligned} \tag{4}$$

Forming the complex conjugate of (2) we have

$$\gamma_1{}'W_1(z_1) + \gamma_2{}'W_2(z_2) + \delta_1{}'\overline{W}_1(\bar{z}_1) + \delta_2{}'\overline{W}_2(\bar{z}_2) = \int_{AP} 2\,(\widehat{nn} - i\,\widehat{ns})\,d\bar{z} \tag{5}$$

and therefore, on differentiation with respect to σ,

$$\begin{aligned} &\gamma_1 W_1(z_1)\,m_1'(\sigma) + \gamma_2 W_2(z_2)\,m_2'(\sigma) - \delta_1\overline{W}_1(\bar{z}_1)\,\overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} - \delta_2\overline{W}_2(\bar{z}_2)\,\overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &= -2\,(\widehat{nn} - i\,\widehat{ns})\,\overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \end{aligned} \tag{6}$$

Note that here, for example, $W_1(z_1) = W_1\,[m_1(\sigma)]$ which is a function of σ alone. This is to be understood for every term in (6).

7.22. Displacement boundary condition

Here we are given the displacement

$$\upsilon_B = u_B + i v_B \tag{1}$$

at every point of the boundary. Then from 7.13 (9), on the boundary, ignoring a possible rigid body displacement,

$$p_1 \text{`}W_1(z_1) + p_2 \text{`}W_2(z_2) + \bar{q}_1 \text{`}\overline{W}_1(\bar{z}_1) + \bar{q}_2 \text{`}\overline{W}_2(\bar{z}_2) = 4\upsilon_B. \tag{2}$$

If as in 7.21 (3) we suppose z, z_1, z_2 expressed as functions of the parameter $\sigma = e^{i\theta}$ we have, on differentiation of (2) with respect to σ

$$\begin{aligned} &p_1 W_1(z_1)\, m_1'(\sigma) + p_2 W_2(z_2)\, m_2'(\sigma) - \bar{q}_1 \overline{W}_1(\bar{z}_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right) \frac{1}{\sigma^2} \\ &\quad - \bar{q}_2 \overline{W}_2(\bar{z}_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right) \frac{1}{\sigma^2} = 4 \frac{d\upsilon_B}{d\sigma}. \end{aligned} \tag{3}$$

Note that in (3) $W_1(z_1) = W_1\,[m_1(\sigma)]$ is a function of σ alone and this applies to every term.

7.3. Determinateness of the complex stresses

Suppose no forces to be operative and the stress to be everywhere zero. Let us seek the most general form of the complex stresses $W_1^{(0)}(z_1)$, $W_2^{(0)}(z_2)$ consistent with this hypothesis.

Since no forces are operative the resultant force on the material within any closed curve C is zero. Therefore from 7.2 (1)

$$\delta_1 \text{`}W_1^{(0)}(z_1) + \delta_2 \text{`}W_2^{(0)}(\bar{z}_2) + \bar{\gamma}_1 \text{`}\overline{W}_1^{(0)}(\bar{z}_1) + \bar{\gamma}_2 \text{`}\overline{W}_2^{(0)}(\bar{z}_2) = K \tag{1}$$

where K is a constant, the same for every circuit C.

Operate with

$$\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} = 4 \frac{\partial^2}{\partial z_2\, \partial \bar{z}_2} \tag{2}$$

and remember that,

$$z_1 = \frac{(\lambda_1 - \bar{\lambda}_2)\, z_2 - (\lambda_1 - \lambda_2)\, \bar{z}_2}{\lambda_2 - \bar{\lambda}_2}, \quad \bar{z}_1 = \frac{(\bar{\lambda}_1 - \bar{\lambda}_2)\, z_2 - (\bar{\lambda}_1 - \lambda_2)\, \bar{z}_2}{\lambda_2 - \bar{\lambda}_2}. \tag{3}$$

Then we get from (1)

$$\begin{aligned} &(\lambda_1 - \bar{\lambda}_2)(\lambda_1 - \lambda_2)\, W_1^{(0)\prime}(z_1) + (\bar{\lambda}_1 - \lambda_2)(\bar{\lambda}_1 - \bar{\lambda}_2)\, \overline{W}_1^{(0)\prime}(\bar{z}_1) = 0, \\ &\lambda_1 (\lambda_1 - \bar{\lambda}_2)(\lambda_1 - \lambda_2)\, W_1^{(0)\prime}(z_1) + \bar{\lambda}_1 (\bar{\lambda}_1 - \lambda_2)(\bar{\lambda}_1 - \bar{\lambda}_2)\, \overline{W}_1^{(0)\prime}(\bar{z}_1) = 0, \end{aligned} \tag{4}$$

by equating the real and imaginary parts to zero.

The determinant

$$\begin{vmatrix} 1 & 1 \\ \lambda_1 & \bar{\lambda}_1 \end{vmatrix} = -2i\beta_1 \neq 0.$$

Hence $(\lambda_1 - \bar{\lambda}_2)(\lambda_1 - \lambda_2)\, W_1^{(0)\prime}(z_1) = 0$ and therefore if $\lambda_1 \neq \lambda_2, \bar{\lambda}_2$

$$W_1^{(0)\prime}(z_1) \equiv 0 \quad \text{and so} \quad \text{`}W_1^{(0)}(z_1) = A_1 z_1 + B_1. \tag{5}$$

Similarly $W_2^{(0)\prime}(z_1) = 0$ and so $\text{`}W^{(0)}(z_2) = A_2 z_2 + B_2$. Here A_1, A_2, B_1, B_2 are arbitrary complex constants.

Inserting these expressions for $\text{`}W_1^{(0)}(z_1)$, $\text{`}W_2^{(0)}(z_2)$ into the formulae, 7.11, which give zero values for $\widehat{yy}$, $\widehat{xy}$, $\widehat{xx}$ we get

$$\begin{aligned} A_1 + \bar{A}_1 + A_2 + \bar{A}_2 &= 0 \\ \lambda_1 A_1 + \bar{\lambda}_1 \bar{A}_1 + \lambda_2 A_2 + \bar{\lambda}_2 \bar{A}_2 &= 0 \\ \lambda_1^2 A_1 + \bar{\lambda}_1^2 \bar{A}_1 + \lambda_2^2 A_2 + \bar{\lambda}_2^2 \bar{A}_2 &= 0 \end{aligned} \tag{6}$$

and therefore

$$\frac{A_1}{\begin{vmatrix} 1 & 1 & 1 \\ \bar{\lambda}_1 & \lambda_2 & \bar{\lambda}_2 \\ \bar{\lambda}_1^2 & \lambda_2^2 & \bar{\lambda}_2^2 \end{vmatrix}} = \frac{-\bar{A}_1}{\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \bar{\lambda}_2 \\ \lambda_1^2 & \lambda_2^2 & \bar{\lambda}_2^2 \end{vmatrix}} = \frac{A_2}{\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_1 & \bar{\lambda}_2 \\ \lambda_1^2 & \bar{\lambda}_1^2 & \bar{\lambda}_2^2 \end{vmatrix}} = \frac{-\bar{A}_2}{\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \bar{\lambda}_1 & \lambda_2 \\ \lambda_1^2 & \bar{\lambda}_1^2 & \lambda_2^2 \end{vmatrix}} = iE \tag{7}$$

Since $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$, we get

$$\frac{A_1}{(\bar{\lambda}_1 - \lambda_2)(\lambda_2 - \bar{\lambda}_2)(\bar{\lambda}_2 - \bar{\lambda}_1)} = \frac{A_2}{(\lambda_1 - \bar{\lambda}_1)(\bar{\lambda}_1 - \bar{\lambda}_2)(\bar{\lambda}_2 - \lambda_1)} = iE. \tag{8}$$

Also from (7) we see by combining the first two terms that E is real. Therefore (5) gives

$$\left.\begin{aligned} \text{`}W^{(0)}(z_1) &= iE m_1 z_1 + B_1, \\ \text{`}W^{(0)}(z_2) &= iE m_2 z_2 + B_2, \end{aligned}\right\} \tag{9}$$

where E is real and arbitrary, B_1, B_2 are completely arbitrary and

$$m_1 = (\bar{\lambda}_1 - \lambda_2)(\lambda_2 - \bar{\lambda}_2)(\bar{\lambda}_2 - \bar{\lambda}_1), \quad m_2 = (\lambda_1 - \bar{\lambda}_1)(\bar{\lambda}_1 - \bar{\lambda}_2)(\bar{\lambda}_2 - \lambda_1). \tag{10}$$

Now let $\text{`}W_1(z_1)$, $\text{`}W_2(z_2)$ be any solution of Problem I for given boundary stresses. Then the general solution is

$$\text{`}W_1(z_1) + \text{`}W_1^{(0)}(z_1), \quad \text{`}W_2(z_2) + \text{`}W_2^{(0)}(z_2), \tag{11}$$

which therefore contains five arbitrary real constants namely E and the real and imaginary parts of B_1, B_2.

If therefore the origin is in the region occupied by the material, we can always choose the constants so that, say,

$$\text{`}W_1(0) = 0, \quad \text{`}W_2(0) = 0, \quad \operatorname{Im}\,[W_1(0)] = 0. \tag{12}$$

In the case of Problem II the boundary condition 7.22 (3) has to be satisfied. Therefore only three of the five constants remain arbitrary. If then the origin is in the region occupied by the material, we could for example choose them so that

$$\text{`}W_1(0) = 0, \quad \operatorname{Im}\,[W_1(0)] = 0.$$

7.31. Cyclic properties of the complex stresses

Given $z = x + iy$, the variables

$$z_1 = x + \lambda_1 y, \quad z_2 = x + \lambda_2 y \tag{1}$$

define complex planes z_1 and z_2. To a curve C in the z-plane there correspond a curve C_1 in the z_1-plane and a curve C_2 in the z_2-plane which are affine maps of C by means of (1).

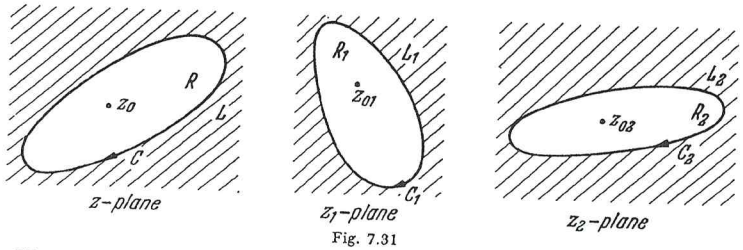

Fig. 7.31

If we write

$$z_1 = x_1 + iy_1, \quad x_1 = x + \alpha_1 y, \quad y_1 = \beta_1 y, \quad \lambda_1 = \alpha_1 + i\beta_1, \tag{2}$$

the JACOBIAN $\dfrac{\partial(x_1, y_1)}{\partial(x, y)} = \beta_1 > 0$ from 7.1 (3). Thus the senses of description of C_1 and C correspond and similarly for C_2 and C.

Thus when z describes C, z_1 describes C_1 and z_2 describes C_2 all in the same sense.

Since the stresses are one-valued it follows from 7.11 (3), (5), that the cyclic functions

$$[\Theta_1]_{C_1} = 0, \quad [\Phi_{1,2}]_{C_1} = 0, \tag{3}$$

and therefore from 7.11 (7) that

$$[(\lambda_1 - \bar{\lambda}_2)(\lambda_1 - \lambda_2) W_1(z_1)]_{C_1} + [(\bar{\lambda}_1 - \bar{\lambda}_2)(\bar{\lambda}_1 - \lambda_2) \overline{W}_1(\bar{z}_1)]_{C_1} = 0 .$$

Since the two terms in this expression are complex conjugates each must be imaginary and so we can write

$$(\lambda_1 - \bar{\lambda}_2)(\lambda_1 - \lambda_2)[W_1(z_1)]_{C_1} = iA, \tag{4}$$

where A is a real constant.

Now $\lambda_2 - \bar{\lambda}_2$ is imaginary and therefore writing for convenience

$$iA = \gamma(\lambda_2 - \bar{\lambda}_2), \ \gamma \text{ real} \tag{5}$$

we have

$$[W_1(z_1)]_{C_1} = \frac{\gamma(\lambda_2 - \bar{\lambda}_2)}{(\lambda_1 - \lambda_2)(\lambda_1 - \bar{\lambda}_2)} = 2\pi i \gamma k_1 \text{ say.} \tag{6}$$

Again from (3) and 7.11 (9) we have

$$(\lambda_1 - \bar{\lambda}_1)(\lambda_1 - \bar{\lambda}_2)[W_1(z_1)]_{C_1} + (\lambda_2 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_2)[W_2(z_2)]_{C_2} = 0 . \tag{7}$$

Combining this with (6) we get

$$[W_2(z_2)]_{C_2} = \gamma \frac{\lambda_1 - \bar{\lambda}_1}{(\lambda_2 - \lambda_1)(\lambda_2 - \bar{\lambda}_1)} = 2\pi i \gamma k_2 . \tag{8}$$

Although (8) can be obtained from (6) by a mere interchange of suffixes it does not follow that γ is the same real constant in both cases, in other words we must use the second of (3).

From (6) we obtain by integration

$$[W_1(z_1)]_{C_1} = 2\pi i(\gamma k_1 z_1 + A_1), \tag{9}$$

where A_1 is an arbitrary complex constant.

Let z_0 be an arbitrary fixed point in the region R determined by describing C in the positive sense and let z_{01}, z_{02} be the corresponding points in the regions R_1, R_2 determined by C_1 and C_2. Then from (9) we must have

$$`W_1(z_1) = (\gamma k_1 z_1 + A_1) \ln(z_1 - z_{01}) + `W_1^*(z_1), \tag{10}$$

where $`W_1^*(z_1)$ is holomorphic in the region between C_1 and any reconcilable contour in L_1.

Similarly we have

$$`W_2(z_2) = (\gamma k_2 z_2 + A_2) \ln(z_2 - z_{02}) + `W_2^*(z_2) . \tag{11}$$

Observe that the arbitrary elements in (10) and (11) are γ, A_1, A_2 but that k_1, k_2 are not arbitrary, being defined by (6) and (8).

7.34. Dislocations

Dislocations have already been defined in 2.61, and in fact the deformation is dislocational if

$$[u + iv]_C = \alpha + i\beta z, \quad \beta \text{ real} \tag{1}$$

where C is an irreducible circuit in the material. Using 7.13 (9) combined with the cyclic properties 7.31 (10) and (11) we have

$$\begin{aligned} 4[u + iv]_C = 2\pi i \{ & p_1(\gamma k_1 z_1 + A_1) + p_2(\gamma k_2 z_2 + A_2) \\ & - \bar{q}_1(\gamma \bar{k}_1 \bar{z}_1 + \bar{A}_1) - \bar{q}_2(\gamma \bar{k}_2 \bar{z}_2 + \bar{A}_2) \} . \end{aligned} \tag{2}$$

For the deformation to be non-dislocational we must have

$$\gamma = 0 \tag{3}$$

$$p_1 A_1 + p_2 A_2 - \bar{q}_1 \bar{A}_1 - \bar{q}_2 \bar{A}_2 = 0 , \tag{4}$$

$$q_1 A_1 + q_2 A_2 - \bar{p}_1 \bar{A}_1 - \bar{p}_2 \bar{A}_2 = 0 , \tag{5}$$

where (5) is simply the complex conjugate of (4).

Observe that provided A_1, A_2 satisfy (4) and (5) the deformation is non-dislocational but that these equations are insufficient to determine

A_1 and A_2. If, however, we know the resultant force $X + iY$ applied to the contour C, we have from 7.2 (1)

$$X + iY = \pi\{\delta_1 A_1 + \delta_2 A_2 - \bar{\gamma}_1 \bar{A}_1 - \bar{\gamma}_2 \bar{A}_2\} \tag{6}$$

$$X - iY = \pi\{-\gamma_1 A_1 - \gamma_2 A_2 + \bar{\delta}_1 \bar{A}_1 + \bar{\delta}_2 \bar{A}_2\} \tag{7}$$

and now (4)—(7) determine A_1, A_2, $\bar{A}_1$, $\bar{A}_2$, in fact

$$\frac{A_1}{\begin{vmatrix} \frac{2iY}{\pi} & 1 & 1 & 1 \\ \frac{-X+iY}{\pi} & \gamma_2 & \bar{\delta}_1 & \bar{\delta}_2 \\ 0 & p_2 & \bar{q}_1 & \bar{q}_2 \\ 0 & q_2 & \bar{p}_1 & \bar{p}_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \bar{\delta}_1 & \bar{\delta}_2 \\ p_1 & p_2 & \bar{q}_1 & \bar{q}_2 \\ q_1 & q_2 & \bar{p}_1 & \bar{p}_2 \end{vmatrix}}$$

where each determinant has been reduced by adding the second row to the first and observing that $\gamma_1 + \delta_1 = \gamma_2 + \delta_2 = 1$.

7.36. Form of the complex stresses at infinity

Assuming that all stresses are bounded at infinity the argument of 2.74 is immediately applicable to show that in the neighbourhood of infinity, in the absence of dislocation, the forms of $W_1(z_1)$, $W_2(z_2)$ derive from

$$W_1(z_1) = A_1 \ln z_1 + B_1 z_1 + \alpha_0^{(1)} + \frac{\alpha_1^{(1)}}{z_1} + \frac{\alpha_2^{(1)}}{z_1^2} + \cdots \tag{1}$$

$$W_2(z_2) = A_2 \ln z_2 + B_2 z_2 + \alpha_0^{(2)} + \frac{\alpha_1^{(2)}}{z_2} + \frac{\alpha_2^{(2)}}{z_2^2} + \cdots \tag{2}$$

The condition that there shall be no dislocation is

$$p_1 A_1 + p_2 A_2 - \bar{q}_1 \bar{A}_1 - \bar{q}_2 \bar{A}_2 = 0\,. \tag{3}$$

Also if the resultant force is $X + iY$ we must have (7.2)

$$\delta_1 A_1 + \delta_2 A_2 - \bar{\gamma}_1 \bar{A}_1 - \bar{\gamma}_2 \bar{A}_2 = \frac{-(X+iY)}{\pi}\,. \tag{4}$$

The equations (3) and (4) together with their complex conjugates determine A_1, A_2, $\bar{A}_1$, $\bar{A}_2$ without ambiguity.

It follows from (1) and (2) that

$$W_1(z_1) = \frac{A_1}{z_1} + B_1 + W_1^*(z_1)\,, \tag{5}$$

$$W_2(z_2) = \frac{A_2}{z_2} + B_2 + W_2^*(z_2)\,, \tag{6}$$

where $W_1^*(z_1)$, $W_2^*(z_2)$ are holomorphic in L and *vanish* at infinity.

It remains to discuss B_1, B_2. From 7.11, (10)—(12), we have for the stresses $\widehat{xx}_\infty$, $\widehat{yy}_\infty$, $\widehat{xy}_\infty$ at infinity

$$\lambda_1^2 B_1 + \lambda_2^2 B_2 + \bar{\lambda}_1^2 \bar{B}_1 + \bar{\lambda}_2^2 \bar{B}_2 = 4\widehat{xx}_\infty \tag{7}$$

$$\lambda_1 B_1 + \lambda_2 B_2 + \bar{\lambda}_1 \bar{B}_1 + \bar{\lambda}_2 \bar{B}_2 = -4\widehat{xy}_\infty \tag{8}$$

$$B_1 + B_2 + \bar{B}_1 + \bar{B}_2 = 4\widehat{yy}_\infty \tag{9}$$

and we have only 3 equations to determine the 4 numbers B_1, $\bar{B}_1$, B_2, $\bar{B}_2$. Thus either B_1 or B_2 remains arbitrary.

It therefore follows that we may assume an arbitrary fourth linear relationship (independent of (7)—(9)) between the unknowns in order to render their values determinate. One such relationship which happens to be highly convenient in reducing the algebra is to suppose that B_1 is real, i. e. to take as our fourth equation

$$B_1 = \bar{B}_1\,. \tag{10}$$

The reader should verify that the determinant of the system (7)—(10) is not zero, and so leads to a solution.

An alternative would be to assign an arbitrary value to the rotation at infinity. This would lead to a fourth equation, replacing (10), but apart from the algebra involved, there is no clear guiding physical principle on which to base such a choice.

7.37. Simple tension

For a plate under simple tension T at angle α to the x-axis we have from 1.24 (2), (3)

$$\Theta = T\,, \quad \Phi = -Te^{-2i\alpha} \tag{1}$$

and therefore

$$\widehat{xx} = T\cos^2\alpha\,, \quad \widehat{yy} = T\sin^2\alpha\,,$$
$$\widehat{xy} = T\sin\alpha\cos\alpha\,. \tag{2}$$

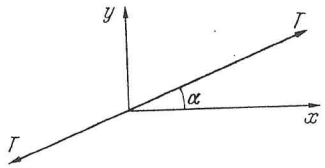

Fig. 7.37

The complex stresses are constants, say

$$W_1(z_1) = B_1\,, \quad W_2(z_2) = B_2\,. \tag{3}$$

Therefore from 7.11 (8), (9)

$$B_1(\lambda_1 - \bar{\lambda}_2)(\lambda_1 - \lambda_2) + \bar{B}_1(\bar{\lambda}_1 - \bar{\lambda}_2)(\bar{\lambda}_1 - \lambda_2)\,,$$
$$= 4T(\cos\alpha + \lambda_2\sin\alpha)(\cos\alpha + \bar{\lambda}_2\sin\alpha) \tag{4}$$

$$(\lambda_1 - \bar{\lambda}_1)(\lambda_1 - \bar{\lambda}_2)B_1 + (\lambda_2 - \bar{\lambda}_1)(\lambda_2 - \bar{\lambda}_2)B_2$$
$$= 4T(\cos\alpha + \bar{\lambda}_1\sin\alpha)(\cos\alpha + \bar{\lambda}_2\sin\alpha)\,. \tag{5}$$

A third equation is needed to determine B_1, B_2 completely. We take, as in 7.36 (10),

$$B_1 = \overline{B}_1 \tag{6}$$

then

$$B_1 = \frac{4\,T\,(\cos\alpha + \lambda_2 \sin\alpha)\,(\cos\alpha + \bar{\lambda}_2 \sin\alpha)}{(\lambda_1 - \bar{\lambda}_2)\,(\lambda_1 - \lambda_2) + (\bar{\lambda}_1 - \bar{\lambda}_2)\,(\lambda_1 - \lambda_2)} \tag{7}$$

$$B_2 = \frac{4\,T\,(\cos\alpha + \bar{\lambda}_1 \sin\alpha)\,(\cos\alpha + \bar{\lambda}_2 \sin\alpha) - (\lambda_1 - \bar{\lambda}_1)\,(\lambda_1 - \bar{\lambda}_2)\,B_1}{(\lambda_2 - \bar{\lambda}_1)\,(\lambda_2 - \bar{\lambda}_2)}\,. \tag{8}$$

For all round tension we take α, and $\frac{\pi}{2} + \alpha$ in turn in (4) and add the results, and treat (5) similarly. The result is to replace the right-hand sides of (4) and (5) by $T\,(1 + \lambda_2\bar{\lambda}_2)$, $T\,(1 + \bar{\lambda}_1\bar{\lambda}_2)$ respectively.

7.38. Standard concentrated force

If in 7.36 (1), (2) we ignore all terms except the first on the right we have the complex stresses for the standard concentrated force $X + iY$ applied at the origin namely

$$'W_1(z_1) = A_1 \ln z_1, \quad 'W_2(z_2) = A_2 \ln z_2 \tag{1}$$

where the force is given by

$$\delta_1 A_1 + \delta_2 A_2 - \bar{\gamma}_1 \bar{A}_1 - \bar{\gamma}_2 \bar{A}_2 = -\frac{X + iY}{\pi} \tag{2}$$

$$\gamma_1 A_1 + \gamma_2 A_2 - \bar{\delta}_1 \bar{A}_1 - \bar{\delta}_2 \bar{A}_2 = \frac{X - iY}{\pi} \tag{3}$$

and the conditions for no dislocation are

$$p_1 A_1 + p_2 A_2 - \bar{q}_1 \bar{A}_1 - \bar{q}_2 \bar{A}_2 = 0 \tag{4}$$

$$q_1 A_1 + q_2 A_2 - \bar{p}_1 \bar{A}_1 - \bar{p}_2 \bar{A}_2 = 0\,. \tag{5}$$

7.4. The half-plane

We consider material occupying the half-plane $y > 0$.

From 7.11 (1), (2) and referring to fig. 4.21

$$\gamma_1 W_1(z_1) + \delta_1 \overline{W}_1(\bar{z}_1) + \gamma_2 W_2(z_2) + \delta_2 \overline{W}_2(\bar{z}_2) = 2\,(\widehat{yy} + i\,\widehat{xy})\,. \tag{1}$$

Here z is in L. In this put zero for $\widehat{yy} + i\,\widehat{xy}$ and write z_1 for $\bar{z}_1$, z_2 for $\bar{z}_2$. Then

$$\gamma_1 W_1(z_1) + \delta_1 \overline{W}_1(z_1) + \gamma_2 W_2(z_2) + \delta_2 \overline{W}_2(z_2) = 0 \tag{2}$$

or

$$\gamma_1 W_1(z_1) + \gamma_2 W_2(z_2) = -\overline{\delta_1 W_1(\bar{z}_1) - \delta_2 W_2(\bar{z}_2)} \quad (z \text{ in } R)\,. \tag{3}$$

Thus (3) continues the left-hand side of (3) from L into R. To see this let z be in R

$$z = x + iy,\ y < 0 \tag{4}$$

$$\bar{z}_1 = x + \bar{\lambda}_1 y$$

$$= (x + \alpha_1 y) - i\,\beta_1 y\,.$$

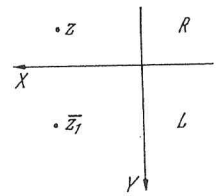

Fig. 7.4

But $\beta_1 > 0$ and $y < 0$. Therefore $-\beta_1 y$ is positive and $\bar{z}_1$ is in L where $W_1(\bar{z}_1)$ is defined.

In (2) write $\bar{z}_1$ for z_1, $\bar{z}_2$ for z_2. Then

$$\gamma_1 W_1(\bar{z}_1) + \delta_1 \overline{W}_1(\bar{z}_1) + \gamma_2 W_2(\bar{z}_2) + \delta_2 \overline{W}_2(\bar{z}_2) = 0 \tag{5}$$

so that if z is in L, $\bar{z}_1$ is in R. Subtract from (1). Then

$$\gamma_1\{W_1(z) - W_1(\bar{z}_1)\} + \gamma_2\{W_2(z_2) - W_2(\bar{z}_2)\} = 2\,(\widehat{yy} + i\,\widehat{xy}) \tag{6}$$

where z is in L and the combination

$$\gamma_1 W_1(\bar{z}_1) + \gamma_2 W_2(\bar{z}_2)$$

is defined by (3). In (6) let z_1 remaining in L tend to t, a point of the boundary. Then $\bar{z}_1 \to t$ from R and therefore

$$\gamma_1\{W_1^L(t) - W_1^R(t)\} + \gamma_2\{W_2^L(t) - W_2^R(t)\} = 2\,[-p(t) + is(t)] \tag{7}$$

where $p(t)$ and $s(t)$ are the pressure and shear on the boundary.

If we start again from the conjugate of (1) namely

$$\delta_1 W_1(z_1) + \bar{\gamma}_1 \overline{W}_1(\bar{z}_1) + \delta_2 W_2(z_2) + \bar{\gamma}_2 \overline{W}_2(\bar{z}_2) = 2\,(\widehat{yy} - i\,\widehat{xy})\,, \tag{8}$$

we get the continuation (analogue of (3))

$$\delta_1 W_1(z_1) + \delta_2 W_2(z_2) = -\overline{\gamma_1 W_1(\bar{z}_1) - \gamma_2 W_2(\bar{z}_2)}\,, \tag{9}$$

and therefore the analogue of (7) namely

$$\delta_1\{W_1^L(t) - W_1^R(t)\} + \delta_2\{W_2^L(t) - W_2^R(t)\} = 2\,[-p(t) - is(t)]\,. \tag{10}$$

Solving (7) and (10) we get

$$(\gamma_1\delta_2 - \gamma_2\delta_1)\,\{W_1^L(t) - W_1^R(t)\}$$
$$= 2\delta_2[-p(t) + is(t)] + 2\gamma_2[p(t) + is(t)]\,, \tag{11}$$

$$(\gamma_1\delta_2 - \gamma_2\delta_1)\,\{W_2^L(t) - W_2^R(t)\}$$
$$= 2\gamma_1[-p(t) - is(t)] - 2\delta_1[-p(t) + is(t)]\,. \tag{12}$$

Write

$$B_1(t) = 2\delta_2[-p(t) + is(t)] + 2\gamma_2[p(t) + is(t)] \tag{13}$$

$$B_2(t) = 2\gamma_1[-p(t) - is(t)] - 2\delta_1[-p(t) + is(t)]\,. \tag{14}$$

Using 7.1 (5) we have

$$B_1(t) = -2i\lambda_2 p(t) + 2is(t) = -2i\,[\lambda_2 p(t) - s(t)] \tag{15}$$

$$B_2(t) = 2i\lambda_1 p(t) - 2is(t) = 2i\,[\lambda_1 p(t) - s(t)] \tag{16}$$

$$\gamma_1\delta_2 - \gamma_2\delta_1 = \gamma_1(1-\gamma_2) - \gamma_2(1-\gamma_1) = -\frac{1}{2}\,i\,(\lambda_1 - \lambda_2) \tag{17}$$

Therefore

$$\frac{B_1(t)}{\gamma_1\delta_2 - \gamma_2\delta_1} = \frac{4\,\{\lambda_2 p(t) - s(t)\}}{\lambda_1 - \lambda_2} \tag{18}$$

$$\frac{B_2(t)}{\gamma_1\delta_2 - \gamma_2\delta_1} = -\frac{4\,\{\lambda_1 p(t) - s(t)\}}{\lambda_1 - \lambda_2}\,. \tag{19}$$

Therefore by the Plemelj theorem, for vanishing at infinity,

$$W_1(z_1) = \frac{1}{2\pi i}\,\frac{4}{\lambda_1 - \lambda_2}\int_A \frac{\lambda_2 p(t) - s(t)}{t - z_1}\,dt \tag{20}$$

$$W_2(z_2) = -\frac{1}{2\pi i}\,\frac{4}{\lambda_1 - \lambda_2}\int_A \frac{\lambda_1 p(t) - s(t)}{t - z_2}\,dt\,. \tag{21}$$

Observe that (21) follows from (20) by interchanging suffixes 1 and 2. Observe also that we have

$$2\,\frac{\lambda_2 p(t) - s(t)}{\lambda_1 - \lambda_2} = \frac{-p(t) + \frac{1}{\lambda_2}\,s(t)}{\frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)}$$

which amounts to replacing, in $-p(t) + is(t)$, i by $\frac{1}{\lambda_2}$ and multiplying by a constant. We are now in possession of a general method for solving boundary value problems for the half plane. This method can be applied to the problems of chapter 4, but now for anisotropic material.

7.41. Force at the edge of a half-plane

Let the force $X + iY$ be applied at the origin, on the edge of a half-plane, (fig. 4.21).

Clearly as in 7.38 the appropriate forms of the complex stresses will be

$$'W_1(z_1) = -A_1 \ln z_1\,, \quad 'W_2(z_2) = -A_2 \ln z_2\,, \tag{1}$$

whence

$$W_1(z_1) = -\frac{A_1}{z_1}\,, \quad W_2(z_2) = -\frac{A_2}{z_2}\,. \tag{2}$$

From 7.2 (1) we must have

$$X + iY = -\frac{1}{2}\,i\,[\delta_1 {'W_1}(z_1) + \delta_2 {'W_2}(z_2) + \bar\gamma_1 {'\overline{W}_1}(\bar z_1) + \bar\gamma_2 {'\overline{W}_2}(\bar z_2)]_{BDC}$$

and since $\arg z_1$ decreases by π on describing BDC, we have

$$\frac{2\,(X + iY)}{\pi} = \delta_1 A_1 + \delta_2 A_2 - \bar\gamma_1 \bar A_1 - \bar\gamma_2 \bar A_2 \tag{3}$$

and the conjugate equation

$$-\frac{2\,(X - iY)}{\pi} = \gamma_1 A_1 + \gamma_2 A_2 - \bar\delta_1 \bar A_1 - \bar\delta_2 \bar A_2\,. \tag{4}$$

Also the stress on the x-axis must be zero and therefore putting $y = 0$ in 7.4 (1) we must have

$$\gamma_1 A_1 + \gamma_2 A_2 + \bar\delta_1 \bar A_1 + \bar\delta_2 \bar A_2 = 0 \tag{5}$$

$$\delta_1 A_1 + \delta_2 A_2 + \bar\gamma_1 \bar A_1 + \bar\gamma_2 \bar A_2 = 0\,. \tag{6}$$

The solution of (3)—(6) gives

$$A_1 = \frac{2\,(X + \lambda_2 Y)}{i\pi\,(\lambda_1 - \lambda_2)}\,, \quad A_2 = \frac{2\,(X + \lambda_1 Y)}{i\pi\,(\lambda_2 - \lambda_1)}\,. \tag{7}$$

Since $\widehat{rr} + \widehat{\theta\theta} = \Theta$, $\widehat{\theta\theta} - \widehat{rr} + 2i\widehat{r\theta} = \Phi e^{2i\theta}$, we have $2\,(\widehat{rr} - i\widehat{r\theta}) = \Theta - \frac{z}{\bar z}\,\Phi$, so that

$$\begin{aligned}
2\,(\widehat{rr} - i\widehat{r\theta}) &= \frac{\gamma_1\delta_1 A_1}{z_1} + \frac{\bar\gamma_1\bar\delta_1\bar A_1}{\bar z_1} + \frac{\gamma_2\delta_2 A_2}{z_2} + \frac{\bar\gamma_2\bar\delta_2\bar A_2}{\bar z_2} \\
&\quad - \frac{z}{\bar z}\left(\frac{\gamma_1^2 A_1}{z_1} + \frac{\bar\delta_1^2\bar A_1}{\bar z_1} + \frac{\gamma_2^2 A_2}{z_2} + \frac{\bar\delta_2^2\bar A_2}{\bar z_2}\right) \\
&= \frac{\gamma_1 A_1}{z_1}\left\{\delta_1 - \frac{\gamma_1 z}{\bar z}\right\} + \frac{\bar\delta_1\bar A_1}{\bar z_1}\left\{\bar\gamma_1 - \frac{\bar\delta_1 z}{\bar z}\right\} \\
&\quad + \frac{\gamma_2 A_2}{z_2}\left\{\delta_2 - \frac{\gamma_2 z}{\bar z}\right\} + \frac{\bar\delta_2\bar A_2}{\bar z_2}\left\{\bar\gamma_2 - \frac{\bar\delta_2 z}{\bar z}\right\}.
\end{aligned}$$

Now

$$z_1 = \gamma_1 z + \delta_1 \bar z\,, \quad \bar z_1 = \bar\delta_1 z + \bar\gamma_1 \bar z\,.$$

Therefore

$$\delta_1 \bar z - \gamma_1 z = 2\delta_1 \bar z - z_1\,, \quad \bar\gamma_1 \bar z - \bar\delta_1 z = 2\bar\gamma_1 \bar z - \bar z_1\,.$$

Therefore using (5)

$$2\,(\widehat{rr} - i\widehat{r\theta}) = \frac{2\gamma_1\delta_1 A_1}{z_1} + \frac{2\gamma_2\delta_2 A_2}{z_2} + \frac{2\bar\gamma_1\bar\delta_1\bar A_1}{\bar z_1} + \frac{2\bar\gamma_2\bar\delta_2\bar A_2}{\bar z_2} = 2\Theta$$

which is real and therefore $\widehat{r\theta} = 0$. Also $2\widehat{rr} = 2\Theta = 2\widehat{rr} + 2\widehat{\theta\theta}$ and therefore $\widehat{\theta\theta} = 0$. Thus the stress is purely radial as in the isotropic case, cf. 4.21. The value of the radial stress is

$$\widehat{rr} = \frac{\gamma_1\delta_1 A_1}{z_1} + \frac{\bar\gamma_1\bar\delta_1\bar A_1}{\bar z_1} + \frac{\gamma_2\delta_2 A_2}{z_2} + \frac{\bar\gamma_2\bar\delta_2\bar A_2}{\bar z_2}\,.$$

7.5. Induced mappings

Let the boundary of the material be the closed curve C and let this curve be mapped on the unit circumference, $\sigma = e^{i\theta}$ in a ζ-plane by

$$z = m(\sigma) . \tag{1}$$

As explained in 7.31, to the circuit C in the z-plane there corresponds a circuit C_1 in the z_1-plane with the same sense of description, see fig. 7.31, and since

$$z_1 = \gamma_1 z + \delta_1 \bar{z} ,$$

it follows from (1) that C_1 is mapped on the circumference γ by the *induced mapping*

$$z_1 = \gamma_1 m(\sigma) + \delta_1 \overline{m}\left(\frac{1}{\sigma}\right) = m_1(\sigma) , \tag{2}$$

while $z_1 = m_1(\zeta)$ maps the region L_1 on the region L determined by γ.

Similarly C_2 is mapped on γ by the induced mapping

$$z_2 = \gamma_2 m(\sigma) + \delta_2 \overline{m}\left(\frac{1}{\sigma}\right) = m_2(\sigma) \tag{3}$$

while $z_2 = m_2(\zeta)$ maps the region L_2 on the region L determined by γ.

7.6. Elliptic hole in an infinite plate

We map the circumference C of the ellipse on that of the unit circle γ by

$$z = \frac{1}{2}(a+b)\,\sigma + \frac{1}{2}(a-b)\,\frac{1}{\sigma} = m(\sigma) , \quad \sigma = e^{-i\theta} \tag{1}$$

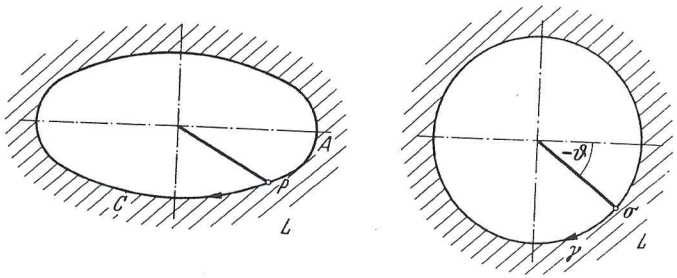

Fig. 7.6

and the outsides will correspond. Also from (1)

$$x = a\cos\theta , \quad y = -b\sin\theta \tag{2}$$

and as θ increases the point P describes C clockwise.

Since $z_1 = \gamma_1 z + \delta_1 \bar{z}$ the induced mappings are

$$z_1 = m_1(\sigma) = \gamma_1 m(\sigma) + \delta_1 \overline{m}\left(\frac{1}{\sigma}\right) = \frac{1}{2}(a - i\lambda_1 b)\,\sigma + \frac{1}{2}(a + i\lambda_1 b)\,\frac{1}{\sigma} \tag{3}$$

$$z_2 = m_2(\sigma) = \frac{1}{2}(a - i\lambda_2 b)\,\sigma + \frac{1}{2}(a + i\lambda_2 b)\,\frac{1}{\sigma} . \tag{4}$$

Now (3) gives

$$\zeta^2(a - i\lambda_1 b) - 2z_1\zeta + (a + i\lambda_1 b) = 0$$

and therefore

$$\zeta = \frac{z_1 + \sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}}{a - i\lambda_1 b} = \frac{a + i\lambda_1 b}{z_1 - \sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}} \tag{5}$$

where we have taken the positive sign for the radical since the exterior of C_1 corresponds to the exterior of γ. Similarly we find

$$\zeta = \frac{z_2 + \sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}}{a - i\lambda_2 b} = \frac{a + i\lambda_2 b}{z_2 - \sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}}$$

and therefore

$$\begin{aligned} \frac{dz_1}{d\zeta} &= m_1'(\zeta) = \frac{\sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}}{\zeta} , \quad z_1 = m_1(\zeta) \\ \frac{dz_2}{d\zeta} &= m_2'(\zeta) = \frac{\sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}}{\zeta} , \quad z_2 = m_2(\zeta) . \end{aligned} \tag{6}$$

7.61. Elliptic hole in an unloaded infinite plate

Let the stress $\widehat{nn}_B + i\,\widehat{ns}_B$ be given on the boundary which in the notation of 7.6 is mapped on the unit circumference γ by

$$z = m(\sigma) . \tag{1}$$

This induces the mappings

$$z_1 = m_1(\sigma) , \quad z_2 = m_2(\sigma) . \tag{2}$$

Therefore the stress boundary condition 7.21 (4) is

$$\begin{aligned} &\delta_1 W_1(z_1)\, m_1'(\sigma) + \delta_2 W_2(z_2)\, m_2'(\sigma) - \bar{\gamma}_1 \overline{W}_1(\bar{z}_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &- \bar{\gamma}_2 \overline{W}_2(\bar{z}_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = 2\,(\widehat{nn}_B + i\,\widehat{ns}_B)\, m'(\sigma). \end{aligned} \tag{3}$$

Similarly starting from the conjugate we get on the boundary

$$\begin{aligned} &\gamma_1 W_1(z_1)\, m_1'(\sigma) + \gamma_2 W_2(z_2)\, m_2'(\sigma) - \delta_1 \overline{W}_1(\bar{z}_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &- \delta_2 \overline{W}_2(\bar{z}_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -2\,(\widehat{nn}_B - i\,\widehat{ns}_B)\, \overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} . \end{aligned} \tag{4}$$

We now consider the case in which the resultant force on the boundary of the hole is zero, and the plate is otherwise unloaded.

Then $W_1(z_1)$, $W_2(z_2)$, $m_1(\zeta)$, $m_2(\zeta)$, $m(\zeta)$ are holomorphic functions of ζ in the region L, and as the plate is unloaded the stresses at infinity are zero, i. e. $W_1(z_1)$, $W_2(z_2)$ vanish at infinity. Therefore multiplying (3) and (4) by $d\sigma/[2\pi i\,(\sigma-\zeta)]$ and integrating round γ we get

$$\delta_1 W_1(z_1)\, m_1'(\zeta) + \delta_2 W_2(z_2)\, m_2'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{2\,(\widehat{nn}_B + i\,\widehat{ns}_B)\, m'(\sigma)\, d\sigma}{\sigma-\zeta}, \quad (5)$$

$$\gamma_1 W_1(z_1)\, m_1'(\zeta) + \gamma_2 W_2(z_2)\, m_2'(\zeta) = -\frac{1}{2\pi i}\oint_\gamma \frac{2\,(\widehat{nn}_B - i\,\widehat{ns}_B)\, \overline{m}'(1/\sigma)\, d\sigma}{\sigma^2(\sigma-\zeta)}, \quad (6)$$

for

$$\overline{W}_1(\bar{z}_1)\,\overline{m}_1'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^2}, \quad \overline{W}_2(\bar{z}_2)\,\overline{m}_2'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^2}$$

are holomorphic in R when ζ is in L and so the corresponding integrals vanish by Cauchy's theorem.

Solving (5) and (6) we get

$$(\gamma_2\delta_1-\gamma_1\delta_2)\, W_1(z_1)\, m_1'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{2F_1(\sigma)\, d\sigma}{\sigma-\zeta}, \quad z_1 = m_1(\zeta) \quad (7)$$

$$(\gamma_2\delta_1-\gamma_1\delta_2)\, W_2(z_2)\, m_2'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{2F_2(\sigma)\, d\sigma}{\sigma-\zeta}, \quad z_2 = m_2(\zeta) \quad (8)$$

where

$$F_1(\sigma) = \gamma_2\,(\widehat{nn}_B + i\,\widehat{ns}_B)\, m'(\sigma) + \delta_2\,(\widehat{nn}_B - i\,\widehat{ns}_B)\,\overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}, \quad (9)$$

$$F_2(\sigma) = -\gamma_1\,(\widehat{nn}_B + i\,\widehat{ns}_B)\, m'(\sigma) - \delta_1\,(\widehat{nn}_B - i\,\widehat{ns}_B)\,\overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}, \quad (10)$$

and (7)—(10) solve the problem.

7.62. Uniform shear on the boundary of an elliptic hole

Applying the method of 7.61 we have $\widehat{nn}_B = 0$, $\widehat{ns}_B = s$ where s is the given shear and therefore from (9), (10)

$$F_1(\sigma) = i\gamma_2 s\, m'(\sigma) - i\delta_2 s\,\overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}$$

$$= \frac{1}{2}\, i s\left\{(a - i\lambda_2 b) - (a + i\lambda_2 b)\frac{1}{\sigma^2}\right\},$$

$$F_2(\sigma) = -i\gamma_1 s\, m'(\sigma) + i\delta_1 s\,\overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}$$

$$= -\frac{1}{2}\, i s\left\{(a_1 - i\lambda_1 b) - (a + i\lambda_1 b)\frac{1}{\sigma^2}\right\}.$$

Therefore from 7.61 (7), (8)

$$W_1(z_1) = \frac{-2s\,(a + i\lambda_2 b)}{\lambda_1 - \lambda_2}\,\frac{1}{\zeta^2 m_1'(\zeta)}, \quad z_1 = m_1(\zeta) \quad (1)$$

$$W_2(z_2) = \frac{2s\,(a + i\lambda_1 b)}{\lambda_1 - \lambda_2}\,\frac{1}{\zeta^2 m_2'(\zeta)}, \quad z_2 = m_2(\zeta). \quad (2)$$

Therefore using 7.6 (5), (6) we have finally

$$W_1(z_1) = \frac{-2s\,(a - i\lambda_1 b)(a + i\lambda_2 b)}{\lambda_1 - \lambda_2}\,\frac{1}{[z_1 + \sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}]\,\sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b)\}}} \quad (3)$$

$$W_2(z_2) = \frac{2s\,(a - i\lambda_2 b)\,(a + i\lambda_1 b)}{\lambda_1 - \lambda_2}\,\frac{1}{[z_2 + \sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}]\,\sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}}. \quad (4)$$

Observe that although (3) and (4) are explicit solutions, (1) and (2) in terms of the parameter ζ are often to be preferred.

7.63. Hair crack opened by pressure

For uniform pressure p on the boundary of an elliptic hole we have $\widehat{nn}_B = -p$, $\widehat{ns}_B = 0$ and therefore by the method of 7.62 we get

$$W_1(z_1) = \frac{2ip\,(b + ia\lambda_2)\,(a - i\lambda_1 b)}{\lambda_1 - \lambda_2}\,\frac{1}{[z_1 + \sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}]\,\sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}}$$

and $W_2(z_2)$ is derived from this by interchanging λ_1 and λ_2. For a hair crack of length $2a$ we put $b = 0$, getting

$$W_1(z_1) = \frac{-2a^2\lambda_2 p}{\lambda_1 - \lambda_2}\,\frac{1}{[z_1 + \sqrt{(z_1^2 - a^2)}]\,\sqrt{(z_1^2 - a^2)}}$$

$$W_2(z_2) = \frac{2a^2\lambda_1 p}{\lambda_1 - \lambda_2}\,\frac{1}{[z_2 + \sqrt{(z_2^2 - a^2)}]\,\sqrt{(z_2^2 - a^2)}}$$

Thus the complex stresses become infinite at the ends of the crack $z = \bar{z} = \pm a$, and elastic failure must therefore start at these points.

7.64. Elliptic hole in a loaded infinite plate

By "loaded" we shall understand the case in which the resultant force on the boundary of the hole is not necessarily zero and the stress at infinity does not necessarily vanish.

In this case we have, in the neighbourhood of infinity,

$$\left.\begin{aligned} {}'W_1(z_1) &= A_1 \ln z_1 + B_1 z_1 + {}'W_1^*(z_1), \\ {}'W_2(z_2) &= A_2 \ln z_2 + B_2 z_2 + {}'W_2^*(z_2), \end{aligned}\right\} \quad (1)$$

where $'W_1^*(z_1)$, $'W_2^*(z_2)$ are holomorphic in the whole region occupied by the material

A_1, A_2 determine the resultant force, B_1, B_2 the state of stress at infinity as explained in 7.36.

In addition we can write the condition for no dislocation given in 7.34.

From (1) we have

$$W_1(z_1) = \frac{A_1}{z_1} + B_1 + W_1^*(z_1),\ W_2(z_2) = \frac{A_2}{z_2} + B_2 + W_2^*(z_2) \tag{2}$$

where $W_1^*(z_1)$, $W_2^*(z_2)$ vanish at infinity, and if we substitute these expressions for $W_1(z_1)$, $W_2(z_2)$ in the boundary conditions 7.61 (3), (4), the results can be written

$$\begin{aligned}&\delta_1 W_1^*(z_1)\, m_1'(\sigma) + \delta_2 W_2^*(z_2)\, m_2'(\sigma) - \bar\gamma_1 \overline{W}_1^*(\bar z_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &\quad - \bar\gamma_2 \overline{W}_2^*(\bar z_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = 2\,(\widehat{nn}_B^* + i\,\widehat{ns}_B^*)\, \overline{m}'(\sigma)\,,\end{aligned} \tag{3}$$

$$\begin{aligned}&\gamma_1 W_1^*(z_1)\, m_1'(\sigma) + \gamma_2 W_2^*(z_2)\, m_2'(\sigma) - \delta_1 \overline{W}_1^*(\bar z_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &\quad - \delta_2 \overline{W}_2^*(\bar z_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -2\,(\widehat{nn}_B^* - i\,\widehat{ns}_B^*)\, \overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,,\end{aligned} \tag{4}$$

where

$$\begin{aligned}&2\,(\widehat{nn}_B^* + i\,\widehat{ns}_B^*)\, m'(\sigma) = 2\,(\widehat{nn}_B + i\,\widehat{ns}_B)\, m'(\sigma) - \frac{\delta_1 A_1}{z_1}\, m_1'(\sigma)\\ &- \frac{\delta_2 A_2}{z_1}\, m_2'(\sigma) + \frac{\bar\gamma_1 \bar A_1}{z_1}\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} + \frac{\bar\gamma_2 \bar A_2}{\bar z_2}\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &- \delta_1 B_1 m_1'(\sigma) - \delta_2 B_2 m_2'(\sigma) + \bar\gamma_1 \overline{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} + \bar\gamma_2 \overline{B}_2 \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,,\end{aligned} \tag{5}$$

$$\begin{aligned}&-2\,(\widehat{nn}_B^* - i\,\widehat{ns}_B^*)\, \overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -(2\,\widehat{nn}_B - i\,\widehat{ns}_B)\, \overline{m}'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &- \frac{\gamma_1 A_1}{z_1}\, m_1'(\sigma) - \frac{\gamma_2 A_2}{z_2}\, m_2'(\sigma) + \frac{\bar\delta_1 \bar A_1}{\bar z_1}\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} + \frac{\bar\delta_2 \bar A_2}{\bar z_2}\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &- \gamma_1 B_1 m_1'(\sigma) - \gamma_2 B_2 m_2'(\sigma) + \delta_1 \overline{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} + \delta_2 \overline{B}_2 \overline{m}_2\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,.\end{aligned} \tag{6}$$

The problem of determining $W_1^*(z_1)$, $W_2^*(z_2)$ is now reduced to the same problem as that of 7.61 and therefore 7.61 (7)—(10) can be used with the notation as modified in this section. When $W_1^*(z_1)$, $W_2^*(z_2)$ have been found we get $W_1(z_1)$, $W_2(z_2)$ from (2).

7.68. Unloaded elliptic hole in a plate under constant tension

The stresses at infinity are defined (cf. 7.36) by the constants B_1, B_2 and therefore for an unloaded hole

$$W_1(z_1) = B_1 + W_1^*(z_1)\,,\quad W_2(z_2) = B_2 + W_2^*(z_2)\,, \tag{1}$$

where $W_1^*(z_1)$, $W_2^*(z_2)$ are holomorphic in L and vanish at infinity.

The boundary conditions 7.21 (4), (6) may then be written

$$\begin{aligned}&\delta_1 W_1^*(z_1)\, m_1'(\sigma) + \delta_2 W_2^*(z_2)\, m_2'(\sigma) - \bar\gamma_1 \overline{W}_1^*(\bar z_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &\quad - \bar\gamma_2 \overline{W}_2^*(\bar z_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -\delta_1 B_1 m_1'(\sigma) - \delta_2 B_2 m_2'(\sigma)\\ &\quad + \bar\gamma_1 \overline{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} + \bar\gamma_2 \overline{B}_2 \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,,\end{aligned} \tag{2}$$

$$\begin{aligned}&\gamma_1 W_1^*(z_1)\, m_1'(\sigma) + \gamma_2 W_2^*(z_2)\, m_2'(\sigma) - \delta_1 \overline{W}_1^*(\bar z_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &\quad - \delta_2 \overline{W}_2^*(\bar z_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -\gamma_1 B_1 m_1'(\sigma) - \gamma_2 B_2 m_2'(\sigma)\\ &\quad + \delta_1 \overline{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} + \delta_2 \overline{B}_2 \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,.\end{aligned} \tag{3}$$

Therefore

$$(\gamma_2\delta_1 - \gamma_1\delta_2)\, W_1^*(z_1)\, m_1'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{f_1(\sigma)\,d\sigma}{\sigma - \zeta}\,,\ z_1 = m_1(\zeta)\,,$$

$$(\gamma_1\delta_2 - \gamma_2\delta_1)\, W_2^*(z_2)\, m_2'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{f_2(\sigma)\,d\sigma}{\sigma - \zeta}\,,\ z_2 = m_2(\zeta)\,,$$

where

$$\begin{aligned}f_1(\sigma) = &-(\gamma_2\delta_1 - \gamma_1\delta_2)\, B_1 m_1'(\sigma) + (\bar\gamma_1\gamma_2 - \bar\delta_1\delta_2)\, \overline{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &+ (\gamma_2\bar\gamma_2 - \delta_2\bar\delta_2)\, \overline{B}_2 \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,,\end{aligned} \tag{4}$$

$$\begin{aligned}f_2(\sigma) = &-(\gamma_1\delta_2 - \gamma_2\delta_1)\, B_2 m_2'(\sigma) + (\gamma_1\bar\gamma_1 - \delta_1\bar\delta_1)\, \overline{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\\ &+ (\gamma_1\bar\gamma_2 - \delta_1\bar\delta_2)\, \overline{B}_2 m_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\,.\end{aligned} \tag{5}$$

Now $f_1(\sigma)$, $f_2(\sigma)$ are of the forms

$$f_1(\sigma) = K_1 + \frac{iM_1}{\sigma^2}\,,\quad f_2(\sigma) = K_2 + \frac{iM_2}{\sigma^2}$$

and therefore

$$(\gamma_2\delta_1 - \gamma_1\delta_2)\, W_1^*(z_1)\, m_1'(\zeta) = \frac{iM_1}{\zeta^2}\,,\quad z_1 = m_1(\zeta)\,, \tag{6}$$

$$(\gamma_1\delta_2 - \gamma_2\delta_1)\, W_2^*(z_2)\, m_2'(\zeta) = \frac{iM_2}{\zeta^2}\,,\ z_2 = m_2(\zeta)\,. \tag{7}$$

From 7.6 (3), (4)

$$m_1'(\sigma) = l_1 - \frac{n_1}{\sigma^2}\,,\quad \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -\bar n_1 + \frac{\bar l_1}{\sigma^2}\,,$$

$$m_2'(\sigma) = l_2 - \frac{n_2}{\sigma^2}\,,\quad \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -\bar n_2 + \frac{\bar l_2}{\sigma^2}\,,$$

where

$$\left.\begin{aligned} l_1 &= \frac{1}{2}(a - i\lambda_1 b), \quad n_1 = \frac{1}{2}(a + i\lambda_1 b) \\ l_2 &= \frac{1}{2}(a - i\lambda_2 b), \quad n_2 = \frac{1}{2}(a + i\lambda_2 b). \end{aligned}\right\} \tag{8}$$

Picking out the coefficients of $1/\sigma^2$ from (4) and (5) we find that

$$M_1 = \frac{1}{2}[(\lambda_1 - \lambda_2)\, n_1 B_1 + (\bar{\lambda}_1 - \lambda_2)\, \bar{l}_1 \bar{B}_1 + (\bar{\lambda}_2 - \lambda_2)\, \bar{l}_2 \bar{B}_2], \tag{9}$$

$$M_2 = \frac{1}{2}[(\lambda_2 - \lambda_1)\, n_2 B_2 + (\bar{\lambda}_2 - \lambda_1)\, \bar{l}_2 \bar{B}_2 + (\bar{\lambda}_1 - \lambda_1)\, \bar{l}_1 \bar{B}_1], \tag{10}$$

and therefore

$$W_1(z_1) = B_1 + \frac{2M_1}{(\lambda_1 - \lambda_2)\,[z_1 + \sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}]\,\sqrt{\{z_1^2 - (a^2 + \lambda_1^2 b^2)\}}} \tag{11}$$

$$W_2(z_2) = B_2 + \frac{2M_2}{(\lambda_2 - \lambda_1)\,[z_2 + \sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}]\,\sqrt{\{z_2^2 - (a^2 + \lambda_2^2 b^2)\}}}. \tag{12}$$

The above applies to any state of (constant) tension at infinity. To apply it to simple tension at angle α to the major axis we substitute the appropriate values of B_1, B_2 from 7.37.

7.69. Hair crack in a plate under tension

For a hair crack of length $2a$ we put $b = 0$ in the results of (7.68) so that $l_1 = n_1 = \frac{1}{2}a = l_2 = n_2$ and therefore

$$M_1 = {}^1/_4\, a\,[(\lambda_1 - \lambda_2)\, B_1 + (\bar{\lambda}_1 - \lambda_2)\, \bar{B}_1 + (\bar{\lambda}_2 - \lambda_2)\, \bar{B}_2]$$

so that

$$W_1(z_1) = B_1 + \frac{\frac{1}{2}a\,[(\lambda_1 - \lambda_2)\, B_1 + (\bar{\lambda}_1 - \lambda_2)\, \bar{B}_1 + (\bar{\lambda}_2 - \lambda_2)\, \bar{B}_2]}{(\lambda_1 - \lambda_2)\,[z_1 + \sqrt{(z_1^2 - a^2)}]\,\sqrt{(z_1^2 - a^2)}}$$

$$W_2(z_2) = B_2 + \frac{\frac{1}{2}a\,[(\lambda_2 - \lambda_1)\, B_2 + (\bar{\lambda}_2 - \lambda_1)\, \bar{B}_2 + (\bar{\lambda}_1 - \lambda_1)\, \bar{B}_1]}{(\lambda_2 - \lambda_1)\,[z_2 + \sqrt{(z_2^2 - a^2)}]\,\sqrt{(z_2^2 - a^2)}}.$$

Owing to the factors $\sqrt{(z_1^2 - a^2)}$, $\sqrt{(z_2^2 - a^2)}$ in the denominators it follows that, in general, the stress tends to infinity at the ends $z = \bar{z} = \pm a$ of the crack whatever the constant state of tension at infinity may be.

7.7. Problem II for an elliptic hole in an infinite plate

The displacement boundary conditions are given by 7.22 (1) and (2), where $u_B + iv_B$ is the given boundary displacement. When we map on the unit circle this displacement is a function of σ

$$u_B + iv_B = \upsilon_B(\sigma) \tag{1}$$

say. This together with the direct and induced mappings give us the boundary conditions of 7.22 (3) namely

$$\begin{aligned} &p_1 W_1(z_1)\, m_1'(\sigma) + p_2 W_2(z_2)\, m_2'(\sigma) - \bar{q}_1 \overline{W}_1(\bar{z}_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &\quad - \bar{q}_2 \overline{W}_2(\bar{z}_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = 4\upsilon_B'(\sigma) \end{aligned} \tag{2}$$

$$\begin{aligned} &q_1 W_1(z_1)\, m_1'(\sigma) + q_2 W_2(z_2)\, m_2'(\sigma) - \bar{p}_1 \overline{W}(\bar{z}_1)\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &\quad - \bar{p}_2 \overline{W}_2(\bar{z}_2)\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} = -4\bar{\upsilon}_B'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}. \end{aligned} \tag{3}$$

Assuming that stresses vanish at infinity and that the resultant applied force is zero we get

$$p_1 W_1(z_1)\, m_1'(\zeta) + p_2 W_2(z_2)\, m_2'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{4\upsilon_B'(\sigma)\, d\sigma}{\sigma - \zeta}, \tag{4}$$

$$q_1 W_1(z_1)\, m_1'(\zeta) + q_2 W_2(z_2)\, m_2'(\zeta) = -\frac{1}{2\pi i}\oint_\gamma \frac{4\bar{\upsilon}_B'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2}\, d\sigma}{\sigma - \zeta}, \tag{5}$$

which we can solve for $W_1(z_1)$, $W_2(z_2)$ as was done in 7.6.

When the resultant applied force is not zero but $X + iY$, and there is a finite constant stress at infinity we write

$$W_1(z_1) = \frac{A_1}{z_1} + B_1 + W_1^*(z_1), \tag{6}$$

$$W_2(z_2) = \frac{A_2}{z_2} + B_2 + W_2^*(z_2), \tag{7}$$

where $W_1^*(z_1)$, $W_2^*(z_2)$ are holomorphic in L and vanish at infinity.

In the left-hand side of (2) and (3) we now can write W^* for W, and modify the right-hand side of (2) to become

$$\begin{aligned} [4\upsilon_B'(\sigma)]^* &= 4\upsilon_B'(\sigma) - \frac{p_1 A_1}{z_1}\, m_1'(\sigma) - \frac{p_2 A_2}{z_2}\, m_2'(\sigma) + \frac{\bar{q}_1 \bar{A}_1}{\bar{z}_1}\, \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &+ \frac{\bar{q}_2 \bar{A}_2}{\bar{z}_2}\, \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} - p_1 B_1 m_1'(\sigma) - p_2 B_2 m_2'(\sigma) + \bar{q}_1 \bar{B}_1 \overline{m}_1'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \\ &+ \bar{q}_2 \bar{B}_2 \overline{m}_2'\left(\frac{1}{\sigma}\right)\frac{1}{\sigma^2} \end{aligned} \tag{8}$$

and a similar modification for the right-hand side of (3). The problem thus reduced to that just solved. The A's and B's are determined as in 7.36.

7.72. Rigid elliptic core welded in an unloaded infinite plate

We envisage an elliptic (or circular hole) in an infinite plate. The hole is filled by a rigid core welded to the plate all round the rim.

This situation arises in the case of a natural knot in a wooden plank, wide compared with the dimensions of the knot, or in the case of a rivet forced into a hole.

We first consider the case of a force $X + iY$ applied to the core at its centre, the plate being otherwise unloaded, and the deformation being non-dislocational.

We can write

$$`W_1(z_1) = A_1 \ln z_1 + `W_1^*(z_1)\,, \quad `W_2(z_2) = A_2 \ln z_2 + `W_2^*(z_2)\,, \tag{1}$$

where $`W_1^*(z_1)$, $`W_2^*(z_2)$ are holomorphic in L and can be taken to vanish at infinity without loss of generality.

The conditions to be satisfied by A_1 and A_2 are therefore given by 7.36 (3) and (4) and their conjugates.

The effect of the force will be to give a translation to the core, and therefore the same translation to the boundary in contact with it. This is a rigid body movement and can therefore be ignored, for it can be nullified by applying the opposite translation to the whole system. Therefore the displacement boundary condition is that of zero displacement which from 7.22 is

$$p_1 `W_1(z_1) + p_2 `W_2(z_2) + \bar{q}_1 `\overline{W}_1(\bar{z}_1) + \bar{q}_2 `\overline{W}_2(\bar{z}_2) = 0\,, \tag{2}$$

$$q_1 `W_1(z_1) + q_2 `W_2(z_2) + \bar{p}_1 `\overline{W}_1(\bar{z}_1) + \bar{p}_2 `\overline{W}_2(\bar{z}_2) = 0\,, \tag{3}$$

at the boundary. On using (1), (2) becomes

$$\begin{aligned} &p_1 `W_1^*(z_1) + p_2 `W_2^*(z_2) + \bar{q}_1 `\overline{W}_1^*(\bar{z}_1) + \bar{q}_2 `\overline{W}_2^*(\bar{z}_2) \\ &= -p_1 A_1 \ln z_1 - p_2 A_2 \ln z_2 - \bar{q}_1 \bar{A}_1 \ln \bar{z}_1 - \bar{q}_2 \bar{A}_2 \ln \bar{z}_2\,. \end{aligned} \tag{4}$$

Now from 7.6 (3) on the boundary

$$\begin{aligned} z_1 &= \frac{1}{2}(a - i\lambda_1 b)\,\sigma + \frac{1}{2}(a + i\lambda_1 b)\,\frac{1}{\sigma} = l_1\sigma + \frac{n_1}{\sigma}\,, \\ z_2 &= \frac{1}{2}(a - i\lambda_2 b)\,\sigma + \frac{1}{2}(a + i\lambda_2 b)\,\frac{1}{\sigma} = l_2\sigma + \frac{n_2}{\sigma}\,, \end{aligned} \tag{5}$$

and so

$$A_1 \ln z_1 = A_1 \ln \sigma + A_1 \ln\left(l_1 + \frac{n_1}{\sigma^2}\right). \tag{6}$$

Now the zeros of $l_1 + n_1/\zeta^2$ are in R, for at a zero

$$-\zeta^2 = \frac{n_1}{l_1} = \frac{(a - b\beta_1) + i b\alpha_1}{(a + b\beta_1) - i b\alpha_1}$$

and therefore

$$|\zeta^4| = \frac{a^2 + b^2(\alpha_1^2 + \beta_1^2) - 2ab\beta_1}{a^2 + b^2(\alpha_1^2 + \beta_1^2) + 2ab\beta_1} < 1\,,$$

since by hypothesis $\beta_1 > 0$.

Therefore $\ln(l_1 + n_1/\zeta^2)$ is holomorphic in L. Therefore if ζ is in L,

$$\frac{1}{2\pi i}\oint_\gamma \frac{\ln(l_1 + n_1/\sigma^2)\,d\sigma}{\sigma - \zeta} = \ln\left(l_1 + \frac{n_1}{\zeta^2}\right) - \ln l_1\,. \tag{7}$$

from 3.13 (6).

Now from (5)

$$\bar{z}_1 = \frac{\bar{l}_1}{\sigma} + \bar{n}_1\sigma = \frac{1}{\sigma}(\bar{l}_1 + \bar{n}_1\sigma^2)\,,$$

and the zeros of $\bar{l}_1 + \bar{n}_1\zeta^2$ are given by

$$-\zeta^2 = \frac{\bar{l}_1}{\bar{n}_1}\,,$$

so that now at a zero $|\zeta| > 1$. Thus the zeros of $\bar{l}_1 + \bar{n}_1\zeta^2$ are in L, and therefore $\ln(\bar{l}_1 + \bar{n}_1\zeta^2)$ is holomorphic in R. Therefore if ζ is in L,

$$\oint_\gamma \frac{\ln(\bar{l}_1 + \bar{n}_1\sigma^2)\,d\sigma}{\sigma - \zeta} = 0\,. \tag{8}$$

We can write (4) in the form

$$\begin{aligned} &p_1 `W_1^*(z_1) + p_2 `W_2^*(z_2) + \bar{q}_1 `\overline{W}_1^*(\bar{z}_1) + \bar{q}_2 `\overline{W}_2^*(\bar{z}_2) \\ &= (-p_1 A_1 - p_2 A_2 + \bar{q}_1\bar{A}_1 + \bar{q}_2\bar{A}_2)\ln\sigma \\ &\quad - p_1 A_1 \ln\left(l_1 + \frac{n_1}{\sigma^2}\right) - p_2 A_2 \ln\left(l_2 + \frac{n_2}{\sigma^2}\right) \\ &\quad - \bar{q}_1\bar{A}_1 \ln\left(\bar{l}_1 + \bar{n}_1\sigma^2\right) - \bar{q}_2\bar{A}_2 \ln\left(\bar{l}_2 + \bar{n}_2\sigma^2\right) \end{aligned}$$

and the coefficient of $\ln\sigma$ is zero from 7.36 (3). Multiply by $d\sigma/[2\pi i(\sigma - \zeta)]$ and integrate round γ. Using (7) and (8), we get

$$p_1 `W_1^*(z_1) + p_2 `W_2^*(z_2) = -p_1 A_1 \ln\left(1 + \frac{n_1}{l_1\zeta^2}\right) - p_2 A_2 \ln\left(1 + \frac{n_2}{l_2\zeta^2}\right)$$

Similarly from (3) we get

$$q_1 `W_1^*(z_1) + q_2 `W_2^*(z_2) = -q_1 A_1 \ln\left(1 + \frac{n_1}{l_1\zeta^2}\right) - q_2 A_2 \ln\left(1 + \frac{n_2}{l_2\zeta^2}\right)$$

and therefore

$$`W_1^*(z_1) = -A_1 \ln\left(1 + \frac{n_1}{l_1\zeta^2}\right), \quad `W_2^*(z_2) = -A_2 \ln\left(1 + \frac{n_2}{l_2\zeta^2}\right).$$

Combining this with (1) and 7.6 (3), (5) we get

$$`W(z_1) = A_1 \ln(l_1\zeta) = A_1 \ln\left[\frac{1}{2}\left(z_1 + \sqrt{z_1^2 - a^2 - \lambda_1^2 b^2}\right)\right].$$

Differentiating, we get finally

$$W_1(z_1) = \frac{A_1}{\sqrt{z_1^2 - a^2 - \lambda_1^2 b^2}}\,, \qquad W_2(z_2) = \frac{A_2}{\sqrt{z_2^2 - a^2 - \lambda_2^2 b^2}}\,, \tag{9}$$

where A_1 and A_2 are determined by 7.36 (3), (4). If we now let a, b tend to zero, we arrive at the standard concentrated force of 7.38, which may therefore be regarded as the limit of a force applied to the centre of an imbedded rigid core whose diameter tends to zero.

Consider now the case where the core is acted upon by a couple of moment M.

Such a couple will twist the core through a (small) angle ε so that the displacement at the boundary will be

$$\upsilon_B = u_B + i v_B = i\varepsilon z = i\varepsilon m(\sigma) = \frac{1}{2}\varepsilon i\left[(a+b)\,\sigma + \frac{a-b}{\sigma}\right]$$

and the boundary condition (2) is replaced by

$$p_1 \text{`}W_1(z_1) + p_2 \text{`}W_2(z_2) + \bar{q}_1 \text{`}\overline{W}_1(\bar{z}_1) + \bar{q}_2 \text{`}\overline{W}_2(\bar{z}_2) = 4 i\varepsilon m(\sigma)\,. \tag{10}$$

Here as no force is operative and the plate is unloaded at infinity $\text{`}W_1(z_1)$, $\text{`}W_2(z_2)$ are holomorphic in L and may be supposed to vanish at infinity.

Multiply (10) by $d\sigma/[2\pi i\,(\sigma-\zeta)]$ and integrate round γ. Then

$$p_1 \text{`}W_1(z_1) + p_2 \text{`}W_2(z_2) = 2\,i\,\varepsilon\,\frac{a-b}{\zeta}\,.$$

Similarly from the conjugate of (10)

$$q_1 \text{`}W_1(z_1) + q_2 \text{`}W_2(z_2) = -2\,i\,\varepsilon\,\frac{a+b}{\zeta}\,.$$

Solving these we get

$$\text{`}W_1(z_1) = \frac{i\,\varepsilon\,L_1}{\zeta}\,,\quad z_1 = m_1(\zeta)\,,\quad L_1 = 2\,\frac{p_2\,(a+b) + q_2\,(a-b)}{p_1 q_2 - p_2 q_1}\,, \tag{11}$$

$$\text{`}W_2(z_2) = \frac{i\,\varepsilon\,L_2}{\zeta}\,,\quad z_2 = m_2(\zeta)\,,\quad L_2 = 2\,\frac{p_1\,(a+b) + q_1\,(a-b)}{p_2 q_1 - p_1 q_2}\,. \tag{12}$$

We have now to relate the moment M to the angle ε, to which end we use 7.2 (2). We have

$$\text{``}W_1(z_1) = \int \text{`}W_1(z_1)\,dz_1 = i\varepsilon L_1 \int \frac{1}{\zeta}\left(l_1 - \frac{n_1}{\zeta^2}\right) d\zeta = i\varepsilon L_1\left(l_1 \ln\zeta + \frac{n_1}{2\zeta^2}\right).$$

Only logarithmic terms contribute to the moment given by 7.2 (2) and therefore

$$M = \frac{1}{2}\varepsilon\pi\,\{L_1 l_1 + L_2 l_2 + \bar{L}_1 \bar{l}_1 + \bar{L}_2 \bar{l}_2\} = N\varepsilon\,, \tag{13}$$

say, and therefore we have ε when M is given or M when ε is given.

7.74. Stretched plate containing a rigid core

When the plate is stretched by constant tension at infinity, a rigid elliptic core supposed welded to the plate will undergo rotation ε and a translation, which as in 7.72 can be disregarded, for it does not affect

stress. Thus we shall have

$$\text{`}W_1(z_1) = B_1 z_1 + \text{`}W_1^*(z_1)\,,\quad \text{`}W_2(z_2) = B_2 z_2 + \text{`}W_2^*(z_2)\,, \tag{1}$$

where B_1, B_2 are determined by 7.36 (7)—(9) in terms of the given constant stress at infinity, and $\text{`}W_1^*(z_1)$, $\text{`}W_2^*(z_2)$ are holomorphic in L and without loss of generality may be taken to be zero at infinity. As in 7.72 the displacement at the boundary is $\upsilon_B(\sigma) = i\varepsilon m(\sigma)$ and therefore at the boundary

$$\begin{aligned} &p_1 \text{`}W_1^*(z_1) + p_2 \text{`}W_2^*(z_2) + \bar{q}_1 \text{`}\overline{W}_1^*(\bar{z}_1) + \bar{q}_2 \text{`}\overline{W}_2^*(\bar{z}_2) \\ &= -p_1 B_1 z_1 - p_2 B_2 z_2 - \bar{q}_1 \bar{B}_1 \bar{z}_1 - \bar{q}_2 \bar{B}_2 \bar{z}_2 + 4 i\varepsilon m(\sigma)\,. \end{aligned} \tag{2}$$

Now from 7.72 (5), on the boundary

$$z_1 = l_1\sigma + \frac{n_1}{\sigma}\,,\quad z_2 = l_2\sigma + \frac{n_2}{\sigma}\,,\quad l_1 = \frac{1}{2}(a - i\lambda_1 b)\,,\quad n_1 = \frac{1}{2}(a + i\lambda_1 b)\,.$$

Multiply (2) by $d\sigma/[2\pi i/(\sigma-\zeta)]$ and integrate round γ. Then

$$\begin{aligned} &p_1 \text{`}W_1^*(z_1) + p_2 \text{`}W_2^*(z_2) \\ &= -\{p_1 B_1 n_1 + p_2 B_2 n_2 + \bar{q}_1 \bar{B}_1 \bar{l}_1 + \bar{q}_2 \bar{B}_2 \bar{l}_2 - 4 i\varepsilon \cdot \tfrac{1}{2}(a-b)\}\,\frac{1}{\zeta} \\ &= -\frac{P_1}{\zeta} \text{ say.} \end{aligned} \tag{3}$$

Similarly from the conjugate of (2)

$$q_1 \text{`}W_1^*(z_1) + q_2 \text{`}W_2^*(z_2) = -\frac{P_2}{\zeta}\,, \tag{4}$$

where

$$P_2 = -\left\{q_1 B_1 n_1 + q_2 B_2 n_2 + \bar{p}_1 \bar{B}_1 \bar{l}_1 + \bar{p}_2 \bar{B}_2 \bar{l}_2 + 4 i\varepsilon\,\frac{1}{2}(a+b)\right\}. \tag{5}$$

Solving (3) and (4) we have

$$\left.\begin{aligned} \text{`}W_1^*(z_1) &= \frac{p_2 P_2 - q_2 P_1}{p_1 q_2 - p_2 q_1}\,\frac{1}{\zeta}\,,\quad z_1 = m_1(\zeta)\,, \\ \text{`}W_2^*(z_2) &= \frac{q_1 P_1 - p_1 P_2}{p_1 q_2 - p_2 q_1}\,\frac{1}{\zeta}\,,\quad z_2 = m_2(\zeta)\,. \end{aligned}\right\} \tag{6}$$

This determines $W_1(z_1)$, $W_2(z_2)$ in terms of known coefficients and ε. To find ε we must express the condition that the moment exerted on the core is zero.

Since

$$\text{`}W_1^*(z_1) = L_1/\zeta\,,\quad L_1 = (p_2 P_2 - q_2 P_1)/(p_1 q_2 - p_2 q_2) \tag{7}$$

$$\text{``}W_1^*(z_1) = \int \frac{L_1}{\zeta}\,dz_1 = L_1 \int \frac{1}{\zeta}\left(l_1 - \frac{n_1}{\zeta^2}\right) d\zeta = L_1 l_1 \ln\zeta + \frac{L_1 n_1}{2\zeta^2}\,.$$

Since the only contribution to the moment arises from the logarithms, for zero moment 7.2 (2) gives

$$L_1 l_1 + L_2 l_2 - \bar{L}_1 \bar{l}_1 - \bar{L}_2 \bar{l}_2 = 0\,, \tag{8}$$

which on substitution furnishes the value of ε.

Examples VII

1. A certain board of oak has the constants

$$l_{11}=\frac{1}{582}\,,\quad l_{12}=-\frac{1}{1800}\,,\quad l_{16}=0$$

$$l_{22}=\frac{1}{219}\,,\quad l_{26}=0\qquad,\quad l_{66}=\frac{1}{132}$$

all measured in mm²/kg. Prove that two roots of the characteristic equation are approximately

$$\lambda_1=1.679\,i\,,\quad \lambda_2=0.971\,i\,.$$

2. In the foregoing example find the new elastic constants when the axes of reference are rotated through 45°.

3. Find the roots of the characteristic equation when the axes of Ex. 1 are turned through 45°.

4. Evaluate the determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

5. Assuming the cyclic function of $'W_1(z_1)$ to be $2\pi i\,(\gamma k_1 z_1 + A_1)$ find the cyclic function of $''W_1(z_1)$ and hence deduce the general form of $''W_1(z_1)$.

6. Find the condition that in anisotropic material the dislocation shall be rotational only.

7. Assuming $B_1=\overline{B}_1$ find explicitly the real and imaginary parts of B_1, B_2 in the case of constant stress $\widehat{xx}_\infty$, $\widehat{xy}_\infty$, $\widehat{yy}_\infty$ at infinity.

8. Calculate explicitly the real and imaginary parts of B_1, B_2 in the case of simple tension T at angle α to the x-axis.

9. Standard concentrated forces $X+iY$, $-X-iY$ act at the points z_0, $-z_0$. Calculate the stress distribution at the origin, and at the point $2z_0$.

10. Three standard concentrated forces of equal magnitude are in equilibrium when applied at the vertices of an equilaterial triangle. Calculate the stress distribution at the mid-point of a side of the triangle.

11. A half-plane is subject to pressure $p\sin\frac{2\pi x}{a}$ over the interval between $z=-a$ and $z=a$ of the boundary. Find the stress at any point.

Forces $X+iY$ at $z=a$, $-X-iY$ at $z=-a$ are applied at the edge of a half-plane. Calculate the stress distribution on the y-axis.

12. In the case of a hair crack in a plate under simple tension in the direction of the crack, prove that the stresses at the ends of the crack remain finite.

13. Obtain the relations between complex variables, say z_1, $\overline{z}_1$, z, by eliminating x, y between

$$x+\lambda_1 y-z_1=0\,,\quad x+\bar{\lambda}_1 y-\overline{z}_1=0\,,\quad x+iy-z=0\,.$$

14. Express $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ by means of the stress combinations Θ_1, $\Phi_{1,2}$, $\overline{\Phi}_{1,2}$.

15. Show how to eliminate from the equations of motion the effect of steady rotation about a fixed point.

16. Determine in what conditions the moment M is given by

$$4M=\mathrm{Re}\,[''W_1(z_1)+''W_2(z_2)+''\overline{W}_1(\overline{z}_1)+''\overline{W}_2(\overline{z}_2)]_C$$

for a closed circuit C.

17. In the case of a plate subjected to tension at infinity defined by constants B_1, B_2, show that the rotation at infinity is zero if

$$(\gamma_1 p_1-\delta_1 q_1)\,B_1+(\gamma_2 p_2-\delta_2 q_2)\,B_2-(\overline{q}_1\delta_1-\overline{p}_1\overline{\gamma}_1)\,\overline{B}_1-(\overline{q}_2\delta_2-\overline{p}_2\overline{\gamma}_2)\,\overline{B}_2=0.$$

18. In the case of an elliptic hole in an infinite plate loaded with hydrostatic pressure p at the edge of the hole, find the hoop stress at any point of the boundary and discuss its maximum value. Find also where the hoop stress is least.

19. Discuss the hoop stress distribution round an unloaded elliptic hole in an infinite plate subject to simple tension T at angle α to the major axis of the ellipse.

Find the points of maximum hoop stress in the cases $\alpha=0$, $\alpha=\pi/2$.

20. Find explicit expressions for A_1 and A_2 in the case of non-dislocational solutions under force $X+iY$.

21. Find the stress distribution when an elliptic hole in an infinite rigid plate has an elliptic plug of anisotropic material forced in to fill the hole, but which is free to slip at the boundary.

22. An infinite plate under simple tension T is weakened by an unloaded elliptic hole. Show that when the tension is parallel to the x-axis,

$$'W_1(z_1)=B_1^{(x)}z_1-\frac{2\,ibT}{\lambda_1-\lambda_2}\,\frac{a-i\lambda_1 b}{z_1+\sqrt{\{z_1^2+(a^2+\lambda_1^2 b^2)\}}}\,.$$

Find the corresponding form of $'W_2(z_2)$ and determine the constants $B_1^{(x)}$, $B_2^{(x)}$.

23. In the preceding example if the tension is parallel to the y-axis, show that

$$'W_1(z_1)=B_1^{(y)}+\frac{2\lambda_2 aT}{\lambda_1-\lambda_2}\,\frac{a-i\lambda_1 b}{z_1+\sqrt{\{z_1^2+(a^2+\lambda_1^2 b^2)\}}}\,.$$

24. In the case of a hair crack of length $2a$ along the x-axis in a plate under simple tension T along the y-axis, show that at a point

$(x, 0)$ of the crack, the stress component $\widehat{yy}$ is independent of the elastic constants and has the value

$$\widehat{yy} = T + \frac{a^2 T}{\sqrt{(x^2 - a^2)}\,(x + \sqrt{(x^2 - a^2)})}\,.$$

25. An infinite plate otherwise unloaded has an elliptic hole, an arc of whose perimeter is subject to uniform hydrostatic pressure p. If the ends of the arc correspond to the points $\sigma_1 = e^{i\theta_1}$, $\sigma_2 = e^{i\theta_2}$ on the unit circumference on which perimeter is mapped, show that the force $X + iY$ on the arc has components

$$X = bp\,(\sin\theta_2 - \sin\theta_1)\,, \quad Y = ap\,(\cos\theta_2 - \cos\theta_1)$$

and find the complex stresses.

Show how the case of a concentrated force may be deduced by a limiting process.

26. At the extremities of the minor axis of an elliptic hole in an otherwise unloaded plate there act forces $iP, -iP$ respectively. Prove that at the extremities of the major axis

$$\widehat{yy} = \frac{P}{\pi b}\,\frac{\beta_1 - \beta_2}{\beta_1\,\beta_1}$$

and examine the isotropic case.

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