STUDY OF STRUCTURES - PIPPARD

THE EXPERIMENTAL STUDY OF STRUCTURES

A.J.S. PIPPARD

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by

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PREFACE

THE theory of structures is largely mathematical and must be taught as such, but treated solely in this way it is unlikely to appeal to the majority of engineering students as a subject of great interest. It is usual, and indeed essential, to amplify the classroom teaching by a considerable amount of drawing-office work and provided the exercises for this purpose are well chosen they are very valuable in counteracting the tendency to regard the subject as merely one more course of lectures in mathematics. The author, however, is convinced that a really live interest can only be induced in most students by complementing the lecture-room and drawing-office work by experiment. Unfortunately it is too often believed that such a treatment of the subject requires considerable laboratory space and expensive equipment, and as a result many students are deprived of the opportunity of invaluable training in one of the most important subjects of their curriculum. While it is true that large laboratories and equipment are for many purposes desirable and for some, particularly in the research field, essential, it is a mistake to suppose that without them an experimental approach to the subject is impossible.

The author and his colleagues have devoted some time to the development of a model-structures laboratory in which the essential theorems of the subject can be illustrated and verified experimentally by the use of small-scale models. Apparatus of this type is simple and any teaching institution should be able either to produce it in its own workshops or to purchase it cheaply, and while it is desirable that it should be always available for students to use at their own pleasure it can, if space be restricted, be stored in cupboards when not required.

The laboratory has proved successful and this book is an attempt to encourage the more general use of such small-scale experimental work in the teaching of theory of structures. It is based on a short course of lectures and demonstrations given by the author at the invitation of the Ministry of Education to the

Summer School for teachers in Technical Colleges, at New College, Oxford, in July 1946. The apparatus described is that used, and for the most part designed and made, in the Civil Engineering Department of the Imperial College of Science and Technology, but is, of course, by no means exhaustive. There is a wide field for ingenuity in devising similar small-scale models, and if this book serves as a stimulus to teachers to build to their own designs and so encourage an experimental development of this branch of study its purpose will be served.

The author has included a number of Experiment Sheets to illustrate the possibilities of the apparatus described.

These contain the general instructions to be given to the student, typical results and comments of interest to the teacher. It is not advisable that the actual students' reports should be presented in this form. There is scope for originality in this matter and any attempt to stereotype the records should be avoided. Many of the experiments need not be written up in detail, but a certain number should be treated fully to give experience in the preparation of an orderly and complete report. The tendency towards verbosity which is too common should be checked, and it is desirable to assign a maximum length to reports; in most cases the limit of 300 words is reasonable.

In conclusion, the author wishes to express his warmest thanks to his two colleagues to whom reference was made earlier, Dr. S. R. Sparkes and Miss L. Chitty, for the energy and enthusiasm they have shown in the development of the model-structures laboratory under conditions of considerable difficulty; to a former student, Mr. J. A. Dunster, who prepared the photographs illustrating the text, and to Mr. A. Hall who drew most of the diagrams.

A. J. SUTTON PIPPARD.

Imperial College,
S.W.7.
July 1946

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CHAPTER 1

THEOREMS AND METHODS

The Functions of Experiment. The old outlook that it mattered little what was taught as long as it was unpopular with the pupil has given place to a more enlightened desire to make study as attractive as possible, and the aim of a teacher now should be to present his subject in such a way as will create interest and arouse imagination. If he succeeds in this he will transform work from sheer drudgery both for himself and his students into a source of mutual pleasure. In technological subjects this is perhaps even more vital than in others, for unless it is achieved the primary object, which is the education of the student's mind, will be replaced by training in the use of more or less simple devices to obtain results of a purely utilitarian nature.

It is the object of this book to describe an attempt to stimulate such an attitude to the study of structural theory by an experimental approach. Such a method serves a double purpose since not only does it create interest in the subject from the start but develops the experimental outlook which is so important for future advanced study or research.

Before proceeding to the main object therefore, it is worth while to consider the function of experiment in structural research.

Some problems are amenable to mathematical treatment without other aid but a great many need experimental work, either alone or in addition to mathematical analysis, before a solution can be found.

The uses of experiment in this connexion are varied and will be briefly considered under the following heads :—

(*a*) Exploratory experiments made before mathematical analysis.

(*b*) Confirmatory experiments made after analysis.

(*c*) Experiments made in conjunction with analysis to provide essential data or to obtain empirical formulas.

(*d*) Analytical experiments to replace computation in specific problems.

(*e*) Experiments of an *ad hoc* nature.

Exploratory Experiments. These are undertaken to study the behaviour of structures or components under test conditions with a view to subsequent mathematical analysis. They are not meant to provide exact data but simply to clarify the problem, and in designing them the aim should be to simplify and idealize the conditions as much as possible. An illustration of this

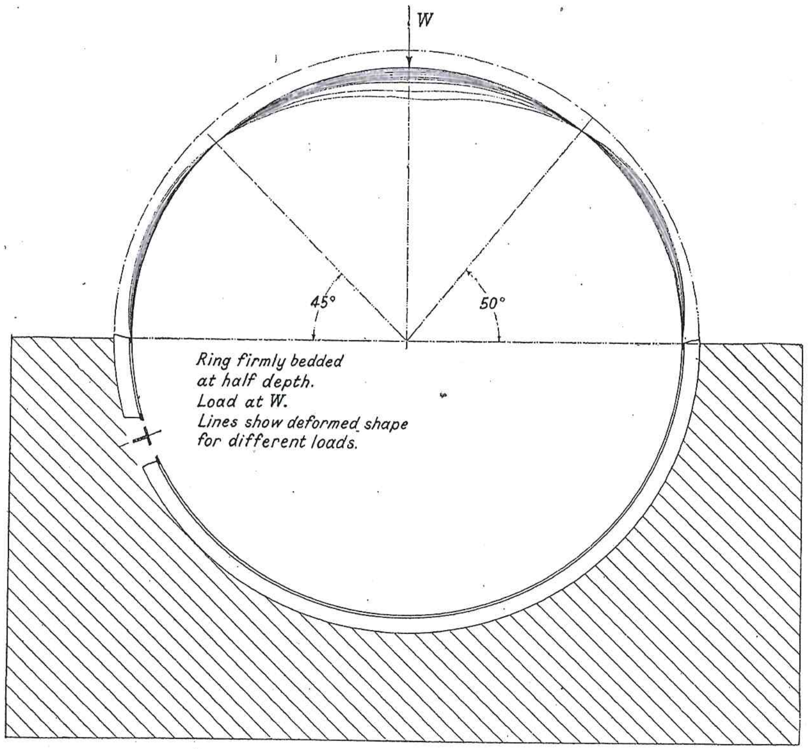


Fig. 1.1

occurred in a research upon the behaviour of the voussoir arch in which simple preliminary experiments on a small model showed clearly the mechanism of failure and directed the whole course of the analysis. Another, rather more recent, instance is provided by a series of tests on the behaviour of rings loaded beyond the elastic range, which showed that two points in the ring remained

undisplaced for all loads up to failure (Fig. 1.1). This experimental fact may help to simplify a difficult analytical investigation.

Confirmatory Experiments. In most practical problems a mathematical treatment is generally only possible when more or less drastic simplifications of the conditions are assumed, and it is generally advisable and often essential to verify the legitimacy of these assumptions. In some cases the assumptions have been so well established in the past that they may be accepted without further verification, but in others experimental tests are imperative before the results of the analysis can be used with confidence. The importance of confirmatory experiments varies considerably with the problem, but prudence would dictate experimental check in any case of doubt.

Data-Providing Experiments. These fall into two groups; in the first the data are needed for the correlation of analytical results with actual cases. An example is provided by the correlation of the Perry strut formula with test figures. The Perry formula was obtained by purely mathematical reasoning a long time before it could be used for design purposes since the term introduced to allow for the departure of the strut from perfection of shape, material and loading could not be given a quantitative value.

This term cannot be specified numerically in a particular case, but a comprehensive study by Professor Andrew Robertson[1] of experimental results from many sources enabled him to evaluate it on a statistical basis and the formula, which was until then only of academic value, became the standard of strut design in this country.

The second group of experiments to provide data include those made in conjunction with a sketchy mathematical theory, or even without any logical background, in an attempt to obtain empirical formulas. Some of the strut tests analysed by Robertson and mentioned above had just such an object and a lot of early work on concrete must be included in this category.

As a last resource it may be necessary to proceed in this way, but it is not generally satisfactory and in some cases the results achieved may be very misleading.

Analytical Experiments. A perfectly valid and well-established mathematical procedure may be available for the solution of a particular problem, but may entail long and laborious computation. In some cases simple experimental methods can be used to obtain results with considerable saving of time; such experiments may either take the place of numerical calculations completely or serve as a check and so save a second set of calculations.

Ad Hoc Experiments. These are perhaps the commonest of all. They are made to determine the actual behaviour or strength of a particular unit and the engineer will always consider such a test as the final court of appeal, whether it is made to check the design of a joint or the proof strength of a completed bridge.

The Experimental Approach in Teaching. If the importance of experimental work for the purposes outlined be accepted, it follows that students should be taught to experiment if only for this reason, but there are in fact even stronger arguments for an experimental approach to the study of structures than this. If the subject is taught simply from the mathematical viewpoint it will make little or no appeal to the imagination of most students of engineering who will fail to appreciate the significance of the results obtained, however important they may be to their future success as designers.

This fact is well recognized and a drawing office course invariably complements the work of the lecture room. This is essential, but the value of such a course depends to a great extent upon the way in which it is linked with the theoretical treatment. Too often it consists solely of problems in design at a stage in the student's development when he can have no conception of the real problems which confront the designer, and the result is disappointing both to the teacher and the student. Some attention to design is necessary, but there is a tendency not only to introduce it too early but to allow it to absorb too much of the drawing office time to the exclusion of work which, because it elucidates the theory of the classroom, is of far more importance in the early stages of study.

Even when the drawing office course is carefully designed to amplify the lectures, however, there is still a danger that the student will not acquire that outlook which is vital for the real understanding of structural behaviour.

A quotation from a book by Professor Hardy Cross [2] and a colleague is pertinent ; they say " Too many students of indeterminate structures hope to progress by acquiring an endless variety of tools and are so busy doing this that they never learn to use them.

" The ability of a designer of continuous structures is measured

chiefly by his ability to visualize the deformation of the structure under load. If he cannot form a rough picture of these deformations when he begins the analysis he will probably analyse the structure in some very awkward and difficult way ; if he cannot picture these deformations after he has made the analysis, he doesn't know what he is talking about. The more or less gentle reader may find the constant repetition of this theme monotonous, but it is the deliberate conclusion of the authors that the most important aspect of the subject is the simple picture of structural deformation."

The present writer agrees with these remarks and is of the opinion that the best, if not the only, way to develop the ability to visualize is by means of experimental work in close association with the lectures and the drawing office.

To summarize ; structural mechanics is an experimental subject as well as a mathematical one, and a real knowledge of the behaviour of members and complete structures under load can only be obtained by analysis and experiment in combination.

The Small-Scale Model-Structures Laboratory. For advanced study and research access to a large and well-equipped laboratory is almost essential, but even before this stage is reached such a laboratory serves a very useful purpose and where space and means permit, should be developed. It is, however, too often assumed that when space and equipment on such a scale cannot be provided there is no alternative but to restrict the teaching of structures to the classroom and the drawing office.

It is the object of the writer to show that this is not the case and that valuable work can be done in the experimental teaching of theory of structures by simple and cheap apparatus so small that it can, if necessary, be stored away in cupboards when not in use. Some of the experiments to be described in illustration of this contention are already fairly well known and in use in a number of laboratories, but others have been designed and made in the Civil Engineering Department of the Imperial College to equip a new model-structures laboratory. This laboratory, shown in Fig. 1.2, is intended to serve a dual purpose ; in the first place it will provide part of the experimental course for students preparing for degrees in engineering which will link up with the work of the drawing office and with the theoretical work of lectures and exercise classes. In the second place it is intended that it shall always be open so that students may experiment as

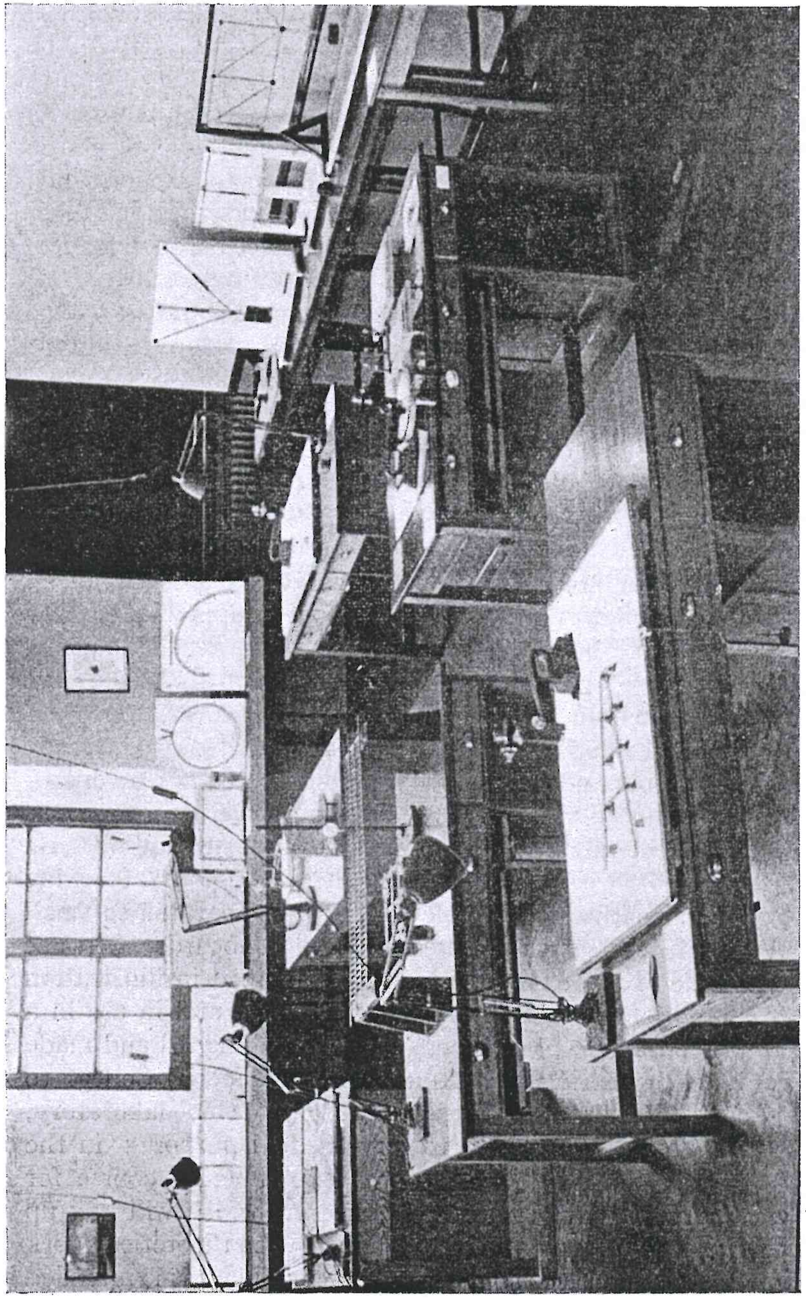


Fig. 1.2

they please at any time. It is hoped that this will assist in the development of the visual sense so strongly advocated in the passage recently quoted. The preparation of students to pass examinations is viewed as a matter of comparatively minor importance; if this subject, or indeed any subject, can be taught so as to stimulate interest and imagination the examination will present no terrors and will be treated as the obvious and normal completion of a course of study which has a clear and important function as part of a professional education.

It is, however, necessary to define the scope of the laboratory and this is conveniently done by reference to the syllabuses of certain examinations. For our purpose, therefore, it will be assumed that its main function is for the education of students working for the degree examination in engineering or for the Higher National Certificate in Civil Engineering. This is not to say that it need be entirely restricted to this purpose; some apparatus to be described is indeed used for more advanced study and for research, but the principal purpose in equipping the laboratory is to present the study of structures as an experimental subject in the undergraduate years.

The small-scale laboratory can be made sufficiently comprehensive to give adequate appreciation of the behaviour of actual structures and to illustrate the basic principles underlying their design and thus provide a sufficient background which is the object of "course-work." If a larger laboratory is available, it can, as already said, be used to considerable advantage, but even in such cases a useful purpose is served by the small-scale equipment and in the many cases where space and equipment for the larger laboratory cannot be provided apparatus of the type to be described can be made to cover most of the essential experimental work. In this connexion it must be said that no claim is made that the equipment described is either exhaustive or the best that can be devised. In the Civil Engineering Department of the Imperial College we have been trying an experiment in the teaching of structures and there are obviously alternative ways of achieving the particular end in view in any particular instance. If the description of our experiment stimulates other teachers to produce better designed apparatus or to extend the scope, the purpose of this book will be amply fulfilled.

In the equipment of the model-structures laboratory, attention has been directed to three main points. In the first place, elementary models are provided to illustrate the behaviour of structural elements and complete structures by a gross exaggeration of the deflexions. A flexible model is impressive in indicating the general shape which the actual structure assumes when loaded and although it is completely out of proportion it shows, in a manner that no other method can do, the essential features of the deformation and the effect of continuity of members or rigidity of joints. The second type of model is used to illustrate a general principle or method of analysis as simply as possible and so impress the fundamentals clearly on the student's mind: such models may be used in lecture demonstrations and will be examined at leisure by students if the laboratory is accessible to them at all times. The third type of model is designed to give quantitative results and to act as a direct link between the classroom, the drawing office and the real structure. Such models enable the student to verify by experiment the truth of the theorems which he has proved mathematically, to check calculations based on those theorems which he has done in exercise classes and graphical constructions which he has made in the drawing office. It may be, indeed has been, argued that when a theorem has been demonstrated mathematically there is no need for experimental evidence, but while this may be true for the mathematician, it must always be borne in mind that the engineer is never really satisfied until he has seen things "work"; for him the experiment is "intended to give artistic verisimilitude to an otherwise bald and unconvincing narrative."[3]

The different types of models will be better appreciated after detailed descriptions of some of them have been given, but before doing this it is necessary, for completeness and clearness, to review briefly certain theorems and analytical methods which are the foundations of the subject. A knowledge of these fundamentals is essential for a proper understanding of the behaviour of structures and the more clearly they can be presented to the student in the early stages the better able will he be to judge the significance and merits of the many special methods he will discover later. These methods used with discrimination are often of considerable value, but they are mainly variations in one form or another of the fundamental theorems.

Most of the work at this stage will be concerned with the behaviour of structures which are not strained beyond the elastic

range and all the theorems are subject to this restriction. The elasto-plastic behaviour of structures has received considerable study during the last few years and while this is a matter of the highest importance it is not to be expected that it will for some time yet be developed sufficiently to enable it to be included to an appreciable extent in the syllabus for initial degrees or National Certificates.

The Laws of Static Equilibrium. The forces acting upon any structure or part of a structure which is at rest or moving with uniform velocity are in equilibrium. In the general case of a solid body or space-frame this condition is satisfied if the sum of the component forces parallel to three axes—usually taken as mutually perpendicular—are zero and the sums of the moments about these three axes are also zero. In the particular case of the plane frame in which the structure and the forces acting upon it lie entirely in one plane the conditions are satisfied if the sums of the component forces along two axes are zero and the moment-sum about any point in the plane is also zero.

If the structure is being accelerated as may occur for example in an aircraft, every part of it is subjected to inertia forces which must be included in the account to balance the applied forces causing the acceleration; when this is done it may be treated as a body at rest and the usual methods of analysis are applicable.

The conditions of equilibrium are applied in different ways in the calculation of reactive forces, in the drawing of stress diagrams, in the determination of stresses by inspection, in Ritter's method of sections and in analysis by tension coefficients.

Hooke's Law. In all stress analysis, other than for simply-stiff frames, and in all calculations of displacements whether for simply-stiff or for redundant structures it is essential to know how the material behaves under load. The fundamental assumption of elastic analysis is conformity with Hooke's Law which states that strains are proportional to the stresses producing them or, in another form, that displacements of any point are proportional to the loads which cause them. This law is very nearly true for a variety of materials, including the important structural ones, and the linear relationship makes it possible to obtain solutions for stress equations which would otherwise be intractable. Most of the theorems that follow are based upon this law and must not be applied where it is untrue. It is important to note that cases arise in which a material may satisfy this law while structures made of it may not exhibit a linear load-displacement relationship. The simplest instance of this is a flexible, eccentrically loaded strut which, although made of a material which obeys Hooke's Law, does not give a straight line for the central deflexion plotted against the load producing it.

Principle of Superposition. This principle is, strictly speaking, a direct corollary from Hooke's law, but it is nevertheless of such importance that it deserves to be treated as a separate theorem. It states that if a body which obeys Hooke's Law be subjected to a number of separate loading systems, the sum of the stresses and displacements at any point due to these are the same as would occur if all the loading systems were applied simultaneously. This principle is of great value in the treatment of certain problems in redundant structures where its judicious use will reduce the number of simultaneous equations to be solved. The general solutions of the encastré arch rib and bow girder are, for example, much more easily obtained by dividing the load system into symmetrical and skew-symmetrical systems and superposing the results obtained for these separately, than is possible by a direct approach.

Principle of Saint Venant. This important principle was stated axiomatically by Saint Venant, but it was not until 1923 that R. V. Southwell [4] gave a formal proof of its validity based upon strain-energy considerations. In the course of that proof he re-stated the principle in a form which enables the engineer to appreciate its importance more readily than in the original statement. Southwell's enunciation was " Forces applied at one part of an elastic structure will induce stresses which, except in a region close to that part, will depend almost entirely upon their resultant action and very little upon their distribution."

The significance of this is not usually immediately apparent to the student but its importance is readily appreciated by reference to such a structural member as an encastré beam. The slopes at the ends of such a member are fixed by couples exerted by the anchorages. The resultant values of these couples are readily calculated but their actual distribution is in most cases indeterminate ; there are an infinite number of combinations of

forces which, applied to the ends of the beams, would give the required values. If the stresses throughout the beam were seriously affected by the distribution of these forces design would be almost impossible, but appeal to the principle stated above allows the assumption that the stresses calculated upon the basis of resultant actions will be very close to the true values, whatever the actual distribution, except in the immediate neighbourhood of the ends. This principle is applicable not only to an elastic solid but also to a framed structure provided that it is sufficiently redundant to allow adequate redistribution of stresses.

Clerk Maxwell's Reciprocal Theorem. This theorem is of outstanding importance, for it is the basis of much of the experimental examination of structures which has a direct value in design and also of many methods of analysis. It has been re-stated in different forms which are useful in particular applications, but the original statement is simple and adequate.

Suppose any body obeying Hooke's Law, whether it be a solid or a framed structure, to be in equilibrium under any number of forces or couples represented by $P_1, P_2 \ldots P_n, M_1, M_2 \ldots M_n$, and let the displacements of these forces and couples in their directions of application be $\delta_1, \delta_2 \ldots \delta_n, \theta_1, \theta_2 \ldots \theta_n$. If these forces and couples are replaced by a second system in equilibrium represented by $P'_1, P'_2 \ldots P'_n, M'_1, M'_2 \ldots M'_n$, acting in each case in the same direction as the corresponding values of the first system, and if these produce corresponding displacements $\delta'_1, \delta'_2, \ldots \delta'_n, \theta'_1, \theta'_2, \ldots \theta'_n$, then Clerk Maxwell's theorem [5] states that

$$P_1\delta'_1+P_2\delta'_2+ \ldots +P_n\delta'_n+M_1\theta'_1+M_2\theta'_2+ \ldots +M_n\theta'_n=$$
$$P'_1\delta_1+P'_2\delta_2+ \ldots +P'_n\delta_n+M'_1\theta_1+M'_2\theta_2+ \ldots +M'_n\theta_n.$$

In its simplest particular form this reduces to the statement that if a unit load acts at any point A on an elastic body and produces a deflexion δ at any other point B, then a unit load applied at B in the direction in which the original deflexion was measured, will produce a deflexion δ at A in the direction of the original load. In this form the theorem shows that the influence line of deflexion for a point in a structure is the same as the deflected line of the structure when unit load is applied at the point for which the influence line was obtained.

Strain-Energy Theorems. When a body is strained, work is done upon it by the straining forces and to the extent that the strain is elastic this energy is recoverable. Castigliano, in 1879, published a treatise [6] in which he developed two theorems concerned with such elastic strain-energy which are perhaps the most important general propositions in the field of structural analysis.

The first theorem states that the first derivative of the strain-energy in an elastic body with respect to an independent external force acting on the body gives the displacement of the point of application of that force in its own line of action. In symbols, if U is the strain-energy of the body, which may be solid or braced, and P is any external load which acts upon it,

$$\frac{\partial U}{\partial P}=\Delta_P,$$

where Δ_P is the displacement of P in its own line of action.

This theorem and the next are restricted to bodies which obey Hooke's Law, are strained within the limits of proportionality and are supported in such a way that the reactive forces do no work. The last condition is satisfied by reactions at fixed points on rigid supports or by frictionless roller bearings on rigid supports. In the first case there is no movement of the points of support and in the second, the whole movement is at right-angles to the reactive force; in neither case has the resultant reaction a component movement in its line of action and so does no work.

This theorem is very useful for the calculation of displacements which may be required either as such or as a step in the analysis of stresses.

The second theorem relates to redundant or redundantly supported structures only and states that the first derivative of the strain-energy of such a structure with respect to an internal redundant force is equal to the initial lack of fit of the member under consideration.

In symbols, if any redundant bar in a structure carries a load R and if the total strain-energy of the structure be U, then

$$\frac{\partial U}{\partial R}=\lambda_R,$$

where λ_R is the amount by which the member was too long or too short when inserted into the unloaded structure. This introduces the important idea of self-straining: if any member

in a statically determinate frame is initially of incorrect length it can be connected to its appropriate joints without producing forces either in itself or in other parts of the structure since the remaining members constitute a mechanism. The only result of such lack of fit is a modification of the final geometrical configuration of the framework. If, however, a just-stiff, *i.e.* statically determinate framework, is to be made redundant by the insertion of an extra bar, such bar must be of the exact length required or it cannot be connected to the appropriate joints without straining. If it is too long by an amount λ it can only be connected by forcing the joints in the structure apart and the redundant bar will, after connexion, be compressed and stresses will be induced in the other members of the frame. If it is too short by an amount λ the effect of forcing it into place will leave it in tension and the other members of the structure will again be stressed. This is known as self-straining, and it is important that the student should appreciate this difference between a non-redundant and a redundant structure.

In the particular case when the redundant bar to be inserted is of the exact length, *i.e.* when there is no self-straining and λ is zero, the second theorem of Castigliano becomes

$$\frac{\partial U}{\partial R}=0$$

and this condition shows that in a loaded structure the force in a redundant bar which was initially of the exact length required is that which makes the strain-energy the minimum possible (consistent with the satisfaction of the conditions of static equilibrium). For this reason, it is commonly known as the principle of least work, but it is only a particular case of the more general second theorem of Castigliano.

Moment-Distribution Methods of Analysis. In 1932 Professor Hardy Cross published a method of analysing continuous girders and frames which revolutionized the treatment of such structures. This method has since been extended to the analysis of braced frames and an analogous method due to R. V. Southwell [8] known as the method of relaxation has been applied by him not only to the solution of numerous problems in structures and elasticity but also to many others in the wider realm of mathematical physics. The extended applications of this new branch of analysis are outside the scope of the present discussion, but its use in the simpler problems of continuous frames should be included in the curriculum of the undergraduate.

The line of attack differs fundamentally from all previous treatments in one important particular; it had been the universal custom prior to the publication of the paper just mentioned to simplify a structure for calculation purposes by assuming it to have pin-joints and then to correct more or less accurately for the effects of fixity. Hardy Cross adopts the opposite assumption; he first imagines all the joints to be clamped in position by extraneous means, so rendering the connecting members fully encastré; the bending moments are then easily calculated, the clamps themselves being assumed to apply the requisite fixing moments. The effect of this is to leave each joint with a resultant couple carried by the clamp which, since the equilibrium conditions for the actual frame are unsatisfied, cannot be the true solution. One clamp is then supposed to be released and the joint thus rendered incapable of carrying an unbalanced couple. This is equivalent to applying to the joint a couple equal to the unbalanced couple but opposite to it in sign, and this is distributed to the adjacent members in proportion to their relative stiffnesses; the joint is now reclamped. The moments thus distributed modify the unbalanced moments already carried by the adjacent clamps. All joints in turn are unclamped and reclamped. The structure is then a stage nearer the equilibrium state but, generally, unbalanced moments will still be carried by the clamps. The process is therefore repeated until all the unbalanced moments have been liquidated, *i.e.* have been distributed to the members of the structure which is then in equilibrium without the aid of external clamps.

It has just been said that when a joint is unclamped the moment carried by it is distributed to the adjacent members in certain proportions; these proportions are obtained as follows.

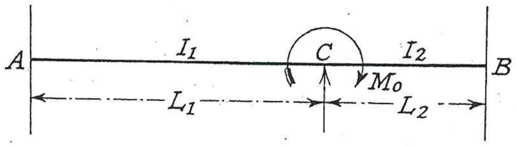


Fig. 1.3

Suppose Fig. 1.3 represents a continuous beam encastré at A and B and continuous over C, the second moments of area of AC and CB being respectively I_1 and I_2 and their spans L_1 and L_2. If a moment M_0 be applied at C it is easily shown that it will be distributed to CA and CB in the following proportions:—

$$M_{CA}=\frac{\frac{I_1}{L_1}}{\frac{I_1}{L_1}+\frac{I_2}{L_2}}\,;\quad M_{B}=\frac{\frac{I_2}{L_2}}{\frac{I_1}{L_1}+\frac{I_2}{L_2}}$$

where M_{CA} is the moment in CA at C and M_{CB} the moment in CB at C.

The terms $\frac{I_1}{L_1}$ and $\frac{I_2}{L_2}$ are known as the stiffnesses of the two sections of the beam.

It may also be shown that the moment induced at the ends A and B are respectively $\frac{1}{2}M_{CA}$ and $\frac{1}{2}M_{CB}$.

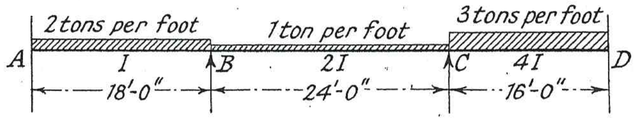


Fig. 1.4

With these results in mind the general method of moment distribution may be illustrated by reference to the continuous beam shown in Fig. 1.4.

In the first place the stiffnesses of the three sections of the beam AB, BC and CD are found to be $\frac{I}{18}$, $\frac{I}{12}$ and $\frac{I}{4}$. If A and C were encastré and a moment applied to B, the distribution would be $\frac{2}{5}$ to BA and $\frac{3}{5}$ to BC from the equations just quoted. Similarly if B and D were encastré and the moment applied to C, the distribution would be $\frac{1}{4}$ to CB and $\frac{3}{4}$ to CD.

The work is best tabulated as shown below:—

Operation	A	B		C		D
1	−54	+54	−48	+48	−64	+64
2	−1·2	−2·4	−3·6	−1·8	0	0
3	0	0	+2·25	+4·45	+13·35	+6·68
4	−0·45	−0·9	−1·35	−0·67	0	0
5	0	0	+0·08	+0·17	+0·51	+0·26
6	−0·02	−0·03	−0·048	−0·024	0	0
7	−55·67	+50·67	−50·65	+50·12	−50·14	+70·94

To begin, the beam is assumed to be encastré at A, B, C, and D and the end moments are calculated. These are $\frac{wL^2}{12}$ and a special convention of sign is convenient; all clockwise moments on the beam are taken as positive and all anti-clockwise moments as negative. Thus for the section AB the numerical value of the end fixing-moments is 54 ft.-tons; at A this acts counter-clockwise on the beam, and at B it is clockwise, so as operation (1) the appropriate value -54 is entered at A and $+54$ to the left of B. Corresponding values for BC and CD are 48 ft.-tons and 64 ft.-tons respectively and are entered appropriately.

The next step is to release the imaginary clamp at B which keeps this joint encastré. Since the net moment applied by this clamp is $+6$ ft.-tons (*i.e.* $54-48$) its release is equivalent to superposing a negative moment of 6 ft.-tons on the beam at B, the ends A and C still being encastré.

From the distribution factors already calculated, $\frac{2}{5}$ of this (*i.e.* $-2{\cdot}4$) will be taken by BA and $\frac{3}{5}$ (*i.e.* $-3{\cdot}6$) by BC; these values are entered at B as operation (2). The effect of applying these moments will be to cause additional moments at A and C of half the values, *i.e.* $-1{\cdot}2$ at A and $-1{\cdot}8$ at C; these are likewise entered. Since C is still assumed to be encastré nothing will be transmitted beyond C and the operation is completed by putting zero values as shown.

Joint C is now released; the unbalanced moment there is $-17{\cdot}8$ (*i.e.* $48-64-1{\cdot}8$) and again from the appropriate distribution factors $+4{\cdot}45$ is taken by CB and $+13{\cdot}35$ by CD. These values are entered as operation (3), together with the "carry-over" moments of half these amounts to B and D. As B is encastré, AB is unaffected by this redistribution. Joint C is now in balance, but there is an unbalanced moment of $+2{\cdot}25$ at B which is distributed as before (operation 4) and this process is repeated until the amounts transferred are negligible. In the present case the approximation has been taken closer than necessary for a practical solution in order to illustrate the process.

Finally, the various moments are summed (operation 7) and it will be seen that the out-of-balance moments are negligible. In this example, since A and D are actually encastré, they are never released.

The direct solution of this problem by the theorem of three moments gives

$$M_A=55{\cdot}66 \text{ ft.-tons}; \quad M_B=50{\cdot}68 \text{ ft.-tons}.$$
$$M_C=50{\cdot}14 \text{ ft.-tons}; \quad M_D=70{\cdot}93 \text{ ft.-tons}.$$

If the sections AB and DC of the continuous beam shown in Fig. 1.4 be assumed to be bent at right-angles to BC, and *if the conditions of support at the joints are kept the same,* the solution obtained for the beam is equally valid for the resulting portal. The significance of the supports shown in Fig. 1.4 must, however, be clearly understood. The constraints at A, B, C, and D, whether in the beam or the portal, are such that no displacement of these points is possible. At A and D, moreover, all angular rotation is prevented, but at B and C there is no such restriction. Points A and D are then said to be both position-fixed and direction-fixed, while B and C are only position-fixed.

In the case of a symmetrical and symmetrically loaded portal there is no tendency for points such as B and C to change position and the hypothetical supports there can be neglected; the solution for such a structure is then obtained exactly as for the continuous beam. If the geometry of the portal or its loading be unsymmetrical, however, it will tend to sway sideways. The forces exerted by the supports in resisting this movement produce a resultant shear across the structure which is easily calculated once the moments have been balanced as already explained. Since in the actual portal there are no supports at B and C it is necessary to neutralize this shear by superimposing an equal and opposite force. This induces additional moments in the members of the portal which are found by distribution as before and a reduced residual shear is then found to exist. This in turn is liquidated and the process is continued until sufficient accuracy has been obtained. This is the procedure which is commonly referred to as "sway correction."

Slope-deflexion Method of Analysis. Beams forming parts of a structure are, by the action of external forces on the structure, subjected to couples and to resultant changes of slopes at the ends; furthermore, one end of a member may sink relatively to the other. The bending moments at the ends of all members can be expressed in terms of these angular changes and displacements and the equations of equilibrium and of continuity provide sufficient data for a complete analysis of the structure. The simplest case is that of a continuous beam and Clapeyron's

Theorem of Three Moments is generally derived by an application of slope-deflexion analysis. The method is, however, applicable to many other types of problem such as portals, building frames, etc. The fundamental equations are easily derived. Let AB be a uniform beam having a flexural rigidity EI and length L and when it is distorted by the straining of the structure of which it is a part let the changes of slope at A and B respectively be ϕ_A and ϕ_B. Also let δ be the displacement of one end relatively to the other. If A is the area of the free bending-moment diagram for the beam, *i.e.* on the assumption that the ends are freely supported, the fixing moments at the end are

$$M_{AB}=\frac{2EI}{L}\left(2\phi_A+\phi_B-\frac{3\delta}{L}\right)-\frac{A}{L},$$

$$M_{BA}=\frac{2EI}{L}\left(\phi_A+2\phi_B-\frac{3\delta}{L}\right)+\frac{A}{L}.$$

Instead of solving the equations analytically the values of ϕ_A, ϕ_B and δ may be obtained experimentally and the values of M_{AB} and M_{BA} directly evaluated; the method will be described later.

CHAPTER 2

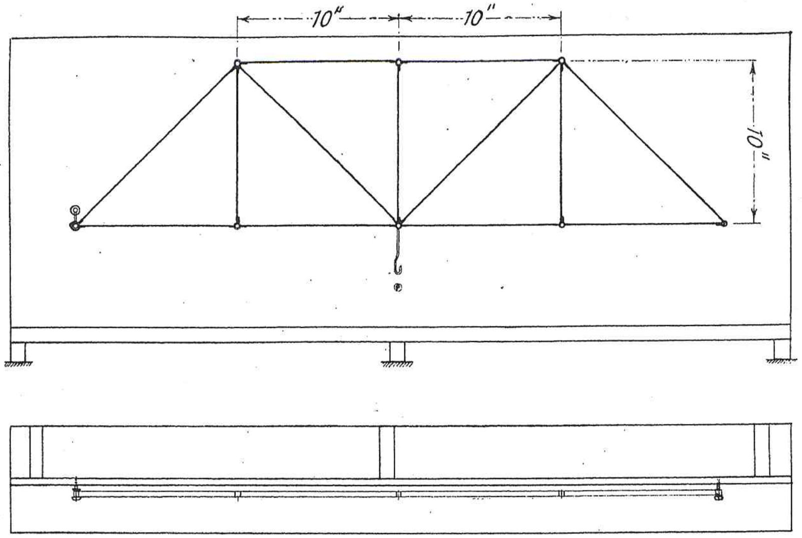
DIRECT EXPERIMENTAL METHODS

Simple Demonstration Models. The different uses of model structures for teaching purposes were referred to in Chapter 1, and in this and subsequent chapters they will be illustrated by reference to apparatus already designed and in use.

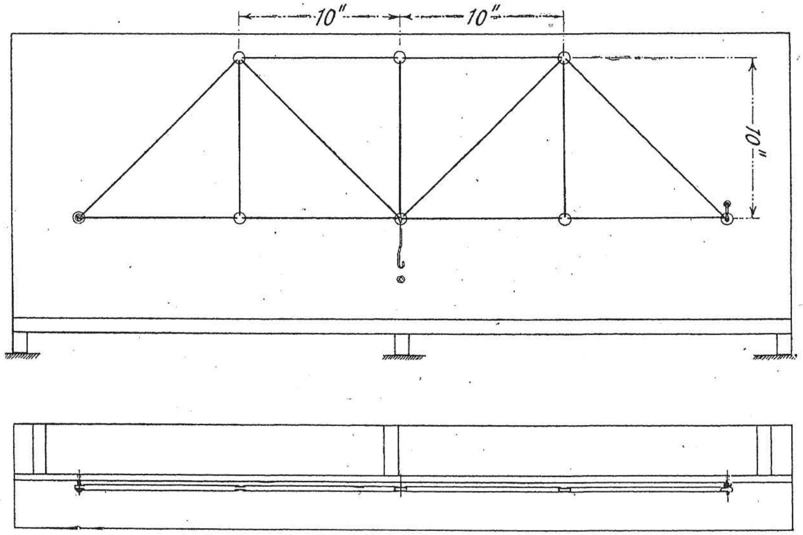
In the first place it is appropriate, since they are the simplest to construct, to deal with models which are only intended to show qualitatively the behaviour of loaded structures with the twofold object of fostering the visual sense of the student and so enabling him in his later design work to obtain a clear picture of the shape of the deformed structure, and to impress upon his mind the physical meaning of theorems and methods of calculation which he is meeting for the first time.

An illustration of this type of model is provided by the simple braced truss shown in Fig. 2.1 (*a*) which is made entirely of spring steel strip $\frac{3}{8}$ in. × 0·018 in. and is very flexible. The joints are simply pinned as shown in the figure and the truss is mounted on a wooden board. A hook at the centre of the span is used to displace the structure and the members are then deformed in a very marked and exaggerated manner, as shown in Fig. 2.2 by photographs of the unstrained and displaced model respectively. The compression members being pinned at the ends, show single curvature deformation.

A second model of the same dimensions is shown in Fig. 2.1 (*b*). This is similar to the first except that its members are connected by small rigid disks which give stiff joints in place of the pins in the first model. The undistorted and displaced forms of these structures are seen in Fig. 2.2. The strut members being encastré now bow with a double curvature and the significance of the so-called secondary stresses, *i.e.* those due to end couples applied by joints, will be readily appreciated by the student. In addition, the difference in the forces required to displace the two models is very perceptible to the touch and the greatly increased stiffness



(a) All joints pinned



(b) All joints rigid

All members of trusses are strip spring steel $\frac{3}{8}$" × ·018"

Fig. 2.1

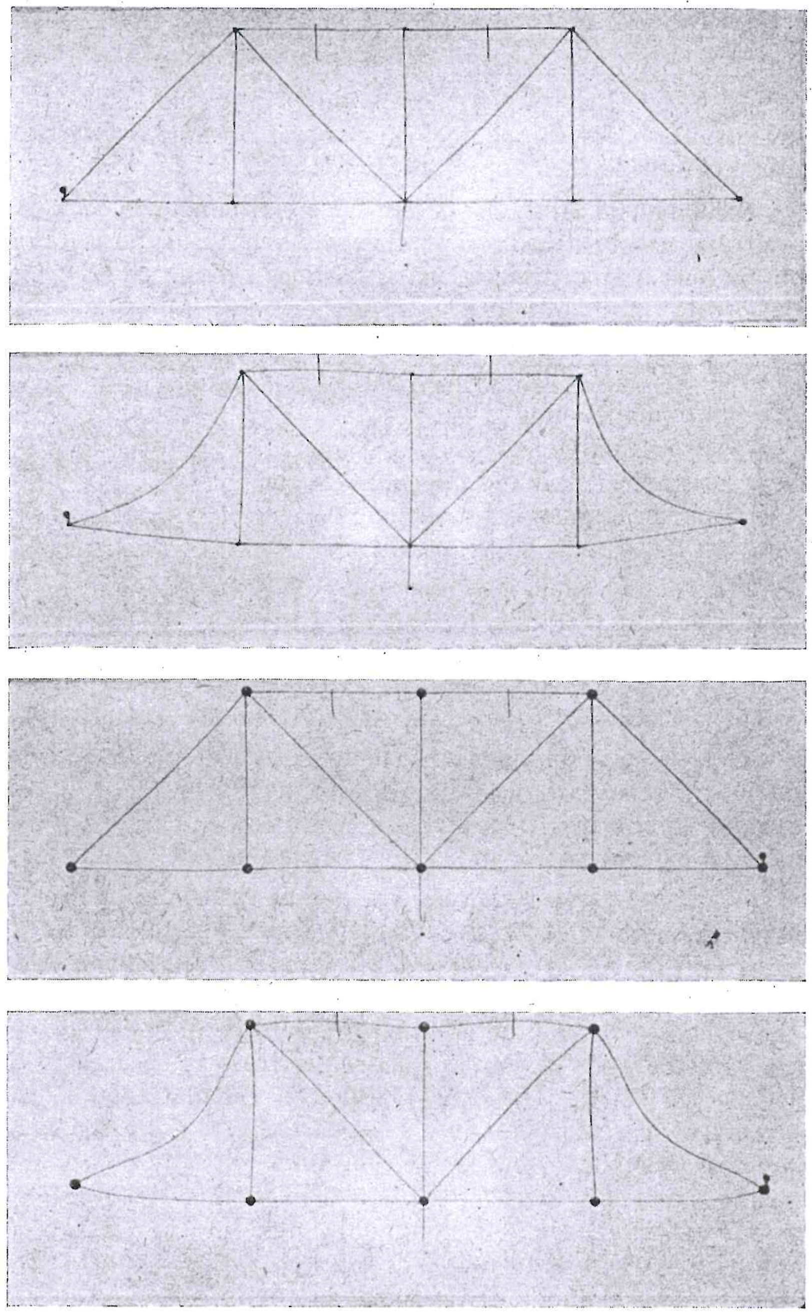


Fig. 2.2

of the second frame due to the rigidity of the joints is evident. These models can be used in lecture demonstrations, but it is essential that students should handle them personally and *feel* the differences. This emphasizes the importance of keeping the model-structures laboratory open as much as possible for the use of students outside their time-table hours.

The Behaviour of Struts. The next model is similar in purpose to the truss just described, *i.e.* to illustrate behaviour rather than to obtain actual quantitative results, although it can be used for this also with quite fair success. It is shown in Figs. 2.3 and 2.4 and consists of a set of four struts similar in all respects except for the end fixing conditions which are—

(1) Two pinned ends ;
(2) Two encastré ends ;
(3) One pinned end and one encastré end ;
(4) One encastré end and one end completely free.

The struts are made of steel strip $\frac{3}{8}$ in. × 0·020 in. and loads can be applied through plungers at the top ends either by hand pressure or by weights. In the former, the differences in the critical loads are easily felt and in the latter a fair quantitative comparison can be made. The important point to be observed, however, is the behaviour of the struts when they are subjected to different loads and then displaced slightly ; this emphasizes their transition from a state of stable equilibrium through a position of neutral equilibrium to a state of instability. The displaced shapes of the members also indicate clearly the idea of " equivalent length."

Simple Experimental Truss. Fig. 2.5 shows the arrangement drawing and details of a simple pin-jointed truss designed particularly for the experimental verification of deflexions calculated in the exercise class or determined graphically in the drawing office. Photographs in Fig. 2.6 show the truss in its unloaded and deflected state. The usual method of calculating the displacement of a single point in a loaded structure is by the first theorem of Castigliano, but if the deflexions of many points are needed it is generally better to obtain them from a Williot-Mohr diagram.[9] This diagram is commonly used in the design of bridge trusses which are carried on more than two supports and in other types of redundantly supported structures, and although this particular aspect of its use is possibly outside the scope of

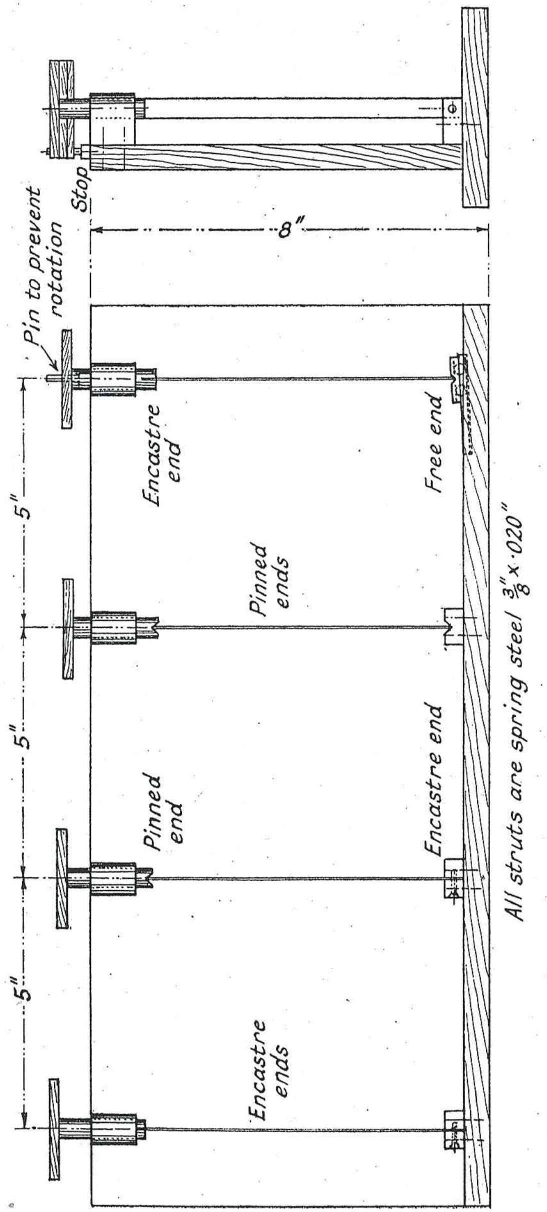


Fig. 2.3

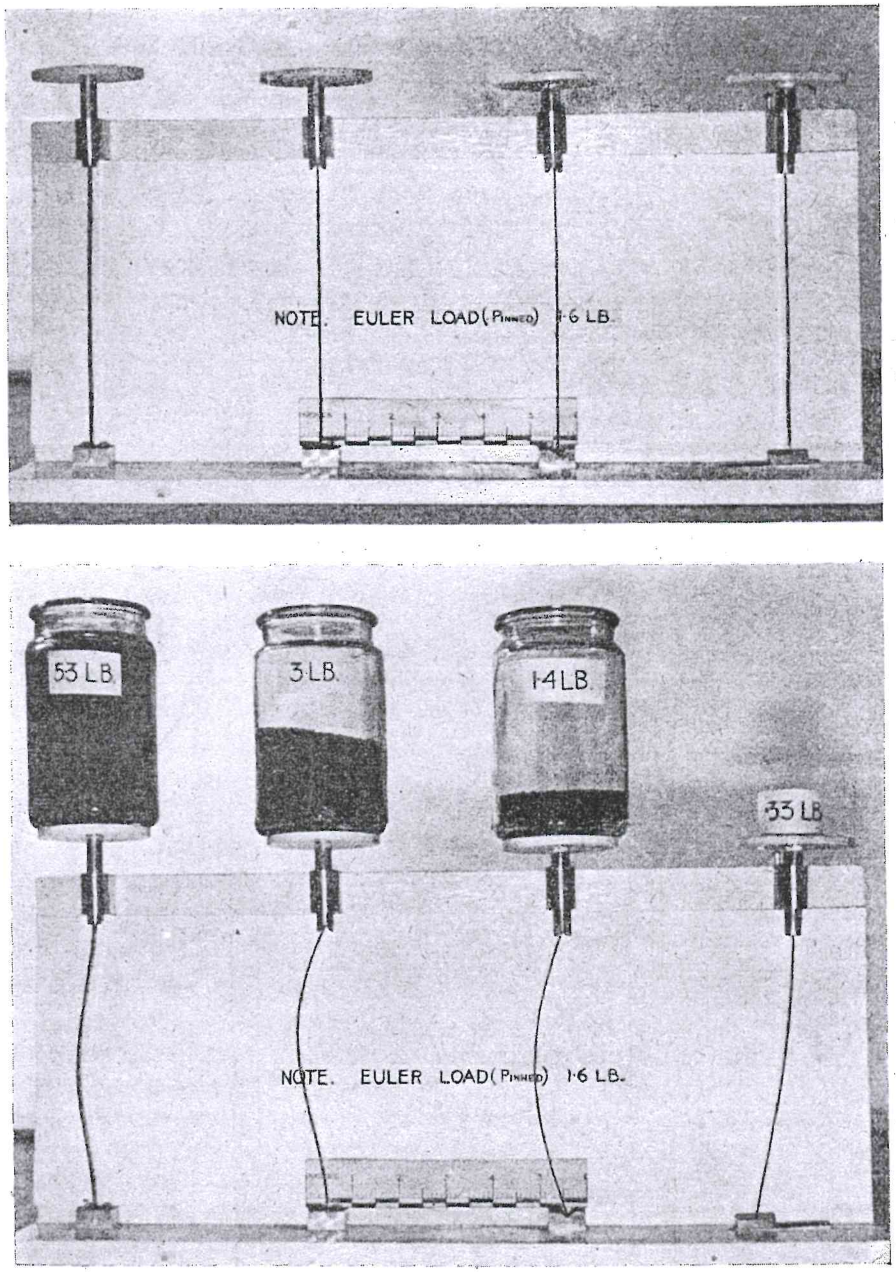


Fig. 2.4

the work which is under discussion the construction of the diagram should certainly be taught early. It is, of course, essentially a drawing office method and the model now to be described affords a good illustration of the possibility of linking the work of the lecture-room, drawing office and laboratory.

Any simple truss would serve the purpose in view, but unless special provision is made the deflexions under load will be so small as to need accurate instrumental measurement. The truss shown has therefore been designed to give deflexions of such magnitudes that they can be measured by an ordinary scale but restricted to such values that the configuration of the structure

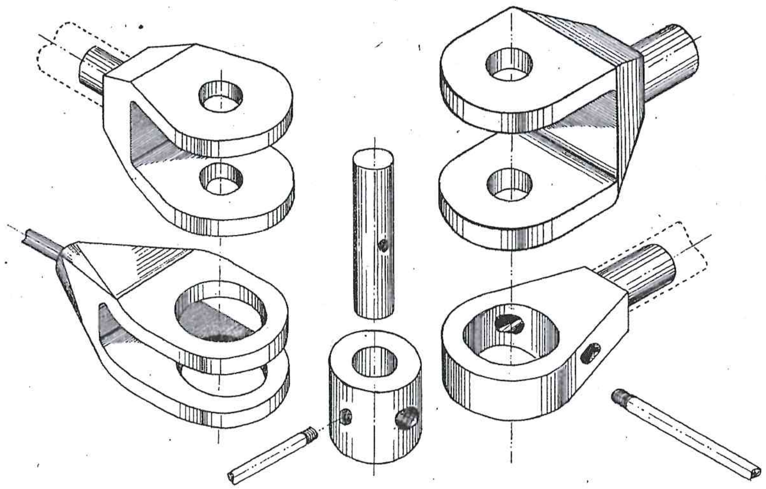


Fig. 2.7

under load is not altered sufficiently to invalidate the calculations. The first essential was to obtain true pin-joints so as to satisfy the usual hypothesis; this condition is met by connecting all members meeting at a joint to the pin by forked ends. A typical joint is shown in detail in Fig. 2.7. These joints require accurate fitting but are not outside the scope of any reasonably equipped workshop, however small. All compression members in the truss are made of duralumin tube $\frac{5}{16}$ in. diameter and 0·056 in. thickness, but every tension member which is stressed under any possible loading of the truss has incorporated in it a small spring with a stiffness of 25 lb./in. The effect is to give an extension of these tension members considerably greater than would occur if they were made of solid rods. It does, in fact, reduce the equivalent

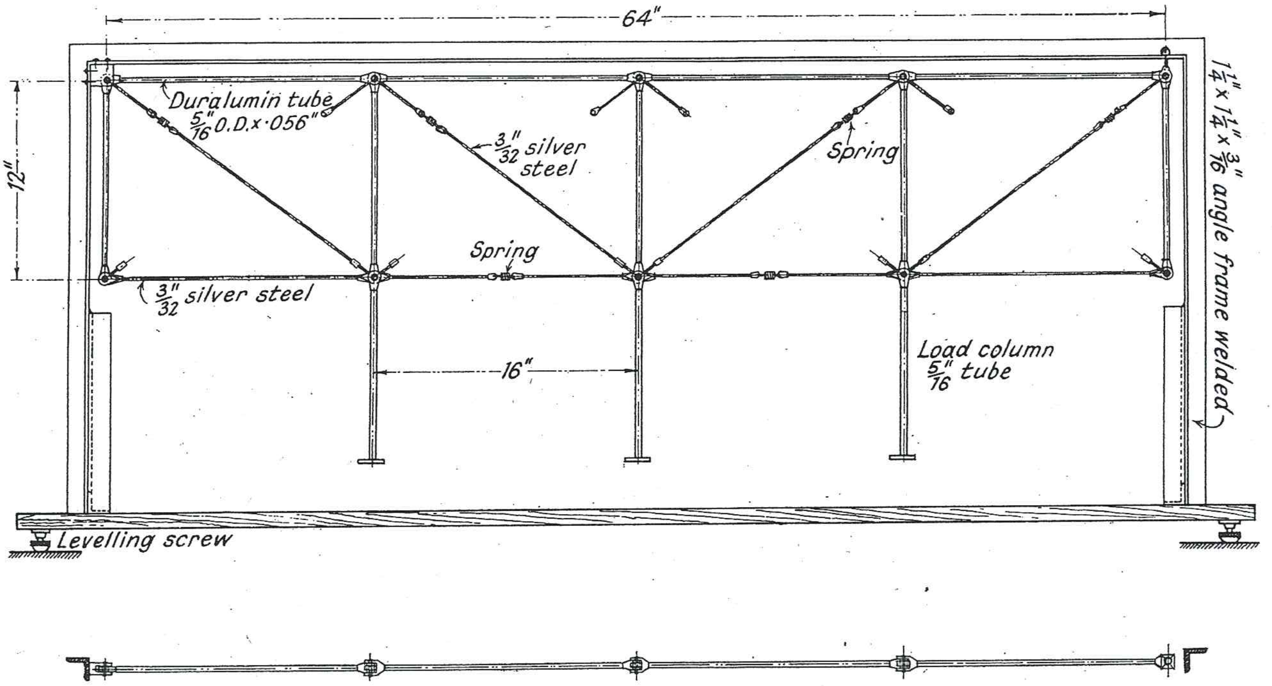

64"
12"
Duralumin tube
5/16" O.D. x ·056"
3/32" silver steel
Spring
Spring
3/32" silver steel
16"
Load column
5/16" tube
1 1/4" x 1 1/4" x 3/16" angle frame welded
Levelling screw

Fig. 2.5

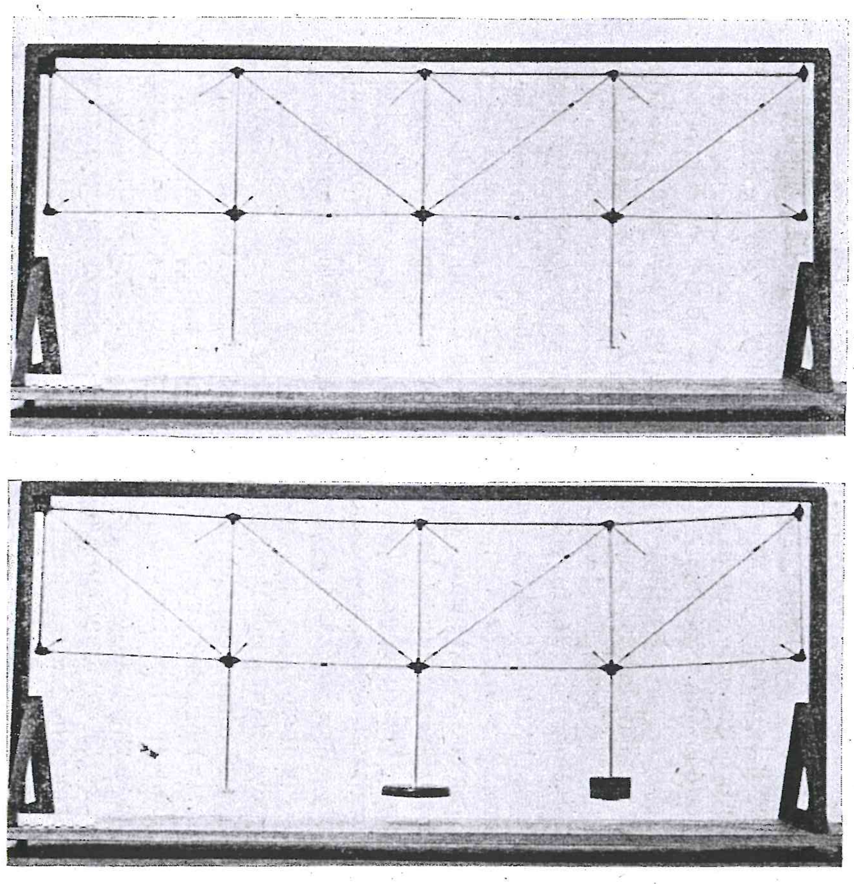


Fig. 2.6

AE of these members to such a very low figure that all other members may be considered to be rigid. The nodal deflexions are large, *e.g.* a central load of 8 lb. on the structure produces a deflexion of about $1\frac{1}{4}$ in. at the loaded point and so can be measured by simple means. Arrangements are made to brace the panels of the structure across either diagonal as required by the loading conditions so that the operative diagonal member is always in tension and this illustrates the principle of counter-bracing. The structure could be modified to include redundancies, but this was not one of the objects in view; the experimental approach to redundant structures is made somewhat differently and will be described later.

The following experiments can be made on this model :—

(*a*) Loads may be applied to any of the lower joints and the nodal displacements measured, thus obtaining the shape of the deflexion polygon for both the top and bottom booms. The Williot-Mohr diagram for the same loads can be constructed in the drawing office and a direct verification of the graphical work thus obtained.

(*b*) Since the extensions of the tension members are for practical purposes confined to the springs, it is only necessary to measure their extensions by callipers or even by a scale to obtain the loads in the bars, which can be compared with those necessarily calculated for the construction of the Williot-Mohr diagram.

(*c*) The model may be used to verify Clerk Maxwell's reciprocal theorem in its restricted form by applying loads to all the bottom joints and measuring the changes in displacements of one joint as loads are varied at another. The load can then be varied at the joint at which the deflexion was measured and the changes in displacement of the original loaded joint should then be the same as those found previously. It is necessary to load all joints initially and so ensure that the same members of the truss are operative under both loading systems.

For the calculation work in the drawing office it is necessary to know the values of AE for the members containing springs. These values may either be given as data from a knowledge of the spring or, better still, can be determined experimentally by the student by suspending a tension member vertically, hanging loads on it and measuring the resultant displacements. The equivalent AE is then found from the slope of the curve obtained by plotting PL against e, where P is the applied load, L is the

overall length of the rod including the spring and e is the measured extension. This will include the extension of the solid part of the member as well as that of the spring, but owing to the very big difference in extensibility between the two sections, this is of no importance and equally good results can be achieved by calibrating the spring alone.

Details of experiments made with this model are given in Experiment Sheets Nos. 1 and 2.

Deflexion of a Beam and Verification of Clerk Maxwell's Theorem. The very simple apparatus shown in Figs. 2.8 and 2.9 provides an exercise in the experimental determination of deflexions and also demonstrates Clerk Maxwell's reciprocal theorem in its most simplified form. A light steel beam is supported without angular restraint at one end and overhangs the other support, upon which it simply rests. If a load is placed at the free end of the beam the deflected form can be obtained by measurements with an Ames' dial and should be carefully plotted. If the load be now transferred to any other point on the beam, X, the deflexion at the free end should, by Clerk Maxwell's theorem, be the same as that found at X when the load was applied to the free end.

If, therefore, the load be applied successively to a number of points on the beam and the displacements of the free end are measured and plotted, the resulting curve, which is the influence line of deflexion for the end of the beam, should coincide with the deflexion curve of the beam under a load at the free end. This coincidence is the basis of a special and useful re-statement of Clerk Maxwell's theorem which is known as Mueller-Breslau's theorem.[10]

The student should also calculate the deflexions and compare the theoretical results with his experimental values (see Experiment Sheet No. 3).

Experimental Application of Slope-deflexion Method. The experimental solution of structural problems using the slope-deflexion equation was referred to in Chapter 1. The technique to be described is due to Professor J. F. Baker.[11]

A model of the frame to be analysed is cut out of thin sheet brass, the sections at every point being made to give the correct proportional second moments of area. This model is mounted horizontally, appropriately to the actual condition in the prototype,

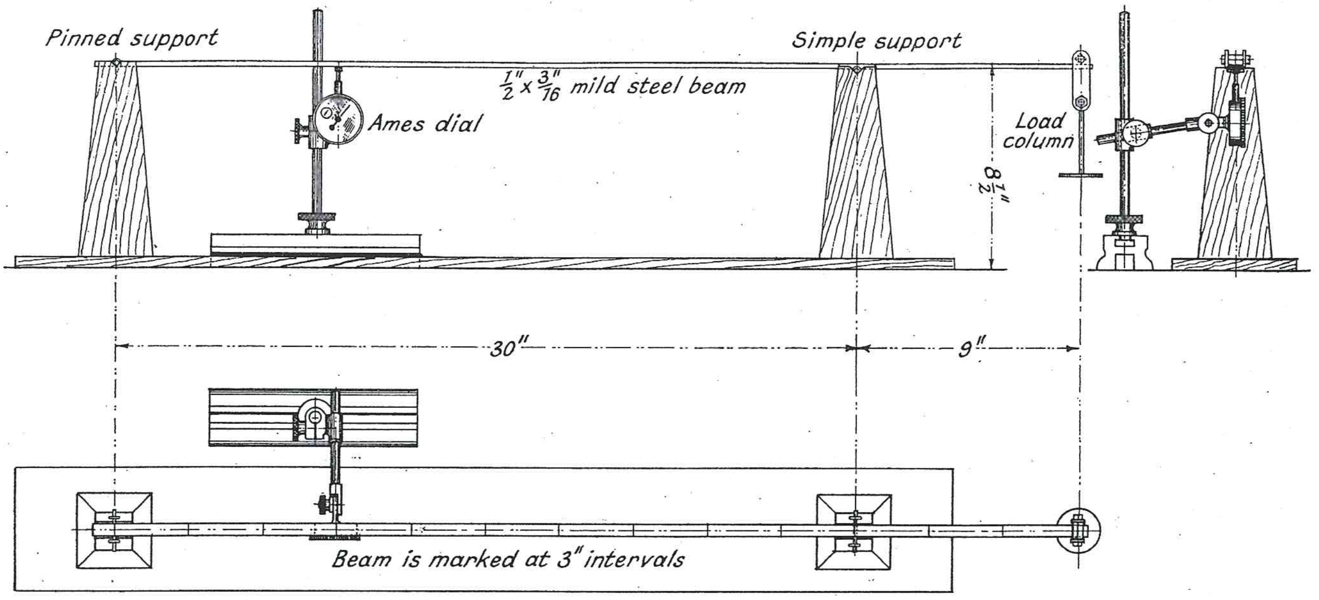

Pinned support
Simple support
1/2" x 3/16" mild steel beam
Ames dial
Load column
8 1/2"
30"
9"
Beam is marked at 3" intervals

Fig. 2.8

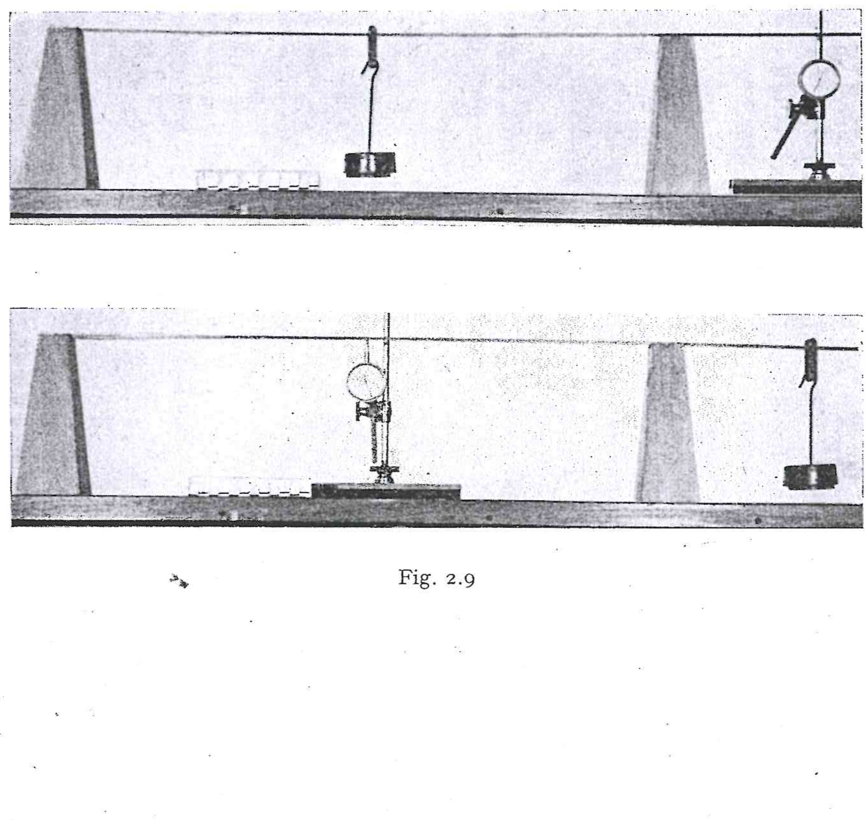


Fig. 2.9

on a drawing board and is supported by a number of balls to reduce friction as much as possible. Loads are applied either by weights over pulleys or by turnbuckles and spring balances. A light pointer about 6 in. long is mounted at each end of the particular member under examination and the movements of both ends of these are measured by a micrometer microscope. From these readings the angles of rotation and the displacements are obtained and can be inserted in the equation given in Chapter 1 (or similar ones for special conditions such as a variable cross-section beam). The end moments are thus determined.

The difficulty in this experiment lies principally in the tendency of the model to buckle out of its plane under load, but this can be counter-acted by placing small weights at appropriate points. The actual buckling is generally too slight to have a direct effect on the result but it is sufficient to throw the microscope out of focus and must therefore be prevented.

An example of the assembly for this experiment is shown in Fig. 2.10 which shows a stiff-jointed rib for the elevator of a large aeroplane. Actual results for this model and a comparison with calculated values are given in Experiment Sheet No. 4.

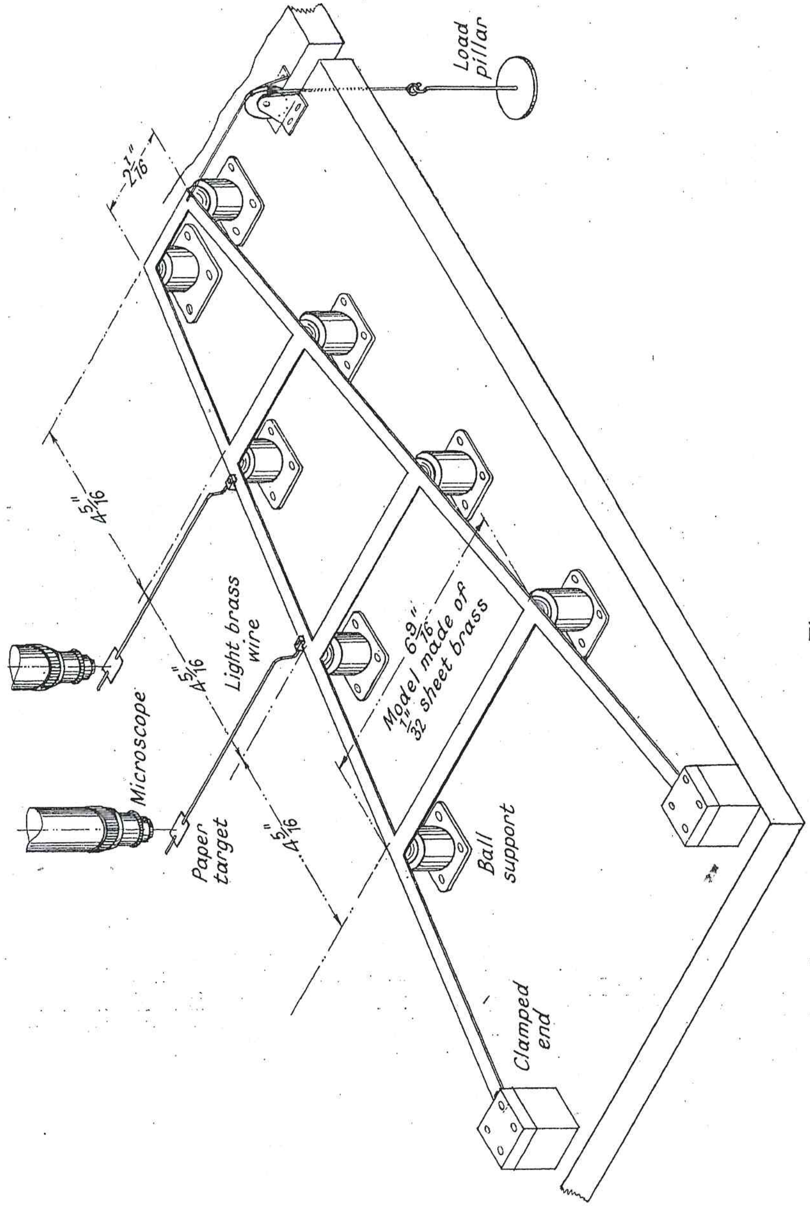


Fig. 2.10

Experiment Sheet No. I

DEFLEXION POLYGON FOR A TRUSS

Instructions

A load of 4 lb. is placed at the quarter point of the lower chord of the truss shown in Fig. 2.6. Calculate the loads in all members, draw the Williot-Mohr displacement diagram and plot the deflexion polygon for the bottom chord. Verify the calculation by experimental measurements.

Results

The extension of all members incorporating a spring is 0·04 in. per lb.

The loads in all members and their extensions are tabulated below.

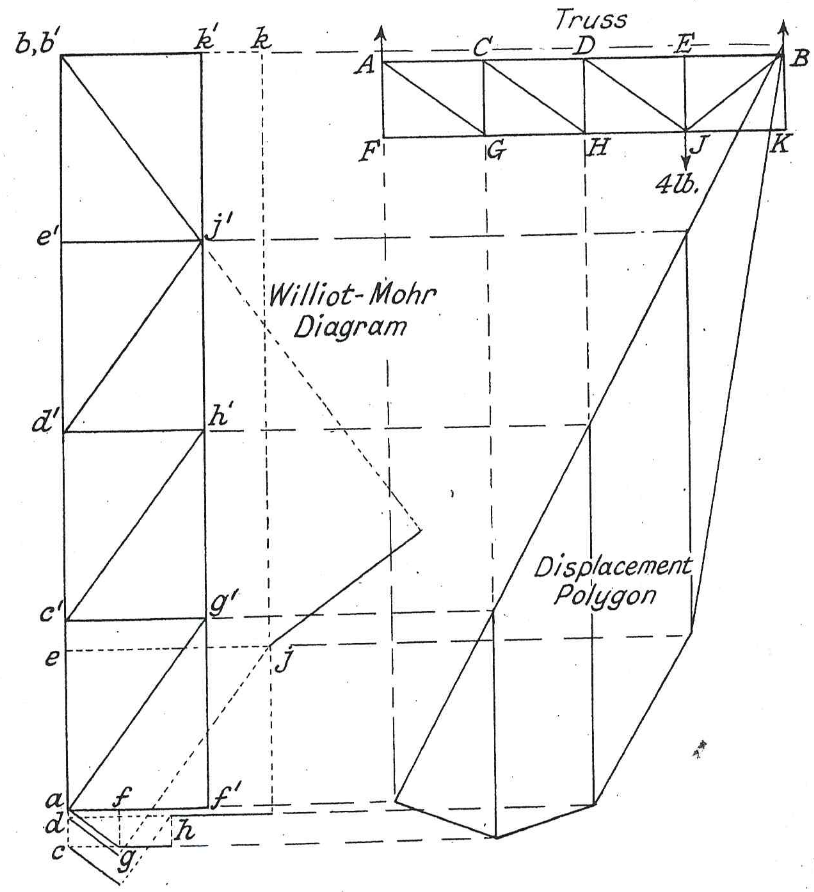
Bar	AC	GH	CD	HJ	DE EB	FG, JK, EJ, BK, AF	CG DH	AG, CH, DJ	JB
Load, lb. ...	−4/3	4/3	−8/3	8/3	−4	0	−1	5/3	5
Extension, in.	0	0·053	0	0·107	0	0	0	0·067	0·20

The Williot-Mohr diagram and deflexion polygon are shown in the figure and the results are as follows :—

Vertical displacement (in.)	G	H	J
From polygon	0·24	0·40	0·43
Measured	0·28	0·40	0·42

Notes

The appropriate diagonals must be braced, *i.e.* DJ and not HE. It is advisable to place a small initial load at J to ensure that all members are fully operative ; the specified weight should then be added and its effects measured. Forces may also be checked by measuring the extensions of the springs.



Experiment Sheet No. 2

VERIFICATION OF CLERK MAXWELL'S THEOREM BY MEANS OF A TRUSS

Instructions

Using the model shown in Fig. 2.5 verify Clerk Maxwell's reciprocal theorem as follows :—

Place 9 lb. on the load column at J* so that all diagonal members are in tension. Apply 6 lb. in 2-lb. increments at column G and measure the increases in deflexion at column J.

Remove all loads.

Then apply 6 lb. in 2-lb. increments at load column J and measure the increase in deflexion at column G.

Results

Load at J (lb.)	0	2	4	6
Deflexion at G (in.)	0	0·13	0·27	0·39
Deflexion per lb. = 0·065 in.				
Load at G (lb.)	0	2	4	6
Deflexion at J (in.)	0	0·13	0·26	0·40
Deflexion per lb. = 0·066 in.				

Notes

It is necessary to do this experiment as instructed since the same wires must be operative in both cases. The initial load of 9 lb. at hanger J ensures that the diagonals are still in tension, and therefore operative, when a 6-lb. load is applied at G.

* The reference diagram is given in Experiment Sheet No. 1.

Experiment Sheet No. 3

DEMONSTRATION OF CLERK MAXWELL'S THEOREM

Instructions

Clerk Maxwell's theorem in its simple form shows that the curve of deflexions for a beam under unit load at any point is the same as the influence line of deflexions for that point. Demonstrate that this is true by obtaining both curves for the free end of the beam shown in Fig. 2.8.

Results

(*a*) *Load at end of beam.*

	Distance from free end of beam (in.)					
	3	9	12	15	18	21
Vertical displacement (in.)	−0·1074	−0·0540	0·0342	0·0595	0·0738	0·0804
	24	27	30	33	36	39
Vertical displacement (in.)	0·0786	0·0706	0·0579	0·0400	0·0214	0

(*b*) *Deflexion of free end of beam.*

	Load position measured from free end of beam (in.)					
	3	9	12	15	18	21
Vertical displacement of free end ...	−0·1074	−0·0541	0·0339	0·0597	0·0739	0·0806
	24	27	30	33	36	39
Vertical displacement of free end ...	0·0789	0·0708	0·0579	0·0405	0·0214	0

Experiment Sheet No. 4

EXPERIMENTAL DETERMINATION OF MOMENTS BY SLOPE-DEFLEXION METHOD

Instructions

Determine the moments at each end of the chord bays of the stiff-jointed rib shown in Fig. 2.10 when a load of 1 lb. acts at the free end. Compare your results with calculated values.

Results

MOMENTS IN IN.-LB.

Chord section		1	2	3	4
Left-hand section	Exp.	0·750	0·467	0·325	0·209
	Cal.	0·790	0·488	0·341	0·240
Right-hand section	Exp.	0·546	0·372	0·276	0·287
	Cal.	0·549	0·384	0·290	0·280

Notes

This experiment takes rather a long time to carry out; it may be advisable therefore only to ask for one bay to be determined experimentally. The technique is described in the present chapter and the necessary equations are given in Chapter 1.

CHAPTER 3

EXPERIMENTAL APPLICATIONS OF CLERK MAXWELL'S THEOREM

Introduction and Description. Apparatus has been described in the previous chapter which enables the student to verify Clerk Maxwell's reciprocal theorem by simple experiments and this chapter will be devoted to a description of some of the many experimental applications of the theorem which are not only useful for educational purposes but are of considerable practical value. On several occasions it has been possible very quickly and cheaply to verify (or otherwise) the accuracy of elaborate calculations made by designers and there is no doubt that some of the methods described should be introduced into every design office. For the preliminary exploration of alternative designs and for final over-all checking they are invaluable and it is pleasing to find that in some, but at present only too rare, instances this aspect of experimental structures is beginning to find adherents in the field of practical design.

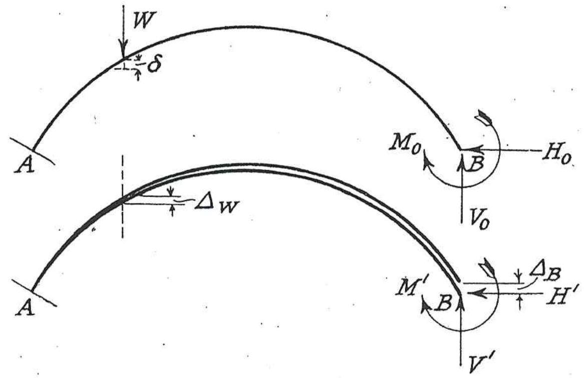


Fig. 3.1

The general principle of these methods is best illustrated by the simple case of an arch with encastré ends. If such an arch is loaded as shown at Fig. 3.1 there are three component redundant reactions at B: a moment M_0, a vertical force V_0 and a horizontal thrust H_0, and when these have been determined the bending-moment diagram for the arch can be drawn. Suppose the displacement of the load in its own line of action to be δ. There is to be no movement of the end B and the problem can be attacked by the aid of Castigliano's first theorem. The solution of the equations

$$\frac{\partial U}{\partial M_0}=\frac{\partial U}{\partial H_0}=\frac{\partial U}{\partial V_0}=0$$

provides the values of the redundant reactions.

Suppose now that the load is removed and that the end B is completely freed and *a purely vertical displacement* Δ_B is imposed upon it by the application of a moment M′, a vertical force V′ and a horizontal force H′ and that, due to these actions, the load W is displaced *in its own line of action* by the amount Δ_W.

The structure has now been assumed to be loaded in two distinct ways which comply with the requirements of Clerk Maxwell's theorem in its general form and the corresponding actions and displacements can be set out as follows:—

System 1.—Original Structure

Action	V_0	H_0	M_0	W
Corresponding displacement	0	0	0	δ

System 2.—Displaced Structure

Action	V′	H′	M′	0
Corresponding displacement	Δ_B	0	0	Δ_W

A direct application of the reciprocal theorem as stated in Chapter 1 gives—

$$V_0\Delta_B+H_0(0)+M_0(0)+W\Delta_W=V'(0)+H'(0)+M'(0)+0(\delta),$$

$$i.e.\ V_0\Delta_B=-W\Delta_W$$

If W is unity
$$V_0=-\frac{\Delta_W}{\Delta_B}$$

Hence, if a model of the structure made to any convenient scale is fixed at A and displaced purely vertically at B by a known amount Δ_B it is only necessary to measure the displacement of the point at which the load is assumed to be applied in the direction of that assumed load in order to determine the value of V_0. It should be noticed that no load is actually applied to the model to correspond with W and the only measurements needed are the displacements. The negative sign in the equation indicates that the deflexion of the point of application of W due to the imposition of Δ_B is in the opposite direction to the action of the load.

If instead of a vertical displacement, purely horizontal or angular displacements, Δ'_B and θ respectively, are applied at B, the same argument gives—

$$H_0=-\frac{\Delta'_W}{\Delta'_B}\qquad M_0=-\frac{\Delta''_W}{\theta}$$

where Δ'_W and Δ''_W are the displacements of the load point corresponding to Δ_B and θ.

Beggs' Method. This is the basis of the method originally devised by Professor G. E. Beggs [12] for the experimental determination of redundant reactions or forces in a structure.

A model representing the structure to be analysed is made of any suitable material (Beggs originally used cardboard but sheet celluloid is now generally preferred). This is suitably mounted and the section at which redundant forces are to be determined is cut and reconnected through the medium of an instrument known as a deformeter. The instrument is in two parts which can be moved relatively to one another by the insertion of accurately ground plugs; one part is attached to each of the sections of the model which are then displaced as required by small, known amounts. The displacements of other points on the model are measured accurately by a micrometer-microscope and from these the values of the redundant forces are found, whether reactions or internal stresses. A deformeter, differing in some particulars from that designed by Beggs, is shown in Fig. 3.2 and in a photograph in Fig. 3.3. The lower or fixed plate of the instrument is screwed to a drawing board and one of the cut sections is clamped to it. The other cut section is clamped to the top plate. The plates are separated by two plugs of equal diameter a resting in grooves with an angle 2θ, and the model is then in its initial, undisplaced state (Fig. 3.2 (a)).

If these plugs are replaced by others of slightly larger diameter $a+\Delta$, a displacement Δ sec θ is applied normal to the line joining

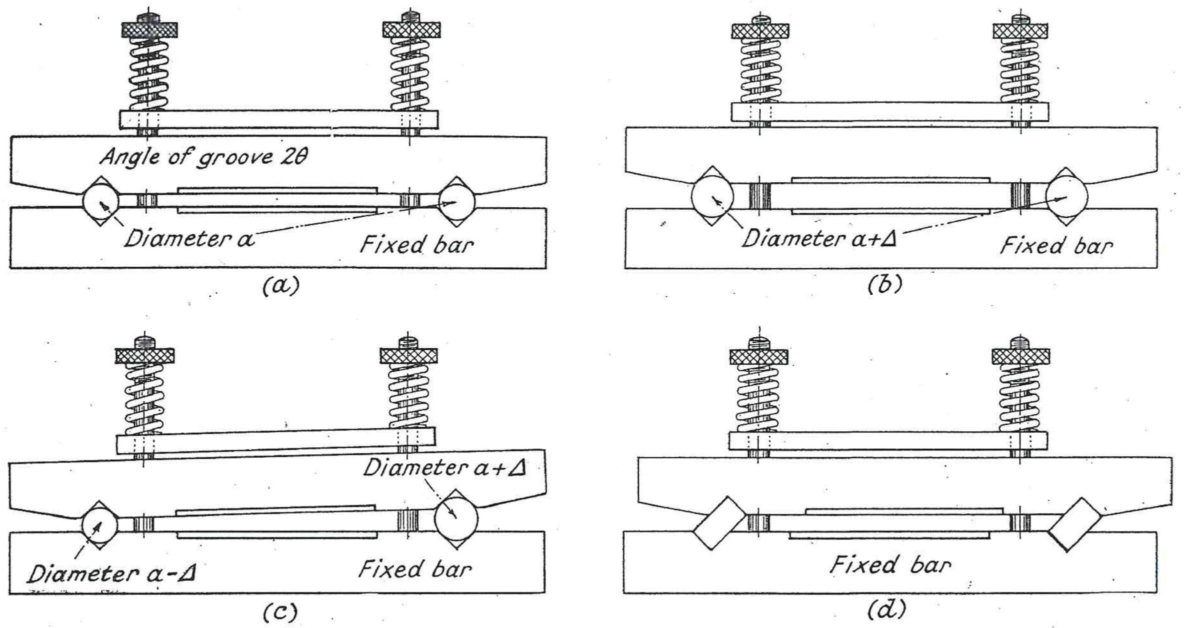

Angle of groove 2θ
Diameter α
Fixed bar
(a)
Diameter α+Δ
Fixed bar
(b)
Diameter α+Δ
Diameter α-Δ
Fixed bar
(c)
Fixed bar
(d)

Fig. 3.2

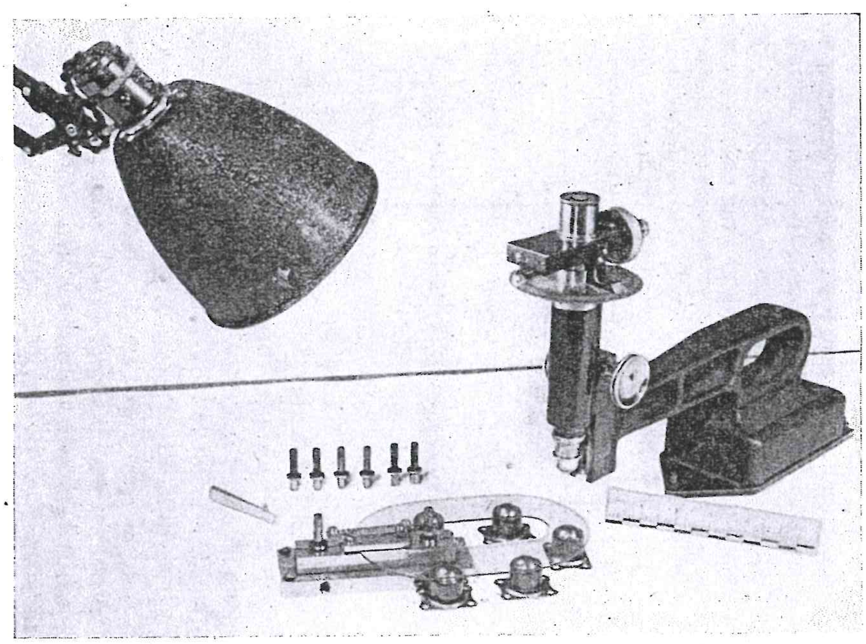


Fig. 3.3

the centres of the plugs (Fig. 3.2 (*b*)), and if plugs of diameter $a-\Delta$ are inserted a normal displacement $\Delta \sec \theta$ in the opposite direction is obtained. A plug of diameter $a+\Delta$ in one notch and of $a-\Delta$ in the other produces an angular movement $\Delta \sec \theta/l$ where $2l$ is the distance between plug centres (Fig. 3.2 (*c*)). The relative displacements of the plates should not be calculated but measured by the micrometer-microscope.

If equal rectangular plugs are inserted in the notches the top plate is moved parallel to the line joining the centres of the plugs as shown in Figs. 3.2 (*d*), the direction of the shear being governed by the direction of slope of the plugs. Examples of the use of this apparatus are given in Experiment Sheets Nos. 9 and 10.

Large Displacement Method. The whole purpose of the deformeter is to impose relative displacements of known amounts at any section of a structure, the displacements being kept small to

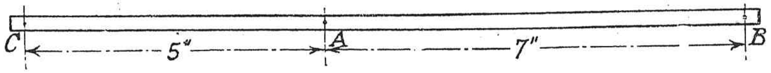


Fig. 3.4

avoid geometrical distortion and overstraining of the model. This restriction on the magnitude of the displacements makes it necessary to use a micrometer-microscope for measuring the resulting movements of points on the model and considerable practice is required to master the technique thoroughly. Errors are introduced if the temperature of the model changes during an experiment as the consequent thermal movements are comparable with those produced by the imposed displacements. In many cases it is difficult or even impossible to avoid these disadvantages and small displacements are essential, but for a number of problems much larger displacements can be imposed without risk of overstrain or undue distortion and movements can then be measured by a finely divided scale and both the deformeter and the micrometer-microscope are unnecessary.[13] It is particularly in such problems that successful use could be made of the method in design offices.

As an illustration of the large displacement method consider the simple case of a beam resting on three supports C, A and B, as shown in model form in Fig. 3.4. This model consists of a strip of celluloid of any suitable dimensions—the results will be the same for any uniform section beam—and it is pinned down

to a sheet of smooth paper to avoid frictional resistance when it is displaced.

If, for example, the influence line for the reaction at C is required, the pin at this point is removed, the end of the beam is displaced by a small amount and pinned down in the new position.

If the vertical displacement given to C is Δ and the vertical displacement of any other point X on the beam is δ, the reaction at C when a unit load acts at X is δ/Δ as shown previously. Thus, by measuring the displacement of a number of points

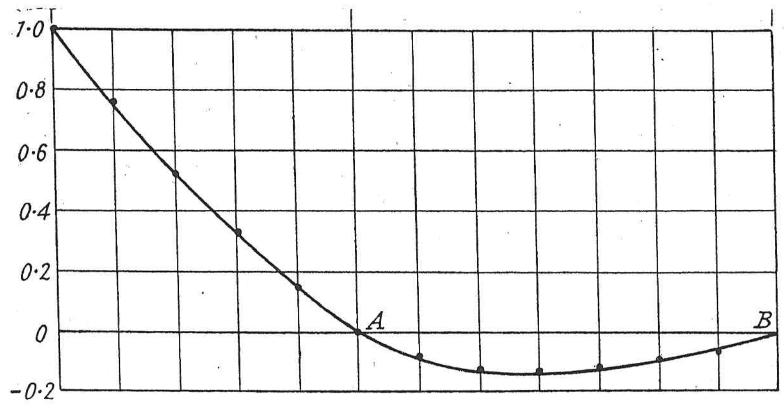


Fig. 3.5

along the beam and dividing them by Δ, the required influence line is obtained. In this particular problem, since the displacements of all points are practically vertical, it is sufficient to draw the displaced configuration of the beam on the paper by a very fine pencil to obtain the influence line, ordinates being measured from the undisplaced line drawn similarly.

Fig. 3.5 shows the experimental line obtained in this way with a number of calculated points plotted on it for comparison. The unit displacement used for this experiment as for many others is 1/48 in., *i.e.* a scale of $\frac{1}{4}$ in. to a foot divided into "inches."

This method is particularly useful in practical cases of portals, arches, rings and beams and examples are given in Experiment Sheets Nos. 5, 6, 7, 8 and 10. From the teacher's viewpoint it is valuable as a simple and effective method of emphasizing the application of a fundamental principle.

Fig. 3.6 shows a selection from a number of models which are in common use in the Imperial College laboratory ; the possible variations are many and obvious, and may range from models of structures either planned or actually built to models of hypothetical frames set simply as problems in the exercise class or drawing office. These impart considerable interest to work which can easily become rather tedious.

Classroom Models. Classroom models to illustrate the method are very useful and easily made ; two such will be described and these, as all other demonstration models, should be handled by students in the laboratory. The first, shown diagrammatically in Fig. 3.7 and by a photograph in Fig. 3.8 is an arch rib of constant section, composed of two quadrants of circles of different radii and pinned at the ends. This was set as an exercise, students being asked to calculate the thrust when a vertical load was placed at the crown, the result being afterwards checked by a small model. The model shown in Figs. 3.7 and 3.8 is a much larger one so that it can be demonstrated to a class.

It is made of celluloid and mounted on a board, the pins consisting of brass pegs $\frac{1}{8}$ in. diameter, the movable one being chained to the board to avoid loss. A datum for measurement of displacements is indicated by a broad black line painted on the board.

Holes are drilled in the board to take the pins, the right-hand end being permanently attached. In addition to the hole for the normal position of the pin at the left-hand support, two others are made at distances of 1 in. on either side of it. This end can, therefore, be displaced 1 in. horizontally either to the left or to the right. The arch is pinned in its normal position and is then displaced first in one direction and then in the other, the movements of the crown being measured. The sum of these movements divided by 2 in. (the total displacement of B) gives an excellent value for the thrust. It should be noted that a movement on each side of the normal is the usual technique adopted as to some extent it counteracts the small errors involved in the slight alterations of configuration due to the imposition of large displacements.

The second model illustrates the application of this method to the determination of the internal reactions in a closed ring.[14] The ring considered is shown diagrammatically in Fig. 3.9 and in the photograph of Fig. 3.10 and is supposed to carry two loads

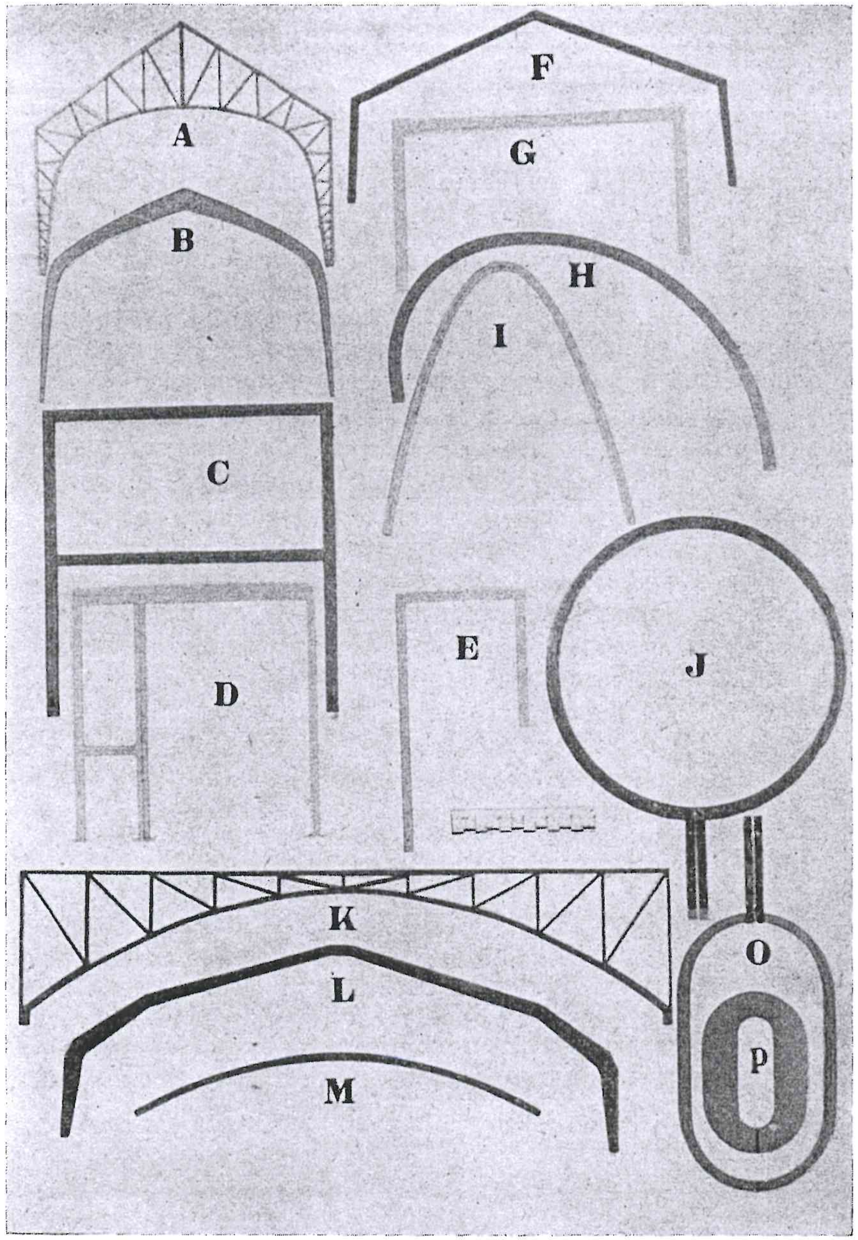
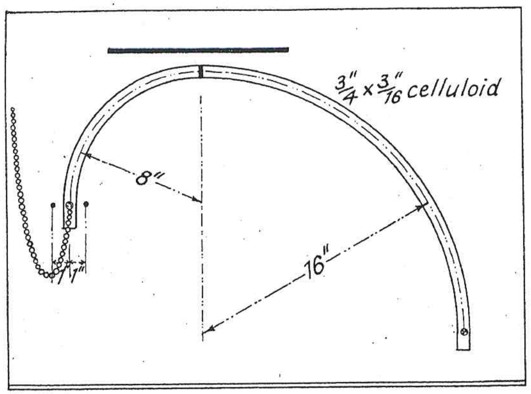


Fig. 3.6



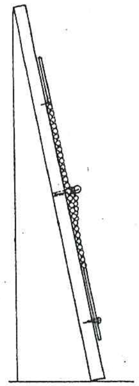


Fig. 3.7

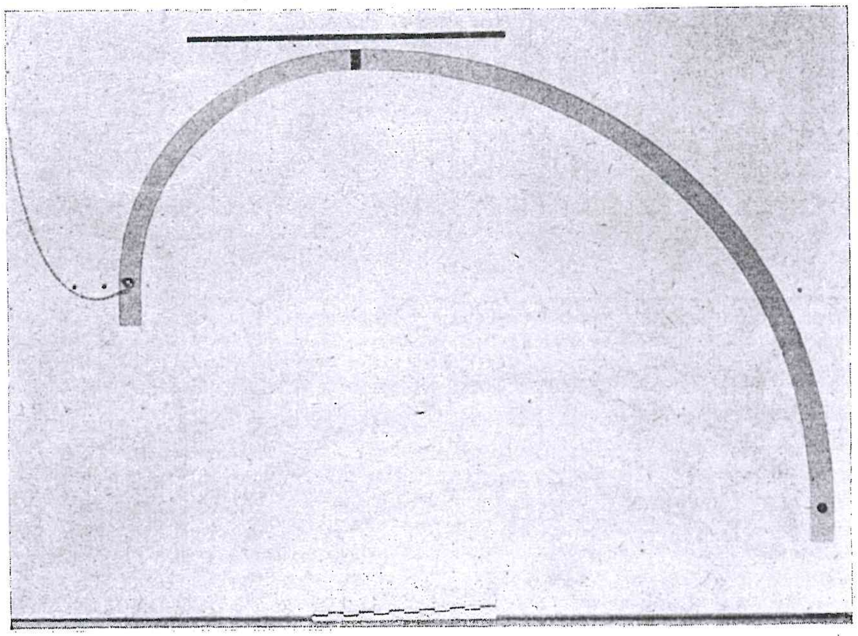


Fig. 3.8

compressing it across a vertical diameter. If the ring is supposed to be cut at one end of this diameter, the resultant actions required to restore the original condition are a couple M_0, a thrust H_0 and a radial shearing force V_0. These can be calculated by strain-energy methods and experimentally verified by the model. One end of the cut section is attached to the board by a pin and arms of length l are left on both cut sections as shown in order that an angular displacement can be easily applied.

By applying a horizontal displacement Δ to the end of the free arm a rotation $\theta = \Delta / l$ is given to the cut section and this produces a vertical movement of the assumed load point at the other end of the diameter. If this movement is δ, the value of M_0 for a unit load is given at once by $-\delta/\theta$. Similarly, by giving purely horizontal and vertical displacements respectively to the cut end the values of H_0 and V_0 are found.

The same technique is used on actual experimental models and to ensure that displacements are strictly horizontal and vertical it is convenient to insert two equal rollers or balls between the arms. For a horizontal displacement the arms are forced in turn into contact with two sets of rollers of different diameters, no rolling being allowed, and for a vertical displacement the free arm is rolled on either set through a suitable vertical distance. The loading need not, of course, be confined to the case given and in the actual laboratory should not be, but for demonstration of the principle the simplest case is preferable.

Experimental Analysis of Bow Girders. The method of large displacements is not only applicable to plane structures, but can be used for space structures of certain types. The general extension of the method is perhaps rather outside the present discussion but one special case might well be treated in a degree course, viz. the semicircular bow girder with a concentrated load. The complete problem of the girder curved in plan is too complicated for the undergraduate but the particular instance mentioned is fairly easy. Such a girder has three redundant support reactions, a bending moment, a torque and a vertical shear. For central loading the first and last of these are readily found but the torque requires a strain-energy analysis; for unsymmetrical loading all three reactions must be determined from the strain-energy equations. A simple model has been made for the experimental

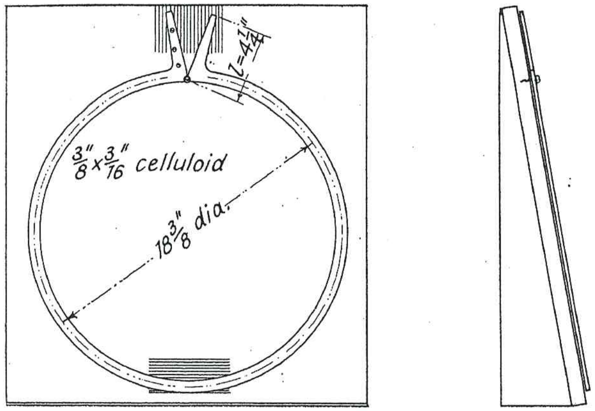


Fig. 3.9

determination of influence lines for the reactions and is shown diagrammatically in Fig. 3.11 and by a photograph in Fig. 3.12. The model shown is made of celluloid of circular cross-section since this section eases the theoretical calculation, which in the general case involves both the flexural and torsional rigidities. The actual shaping of this section requires some skill, but the purpose would be equally well served by a steel rod of small diameter accurately bent to a semicircular plan form.

The ends of the specimen are left rectangular in shape and can be fixed by wedges into slots formed by blocks of wood screwed to the back board; two sets of these slots are provided at different levels. In any test one end of the girder is rigidly fixed in its normal position. For torsional measurements the model is first set up in the upper pair of slots in its unstrained shape and the height of any point on it from the base of the model is measured by a scale. The wedges at one end are then replaced by a pair which apply an angular displacement of 15° in a plane normal to the end of the model, *i.e.* corresponding to a torque. The height of the point under consideration is again measured and the ratio of its vertical displacement to the angular movement of the end gives the value of the fixing torque for a unit load on the girder at that point. To find the vertical reactions the "free" end of the model is transferred to the lower slot and wedged so that it has no rotational displacement in either plane. The end is thus given a purely vertical displacement and another height measurement allows the shear reaction to be calculated.

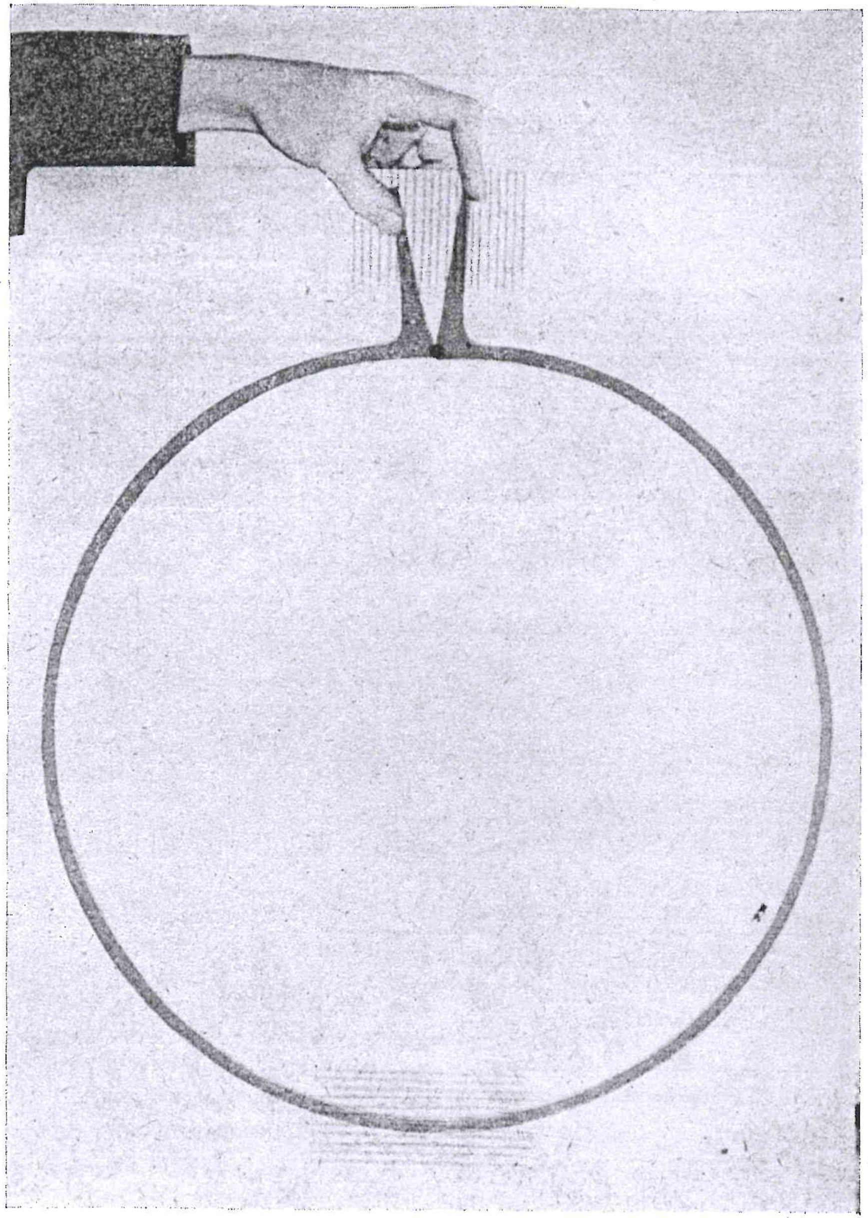


Fig. 3.10

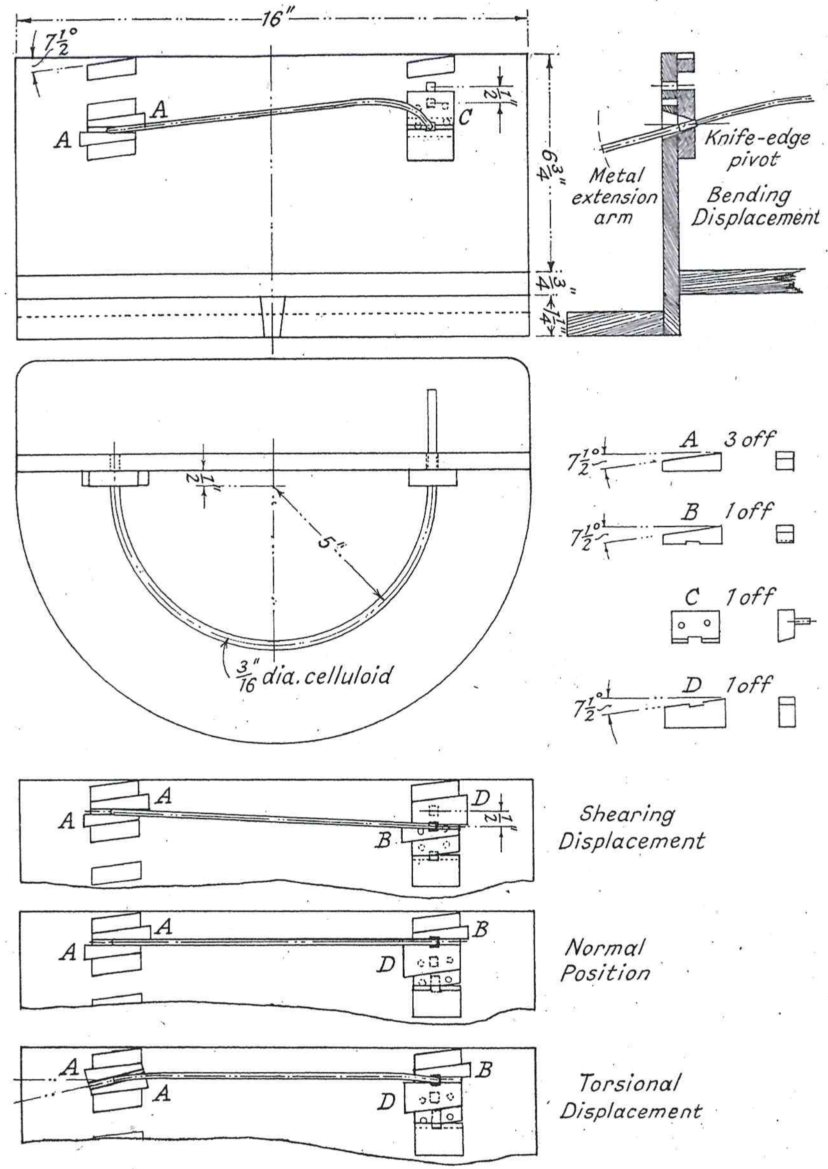


Fig. 3.11

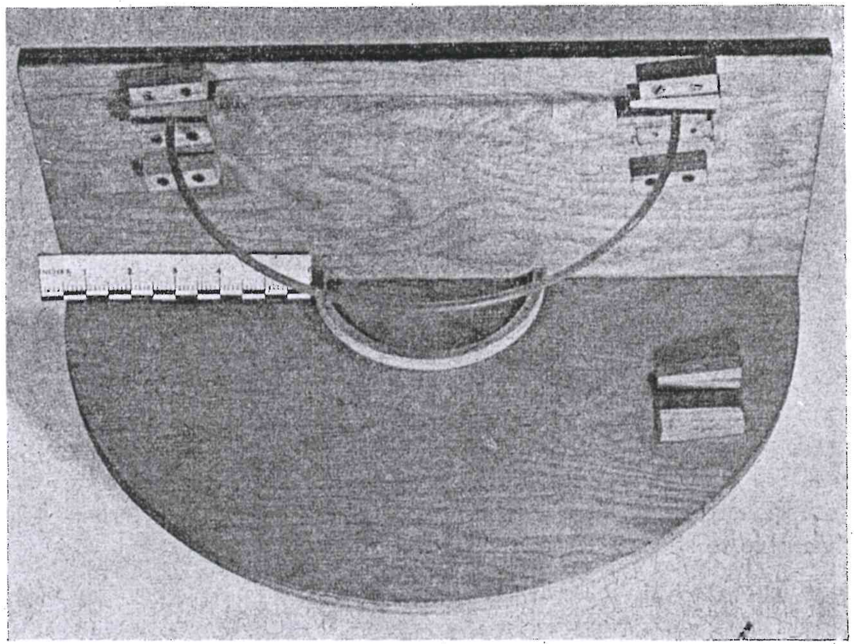


Fig. 3.12

A rotation corresponding to a bending moment is applied at the free end by a pair of specially shaped wedges which provide a knife-edge axis. A square bar made of metal to ensure rigidity is attached to the model girder and projects through a hole in the back board. This bar is deflected to give an angular movement about the support, the magnitude being determined from the difference in height of the end of the bar before and after its deflexion. The corresponding displacement of any point on the girder is measured as before and the ratio of this displacement to the angular rotation of the bar is the bending moment at the support when a unit load is placed at the point where the displacement was measured. Complete influence lines of moment, torque and shearing force at the support can be found by taking readings at a sufficient number of points around the model girder. An example of an experiment on this model is given in Experiment Sheet No. 13.

The practical use of such a method is illustrated in Fig. 3.13 which is a photograph of a model made to represent one section of the Penguin Pool Ramp at the Zoological Gardens, Regents Park. This slender structure is made of reinforced concrete, the cross-section being trapezoidal, and the construction of the model in celluloid gave scope for some ingenuity. The method used for applying displacements was more elaborate than that just described, a special end fitting made of brass being found more suitable than a simple system of wedges. This problem is too complicated for the undergraduate but is mentioned here as it is desirable to show actual practical applications when possible.

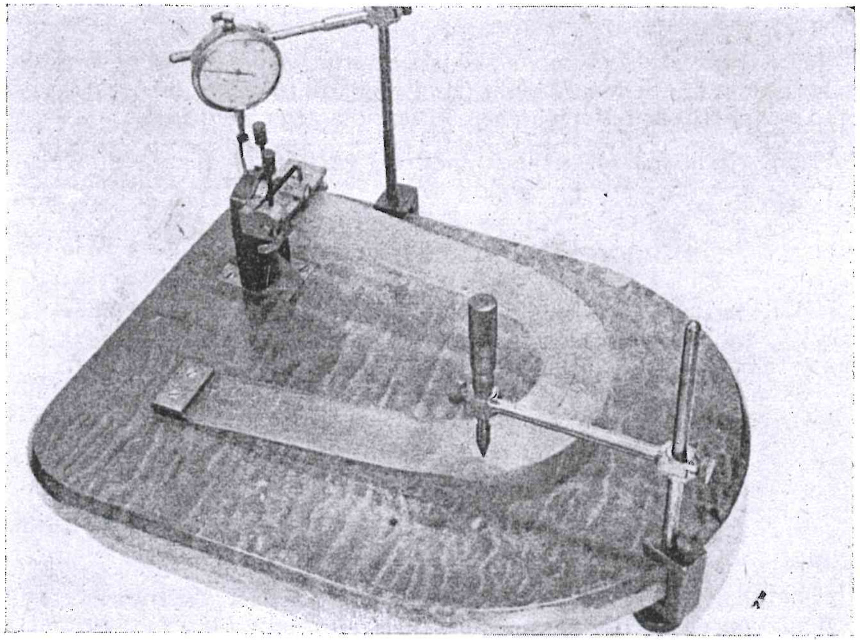


Fig. 3.13

Experiment Sheet No. 5

THRUST IN ARCH RIB

Instructions

In Fig. 3.7 is shown a two-pinned arch rib formed of two quadrants of circles, the radius of the right-hand one being twice that of the other. Use the method of large displacements to determine the horizontal thrust in the rib when a load W acts vertically at the crown. Compare the result with that obtained analytically.

Result

(*a*) *Analytical.* By moments about the supports the vertical reactions at the left- and right-hand pins are found to be $\frac{1}{3}(2W-H)$ and $\frac{1}{3}(W+H)$ respectively, where H is the horizontal thrust.

Using strain energy methods, the total value of $\frac{dU}{dH}$ for the rib when equated to zero gives

$$\frac{H}{W}=\frac{3(6-\pi)}{2(9\pi-13)}=0{\cdot}281.$$

(*b*) *Experimental.* Using the actual demonstration model shown in Fig. 3.7 and displacing the left-hand pin from its extreme left-hand to its extreme right-hand position the following values were obtained.

Horizontal displacement of left-hand pin = 1·98 in.
Vertical movement of crown = 0·56 in.

$$\therefore \frac{H}{W}=\frac{0{\cdot}56}{1{\cdot}98}=0{\cdot}28.$$

Notes

The analysis of this problem is a useful exercise in strain energy calculation, since the unknown H appears in the vertical reactions. The experimental check has been done on the large lecture-room demonstration model, but a smaller model can be used as shown in Experiment Sheet No. 6.

Experiment Sheet No. 6

INFLUENCE LINE OF THRUST FOR AN ARCH

Instructions

The model arch rib shown in Fig. 3.6, H, is similar to the demonstration model of Fig. 3.7, but has a span of 15 in. only. Use this model to determine the influence line of thrust.

Results

The ordinates of the influence line both calculated and experimental are given in the table. Distances are measured horizontally from the left-hand support.

Distance (in.)	0	1	2	3	4	5	7	9	11	13	15
Calculated	0	0·074	0·140	0·196	0·244	0·281	0·323	0·322	0·266	0·161	0
Experimental	0	0·08	0·12	0·20	0·24	0·28	0·32	0·32	0·26	0·16	0

Notes

In Experiment Sheet No. 5 the horizontal thrust was found from the demonstration model when a load was placed at the crown. Calculation of the thrust for a load at any other point presents no great mathematical difficulties but requires considerable and sustained effort. If the student attempts the problem he will more readily appreciate the advantage of the experimental method since the whole influence line can be obtained with two settings of the model. No more difficulty is experienced with the load at any other point than when it is applied at the crown. A comment from a student's laboratory book after doing this experiment is noteworthy. It was "The time-saving value of this method is apparent. Obtaining a correct solution for the double-radius rib by analysis occupied 6 undergraduate hours and 5 post-graduate (Cambridge) hours. The experiment occupied 15 minutes."

Experiment Sheet No. 7

THE HORIZONTAL THRUST IN A PARABOLIC ARCH

Instructions

In calculating the thrust in a two-pinned parabolic arch rib of constant section it is usual to assume that the second moment of area varies as the secant of the angle of slope of the rib. Calculate the thrust on this assumption for two parabolic arches, each carrying a vertical load at the crown.

(*a*) when the rise to span ratio is 1 : 1;

(*b*) when the rise to span ratio is 1 : 6·35.

Using the method of large displacements, obtain experimental values for the thrust in each case by means of celluloid models and compare the results with those obtained by calculation.

Results

Model (*a*) is shown in Fig. 3.6, I. The results obtained by calculation and experiment are given in the table.

Arch	Calculated H	Experimental H	Per cent. difference
Rise 1 : 1	0·195	0·189	3·2
Rise 1 : 6·35	1·24	1·24	0

Notes

This experiment should impress upon the student the value of the assumption so commonly made. The technique is as described in this chapter.

A useful addition to this exercise, as it includes drawing-office work, is to obtain the true thrust for both arches (*i.e.* on the assumption of uniform cross-section) by integrating the strain-energy equations graphically. The graphical treatment of equations which are difficult to integrate directly is of considerable importance and this is a problem in which it can be given an added interest.

Experiment Sheet No. 8

DISTRIBUTED LOAD ON A PORTAL

Instructions

A portal frame has vertical legs and a horizontal top beam. One leg is half the length of the other and the span is the same as the length of the longer leg. The longer leg carries a distributed horizontal load which varies in intensity from zero at the top to w at the bottom. The legs are pinned at their bases to rigid supports. Determine analytically and experimentally the horizontal thrust on the pins.

Results

By using strain-energy methods of calculation the thrust is found to be $0{\cdot}302wL$, where $2L$ is the length of the longer leg.

The experimental values determined by five students were respectively $0{\cdot}300wL$, $0{\cdot}298wL$, $0{\cdot}314wL$, $0{\cdot}300wL$ and $0{\cdot}311wL$, the average of these being $0{\cdot}305wL$

Notes

The experimental determination was made by pinning a celluloid model of the frame to a sheet of smooth paper and applying a "large" horizontal displacement to one support. The horizontal distances between the original and final positions of the longer leg were measured at a number of points and these divided by the amount of the pin displacement gave points on the influence line of thrust for loads on the leg.

The ordinates of this curve were multiplied by the proportional loading and a new curve obtained, the integral of which was the required thrust.

Experiment Sheet No. 9

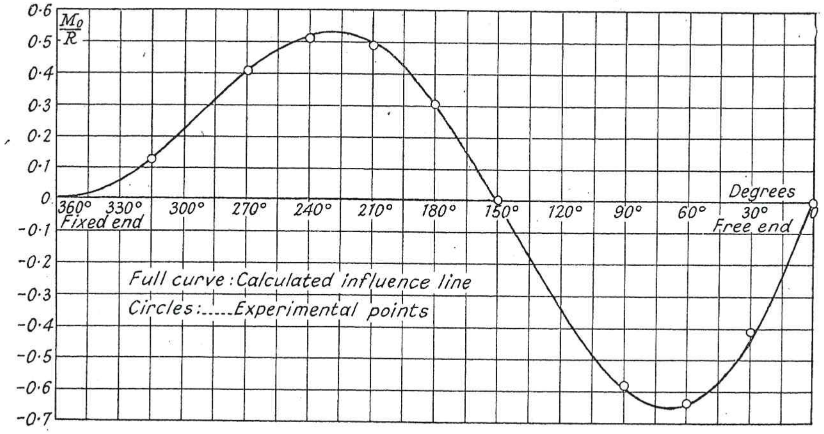
INFLUENCE LINE FOR BENDING MOMENTS IN A CIRCULAR RING BY BEGGS' DEFORMETER

Instructions

Using Beggs' deformeter determine the influence line of bending moments at a section of a circular ring when subjected to radial loads. Compare your results with calculated values.

Results

The calculated influence line is shown in the diagram and experimental points are plotted for comparison.



Notes

The model was of celluloid 0·08 in. thick, mean diameter 10 in. and width 0·5 in. It was cut at one section and the deformeter attached as described in this chapter, one bar being screwed to a drawing board.

Experiment Sheet No. 10

THE RESULTANT ACTIONS IN A LOADED LINK

Instructions

A variable section link is represented by the flat celluloid model 0·08 in. thick shown in Fig. 3.6, P, in which the second moments of area are proportional at every section to those of the actual link and the centre line is correctly to scale.

Using Beggs' apparatus determine the bending moment at one of the load points.

The model shown in Fig. 3.6, O, also represents the link; the scale of widths is, however, much reduced from that of the first model, while the scale of the outline is increased.

Using Beggs' apparatus and also the method of large displacements, determine the values of the bending moments at the same point as for the true scale model.

Compare results with each other and also with the theoretical value calculated by strain-energy methods.

Results

Average of six tests on model P by Beggs' deformeter, M=0·439W.
Average of six tests on model O by Beggs' deformeter, M=0·434W.
Average of three tests on model O by large displacements, M=0·445W.
Value calculated by strain energy method, M=0·460W.

Notes

This experiment illustrates the wide freedom in choice of scales in preparing models for analysis. The best result was achieved on the second and very flexible model without the use of the deformeter, the difference between this result and the calculated value being about 3 per cent. This type of experiment, on easily made models, gives direct verification of calculated results and produces confidence in the future application of model analysis to incalculable problems.

Experiment Sheet No. 11

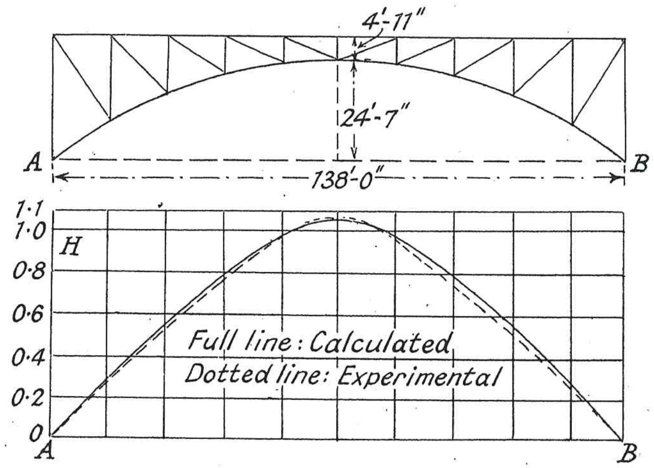
THE INFLUENCE LINE OF THRUST FOR A SPANDREL-BRACED ARCH

Instructions

In the *Proceedings of the Institution of Civil Engineers*, vol. clxvii (1906–7, Part I), p. 343, Mr. Ralph Freeman described the design of a two-hinged spandrel-braced arch. The calculated influence line of thrust is given and this is to be checked experimentally by a celluloid model using the method of large displacements. The calculated and experimental lines should be plotted on the same figure.

Results

The figure shows diagrammatically the outline of the arch structure and the two influence lines. The actual model is shown in Fig. 3.6, K.



Notes

Since the structure is fully braced the areas of all members in the model are made proportional to those of the prototype. The model scale is 1 in. to 5 ft. A reasonable displacement of one end of the arch in the line of thrust is 0·25 in. It will be found that larger displacements tend to cause buckling of the model.

Experiment Sheet No. 12

A COMPARISON OF LATTICED AND SOLID WEB BRACING

Instructions

Figs. 3.6, A, and B show celluloid models of two portal roof trusses for a small hall. The shear bracing in model A is of latticed construction and the model has therefore been made with the areas of the bars proportional to those of the full-scale structure. Model B has a solid web to resist shear and the model has been designed so that the second moment of area at every section is proportional to that of the full-scale structure. The trusses are to carry the same loads.

Determine experimentally for both models the horizontal thrust at the pinned supports when a unit vertical load acts at the crown.

Results

The test was made using the large displacement method, the unit of measurement being $\frac{1}{48}$ in.

Model A. Mean value of three horizontal displacements of a support $=9{\cdot}4$ units.

Mean of three corresponding vertical displacements at crown $=1{\cdot}2$ units.

$$\therefore \text{Thrust} = \frac{1{\cdot}2}{9{\cdot}4} = 0{\cdot}128.$$

Model B. Mean value of three horizontal displacements of a support $=16.5$ units.

Mean of three corresponding vertical displacements at crown $=2{\cdot}6$ units.

$$\therefore \text{Thrust} = \frac{2{\cdot}6}{16{\cdot}5} = 0{\cdot}158.$$

Experiment Sheet No. 13

TORSIONAL REACTION IN A BOW GIRDER

Instructions

Using the apparatus shown in Fig. 3.11 determine the torsional reactions at the supports of the semi-circular bow girder when a concentrated load acts at the centre of the span. Compare your results with the calculated value.

Results

The model was adjusted to its normal unstrained position and the height from the base at the mid-span measured by a steel rule. The normal wedges at one support were then replaced by the torque wedges which applied a twist of $\theta=15°$. The deflexion of the centre point was measured and found to be $\delta=0{\cdot}25$ in. Hence, the reaction torque produced by a central load W was

$$\frac{\delta}{\theta}W=\frac{0{\cdot}25\times 57{\cdot}4}{15}W=0{\cdot}95W.$$

In general there are three redundant reactions at the supports, but in the symmetrical case under consideration the vertical shear is clearly 0·5W and the fixing moment, obtained by taking moments about the diameter through the supports, is $\frac{WR}{2}$, where R is the radius of the semicircle.

The reaction torque T is then calculated by solving the equation obtained from $\frac{dU}{dT}=0$ and this gives

$$T=WR\left(\frac{1}{2}-\frac{1}{\pi}\right)=5(0{\cdot}5-0{\cdot}318)=0{\cdot}91W.$$

Notes

This exercise involves the simplest calculation possible with a bow girder. If the load is not at the centre it is necessary to solve the three simultaneous equations $\frac{dU}{dM}=\frac{dU}{dT}=\frac{dU}{dF}=0$ to determine the theoretical values of the reactions.

CHAPTER 4

STRAIN-ENERGY AND DISTRIBUTION METHODS

Self-straining. Reference was made in Chapter 1 to the possibility of self-straining in a redundant framework and it is of advantage to illustrate it by a simple model such as that shown diagrammatically in Fig. 4.1 and in the photograph of Fig. 4.2. A rectangular pin-jointed frame made of duralumin tubes is hung from two supports and two diagonal members are provided for insertion. These are $\frac{1}{4}$ in. shorter than the exact length required to maintain the frame in its intended rectangular shape and incorporate springs similar to those used in the model of Fig. 2.6. When one of them is placed in position the structure is simply-stiff but is distorted from the designed shape without the application of force and there is no stress in any member of the frame. This is, or should be, obvious without a demonstration, but it emphasizes an essential point. If the other bar, also incorporating a similar spring to that already in position, is now inserted it is necessary to stretch the spring before connexion can be made. The frame then assumes the rectangular shape and the springs in the two diagonals are equally extended, showing equal tensions in these members.

Three-Wire Suspension. One of the simplest problems of redundancy is that of the three-wire suspension and a model for verifying the results of calculation is shown in Figs. 4.3 and 4.4. It consists of three rods suspended from fixed supports and meeting at a point from which a vertical load can be hung. Any or all of the rods may incorporate springs to decrease their stiffnesses to values which will give displacements which can be easily measured by ordinary scales. These springs may be attached as in the truss previously described, but in the model shown they are in the form of simple spring balances so that forces can be measured directly. One purpose of this experiment is to make a comparison with the results of calculation

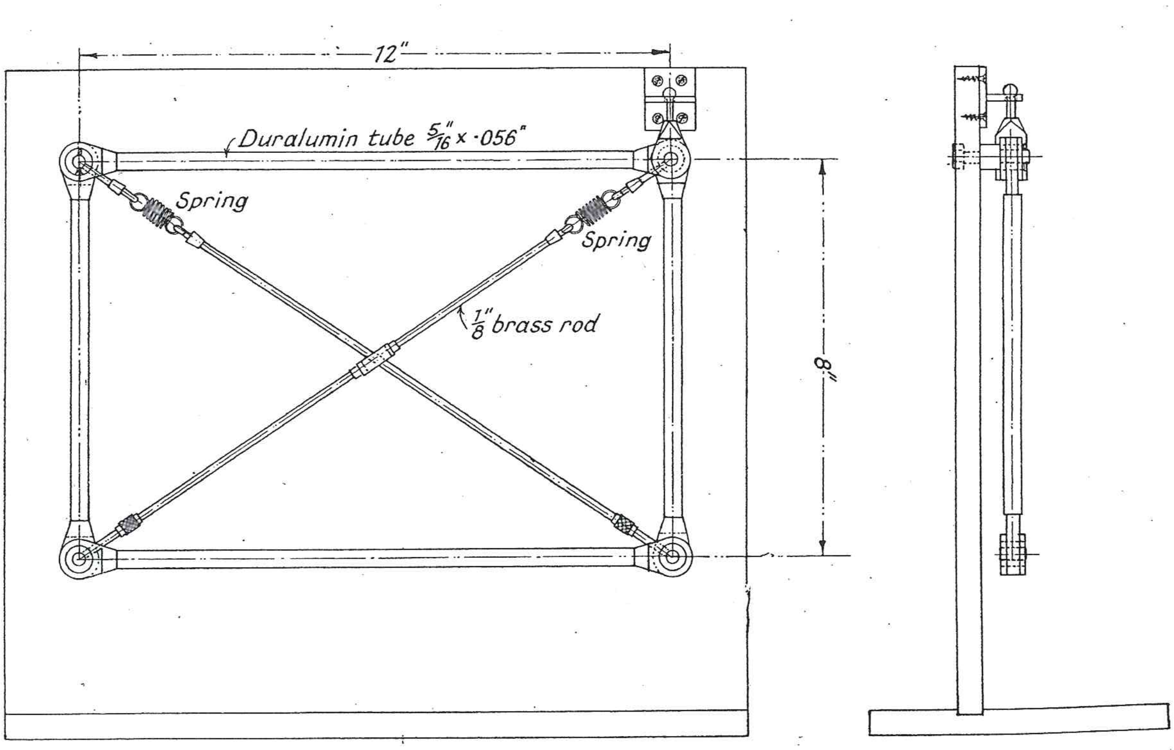


Fig. 4.1

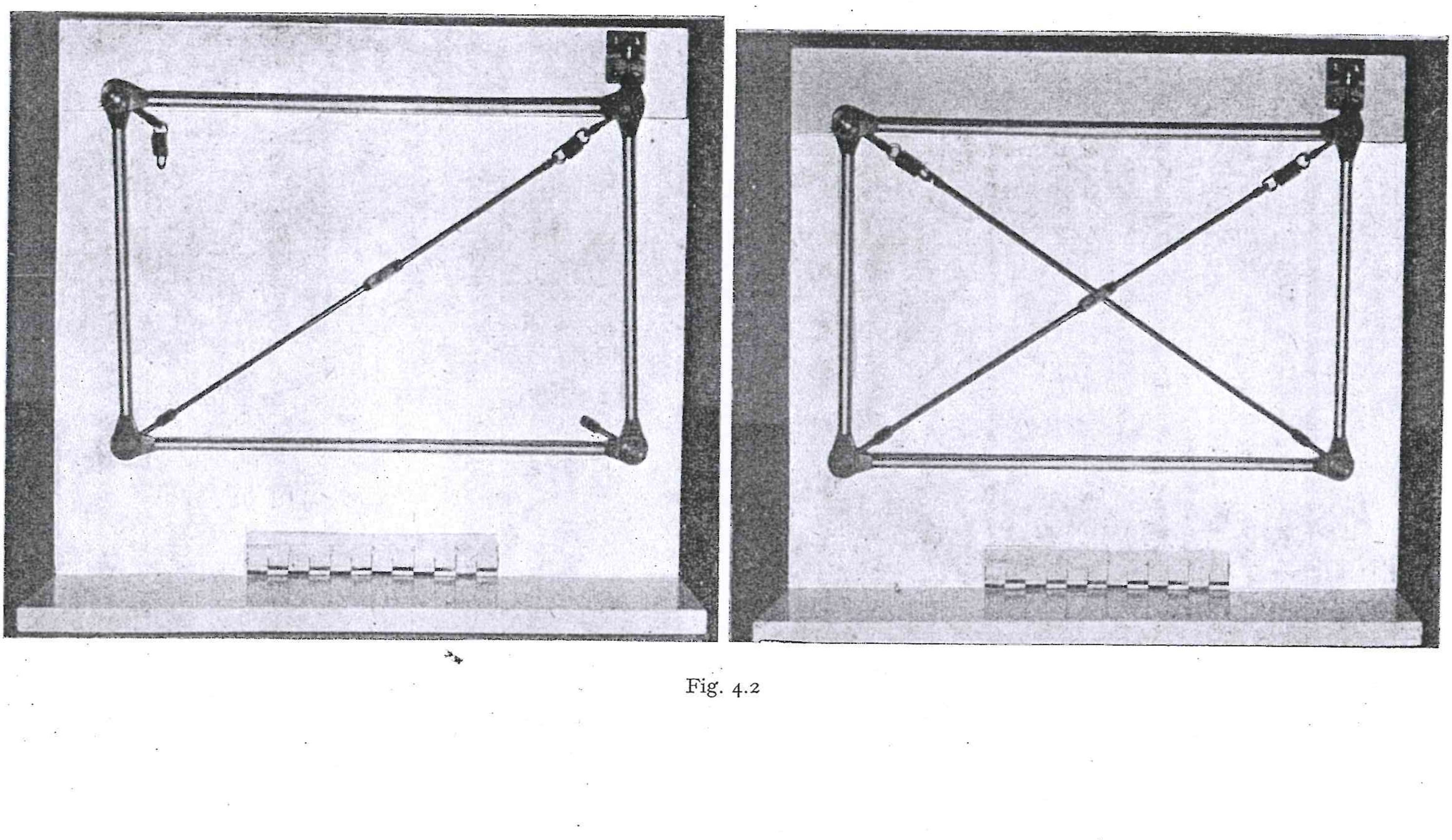


Fig. 4.2

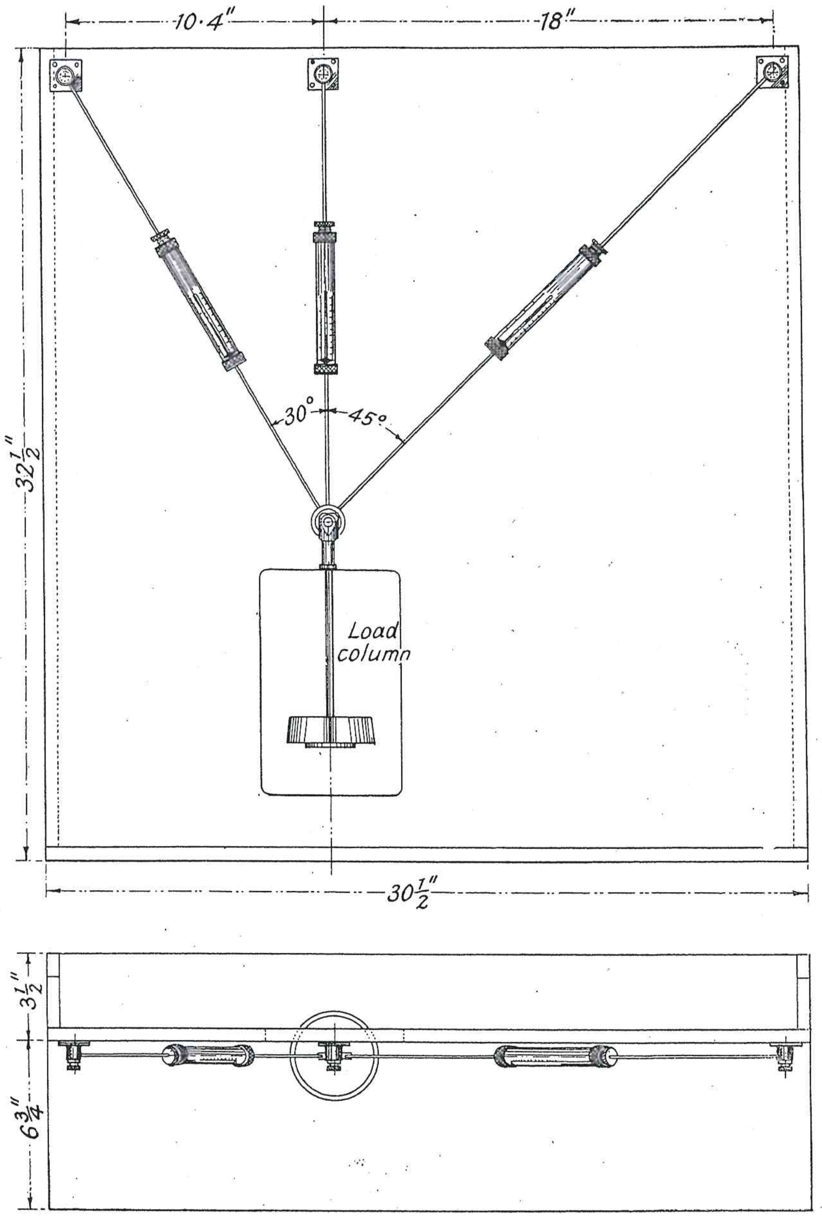


Fig. 4.3

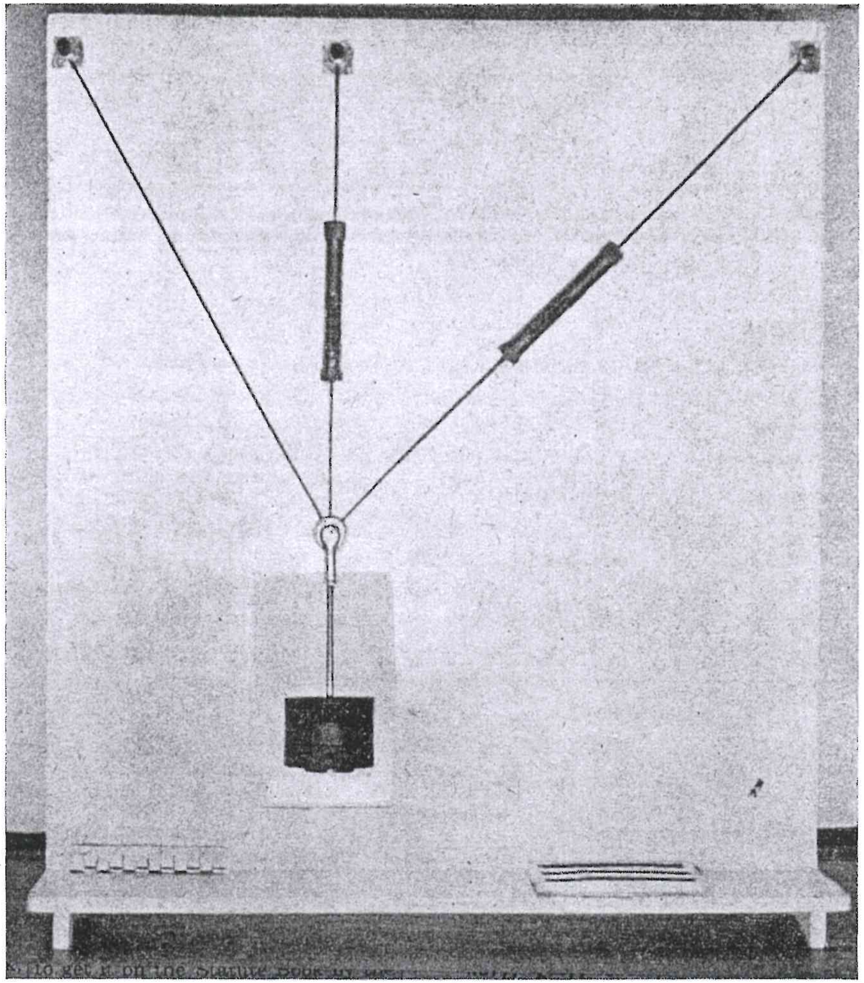


Fig. 4.4

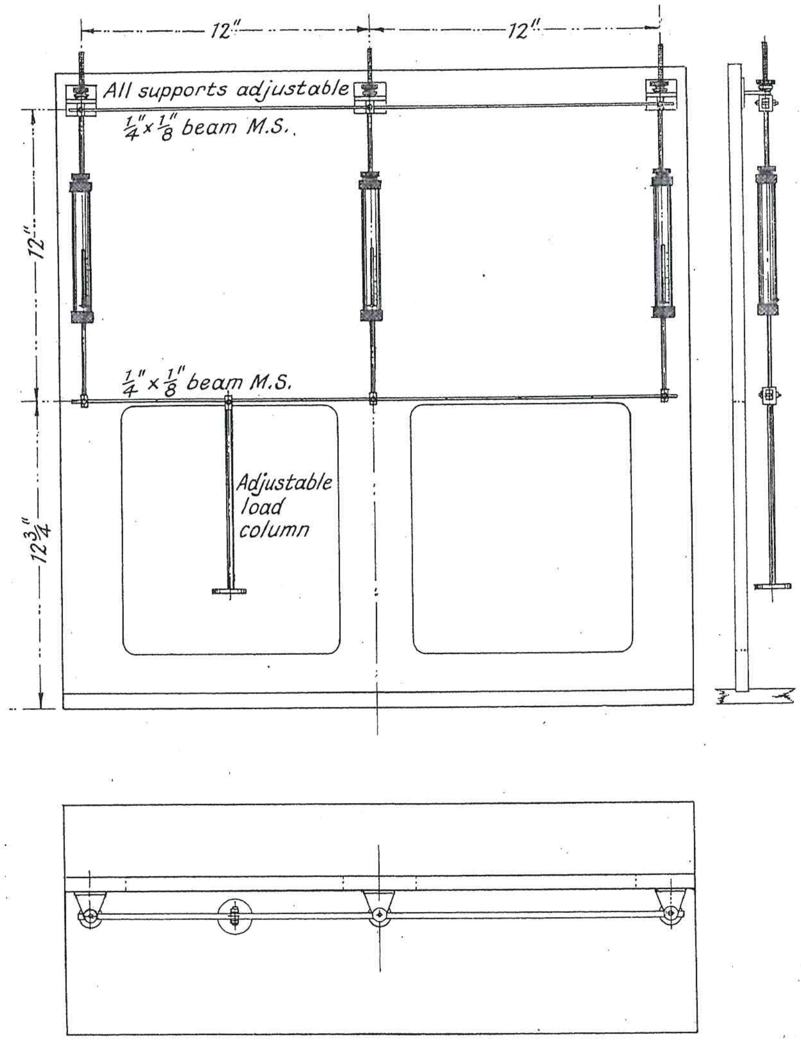

12"
12"
All supports adjustable
1/4" x 1/8" beam M.S.
12"
1/4" x 1/8" beam M.S.
Adjustable
load
column
12 3/4"

Fig. 4.5

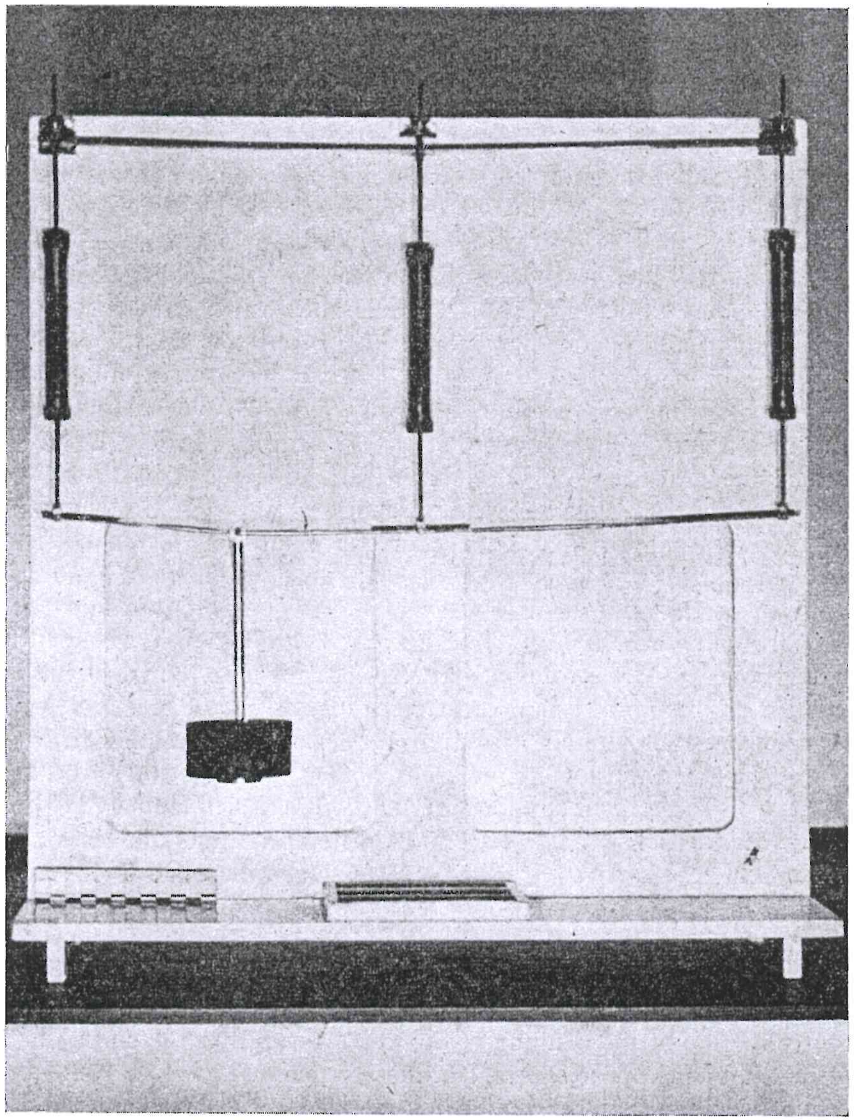


Fig. 4.6

and it is convenient to have springs in all three members so that all the forces can be measured. Another and more important point which the experiment illustrates is in connexion with the displacements of the loaded point. It is a common and erroneous assumption—usually by implication rather than by statement—that the displacement of a loaded point is in the direction of the load and this error cannot be appreciated visually unless the displacements are of sufficient size to make it obvious. It is advisable therefore for the student to calculate the vertical and horizontal displacements of the loaded point when one of the outside rods is rigid compared with the other two, *i.e.* when it is not provided with a spring. The horizontal displacement of the point under a vertical load is then very evident in the experiment and once it has been seen it is unlikely that it will be forgotten. The springs are easily inserted and a number, of varying stiffnesses, should be provided. Displacements are measured with sufficient accuracy by an ordinary scale. Experiment Sheet No. 14 gives both analytical and experimental results.

Beam on Multiple Elastic Supports. Another problem in redundancy is provided by the elastic beam carried on elastic supports; this introduces the idea of strain energy due to bending as well as that due to direct tensions and compressions and the apparatus shown in Figs. 4.5 and 4.6 is devised to give several variations of the problem. It consists of two equal flexible beams of light steel bar connected together by three vertical suspension rods similar in design to those just described in the three-wire suspension experiment. The upper beam is connected to three fixed supports at the points of attachment of the verticals in such a way as to allow different support conditions. In the first place if the control screws shown on the drawing are tight the upper beam is inoperative and the problem is simply that of a beam on three elastic supports. By adjusting the screws any support point can be allowed to sink by a known amount and the problem is complicated by this and by the introduction of the bending of the upper beam into the equations. Finally, one support can be completely released and the problem then becomes that of two beams coupled by elastic connexions and simply supported. A weight can be applied to any point on the lower beam and loads and deflexions can be measured for comparison with calculations made in the exercise class. Experiment Sheets Nos. 15 and 16 give examples.

Experimental Demonstration of Moment Distribution. The apparatus now to be described was designed to illustrate the principle of moment distribution which has been previously outlined in Chapter 1. It is shown in detail in Figs. 4.7 and by a photograph in Fig. 4.8.

The case is that of a continuous beam encastré at the ends and freely supported over two intermediate supports. The beam is made of flexible strip steel and to allow the development of large deflexions it is allowed to slide freely through the end supports, but is restrained in direction. The intermediate supports are in the form of bearings which are quite free to rotate or can be clamped at will. Spring controls are provided for this purpose at the back of the model; normally each bearing is clamped, but pressure on a trigger releases it and the beam can then assume its unrestrained shape. The slopes of the beam at the supports are indicated by pointers on graduated scales. A weight can be hung from any point of the beam and the shape of the bent strip is made visible to a class by light pointers attached at close intervals. The strip is first adjusted without any load and all supports are clamped; the load is then placed at any point on the beam. This corresponds to the first state assumed in calculation; all members are encastré and unbalanced moments at every support are carried by the external clamps. One intermediate joint is then released by pressure on the trigger, and the moment carried by the clamp is distributed to the adjacent joints. The clamp having assumed a new equilibrium position, indicated by a big movement of the pointer, is then locked by releasing the trigger. The other intermediate joint is next released and the clamp moment is distributed to the adjacent supports. The pointer attached to it indicates the change of angle undergone in this process. This joint is re-clamped and the first one is again released. A much smaller movement of the pointer indicates that the unbalanced moment to be distributed was correspondingly less than before. The second joint is again released and this is accompanied by a small angular change. The third release causes nearly imperceptible movements, showing that balance has been practically reached by two relaxations of the clamps, and the system is in equilibrium without the application of external moments. The various steps are shown in the photograph of Fig. 4.8.

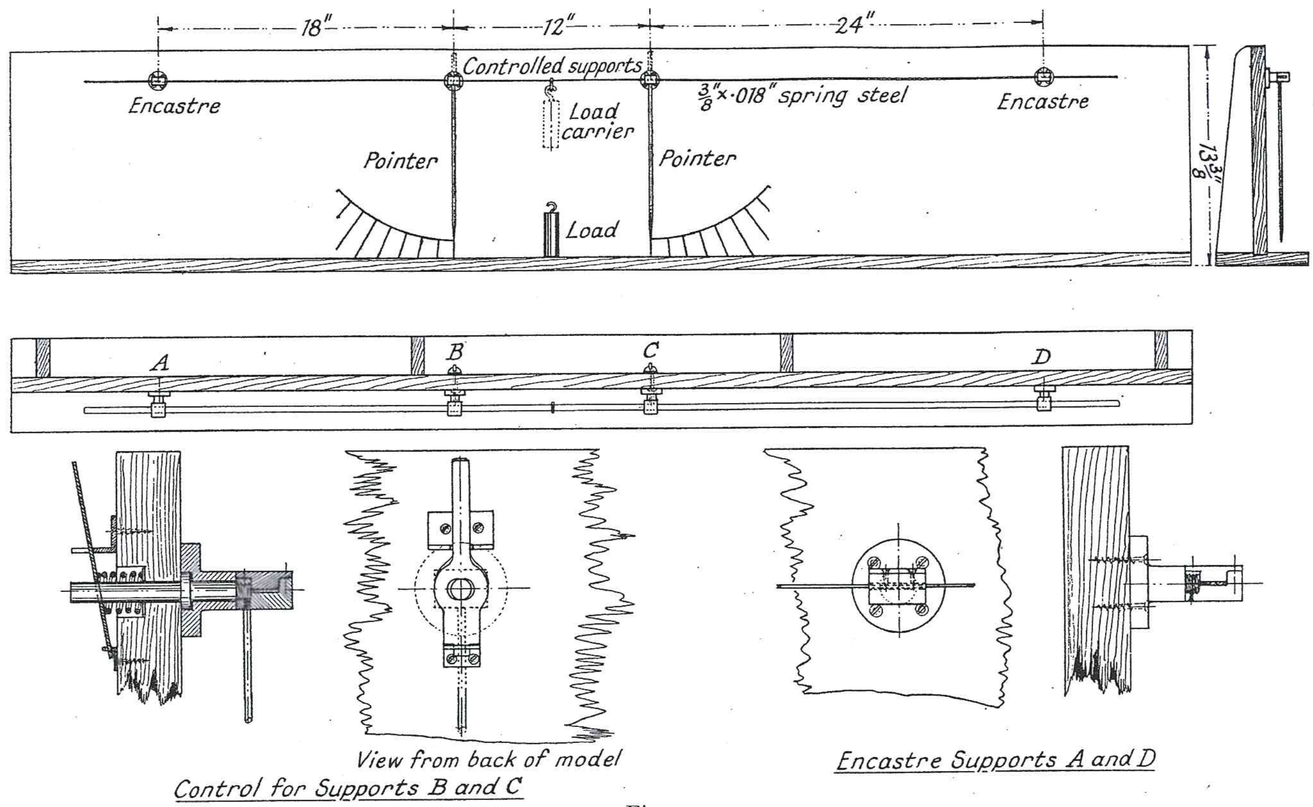

18"
12"
24"
Controlled supports
Encastre
Load carrier
3/8" x ·018" spring steel
Encastre
Pointer
Pointer
Load
13 3/8"
A
B
C
D
View from back of model
Control for Supports B and C
Encastre Supports A and D

Fig. 4·7

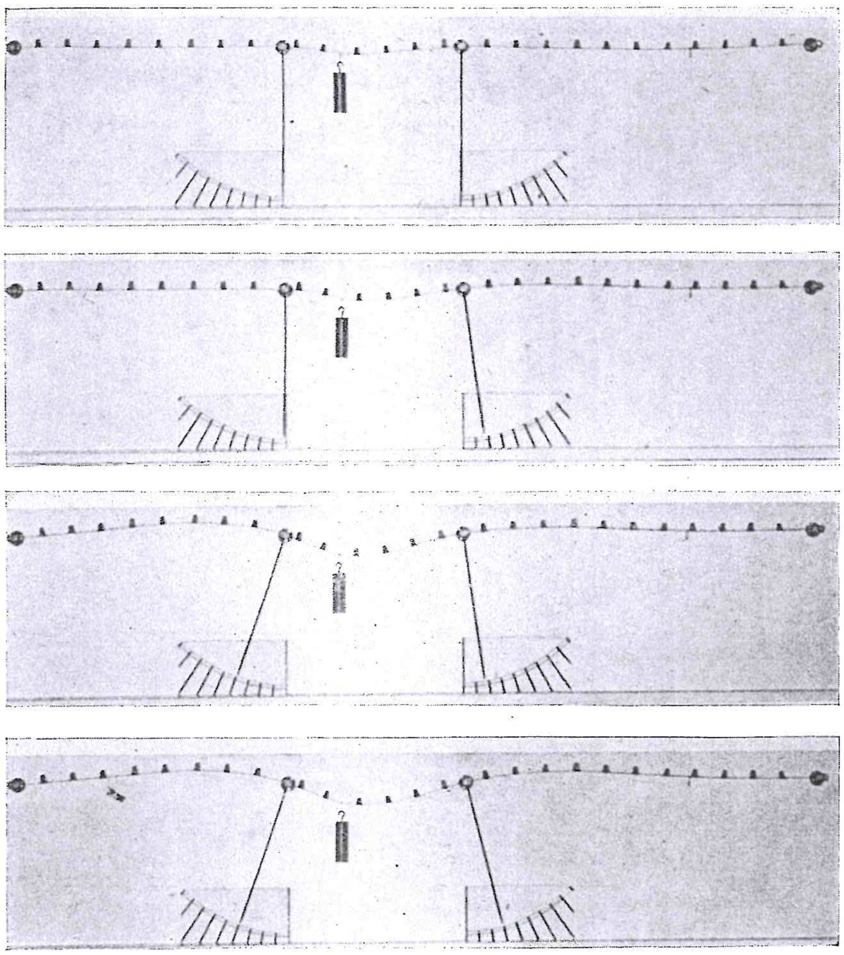


Fig. 4.8

Side Sway in Portals. The presence of side sway in a portal frame was mentioned in Chapter 1 and the flexible model shown in Figs. 4.9 and 4.10 demonstrates it clearly. The portal is made of flexible steel strip, the feet being encastré. The top joints of the frame are supported by removable pins. The pins are inserted and a load is hung at any point on the horizontal member; the pins now take any shearing force which exists. When the pins are removed the portal is no longer in equilibrium unless the load is central and it sways into its new stable position, the amount being indicated by a pointer at the centre of the cross-member.

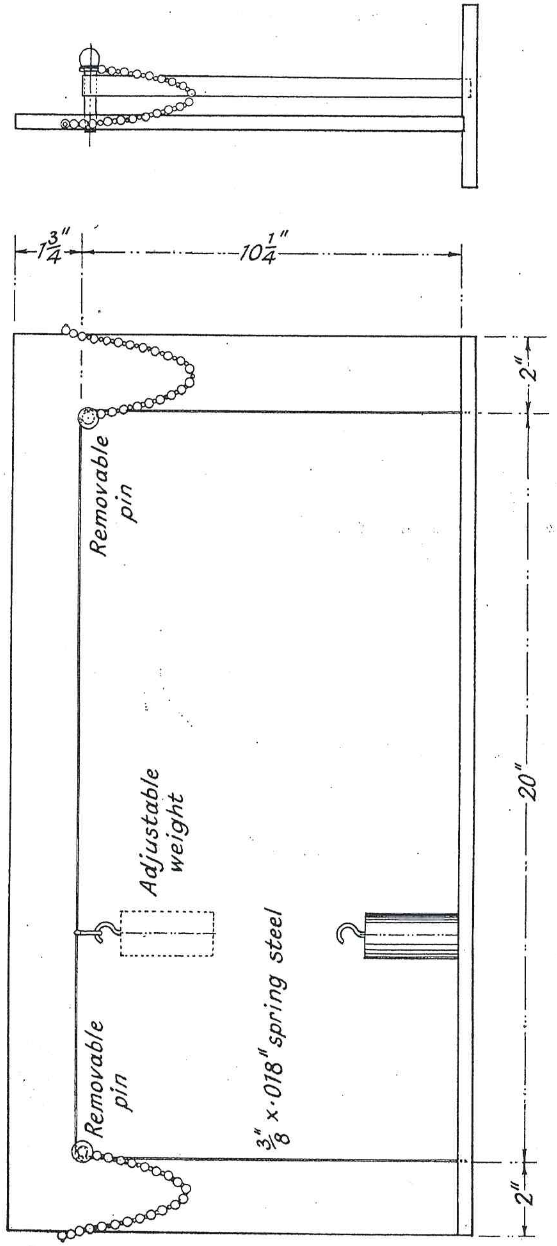


Fig. 4.9

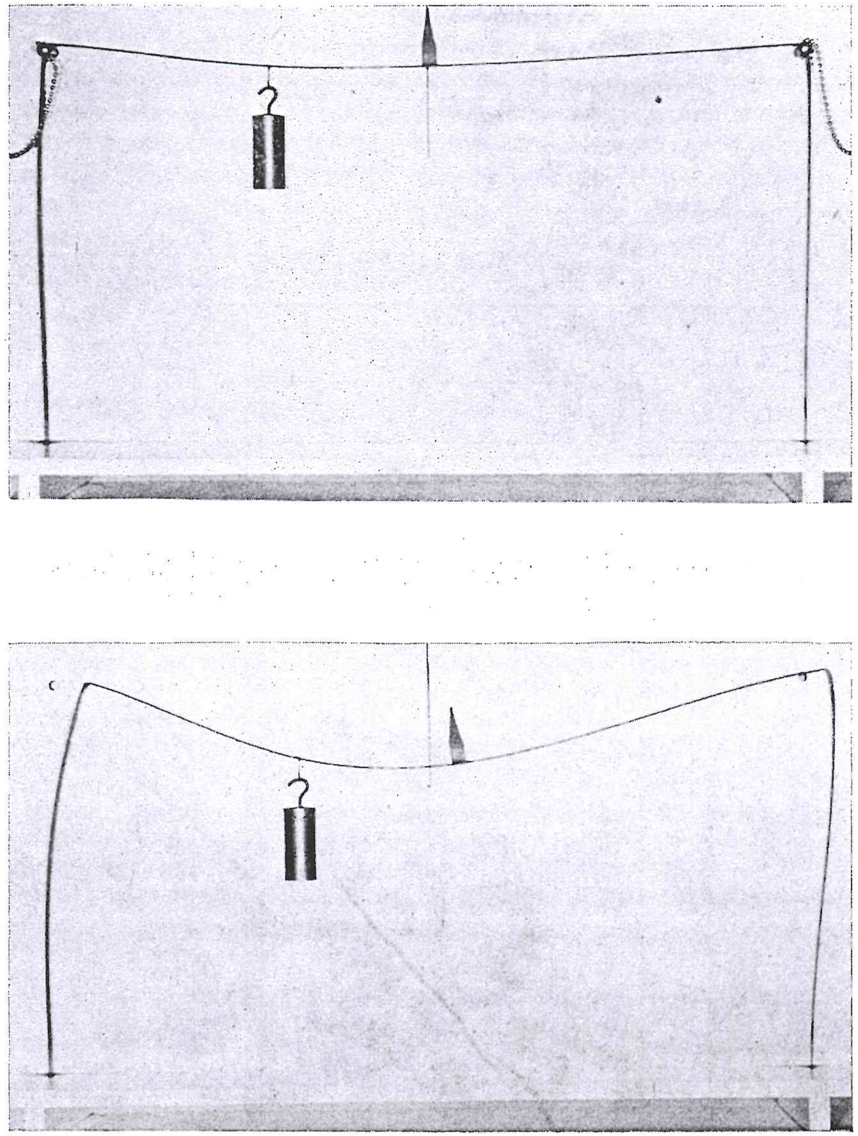


Fig. 4.10

Interconnected Bridge Girders. When a bridge has more than two main trusses the cross-girders distribute loads placed on the deck to these trusses in proportions which cannot be calculated from purely statical considerations. The actual analysis of the general problem is too advanced for the average undergraduate [15] but a general idea of the effects of interconnexion of the trusses can be obtained by the simple model shown in Figs. 4.11 and 4.12. This consists of four beams cross-connected by members riveted to them, the structure being sufficiently flexible for deflexions of the beams to be measured by an Ames' dial with considerable accuracy. If a load is placed anywhere on the model and the deflected forms of the beams are plotted the distribution due to interconnexion is appreciated. Such a set of curves are given in Fig. 4.13 and calculated points are also shown for comparison. A further example is given in Experiment Sheet No. 17.

A much simpler model, shown in the diagram of Fig. 4.14, is a good introduction to the type of problem which is more elaborately illustrated by that just described. This is the case of three equal beams connected by a single cross-piece of the same dimensions as the beams and its analysis should be well within the capacity of the student. If a load be placed at A on the diagram the calculated deflexion at B is zero and this is evident from the model.

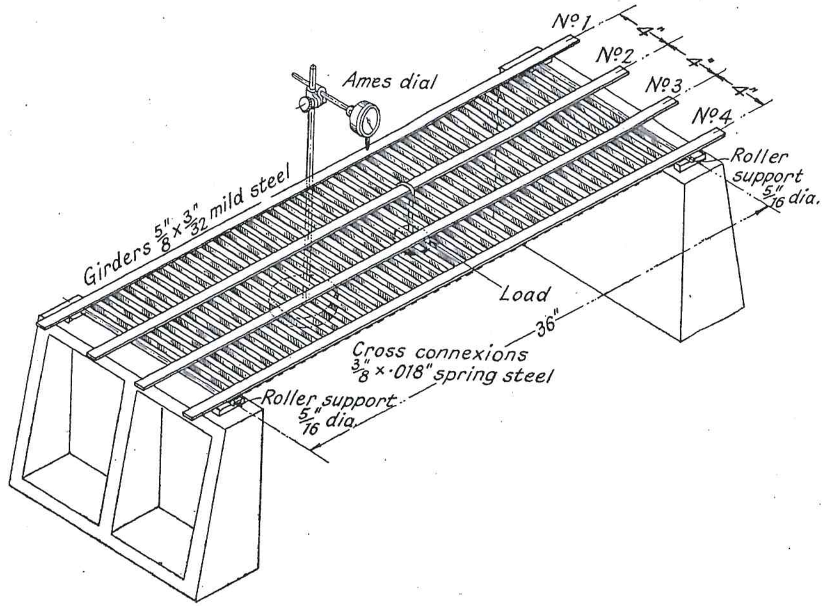


Fig. 4.11

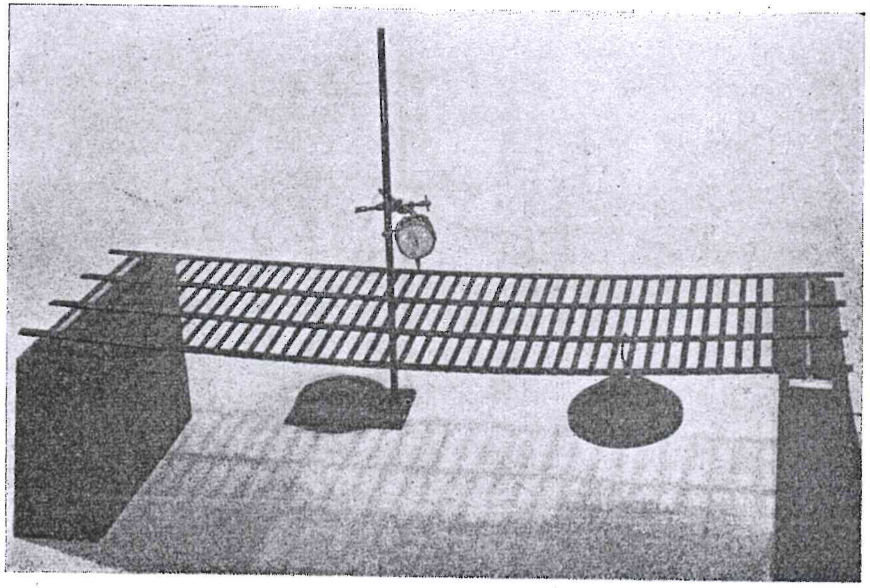


Fig. 4.12

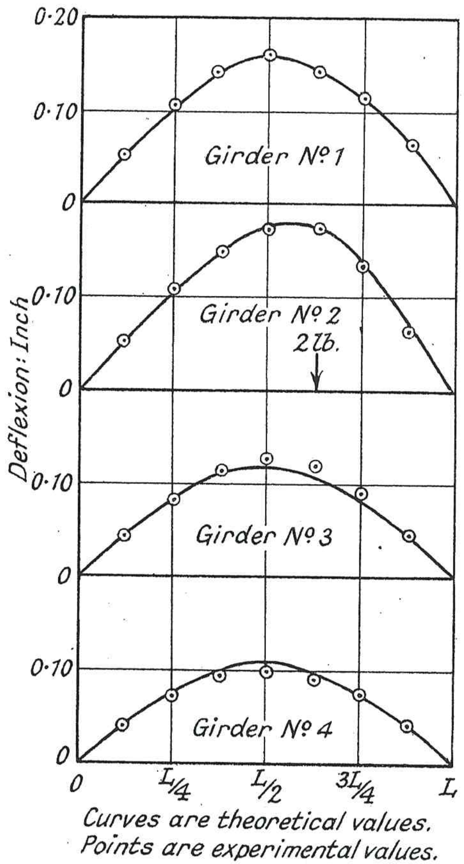


Fig. 4.13

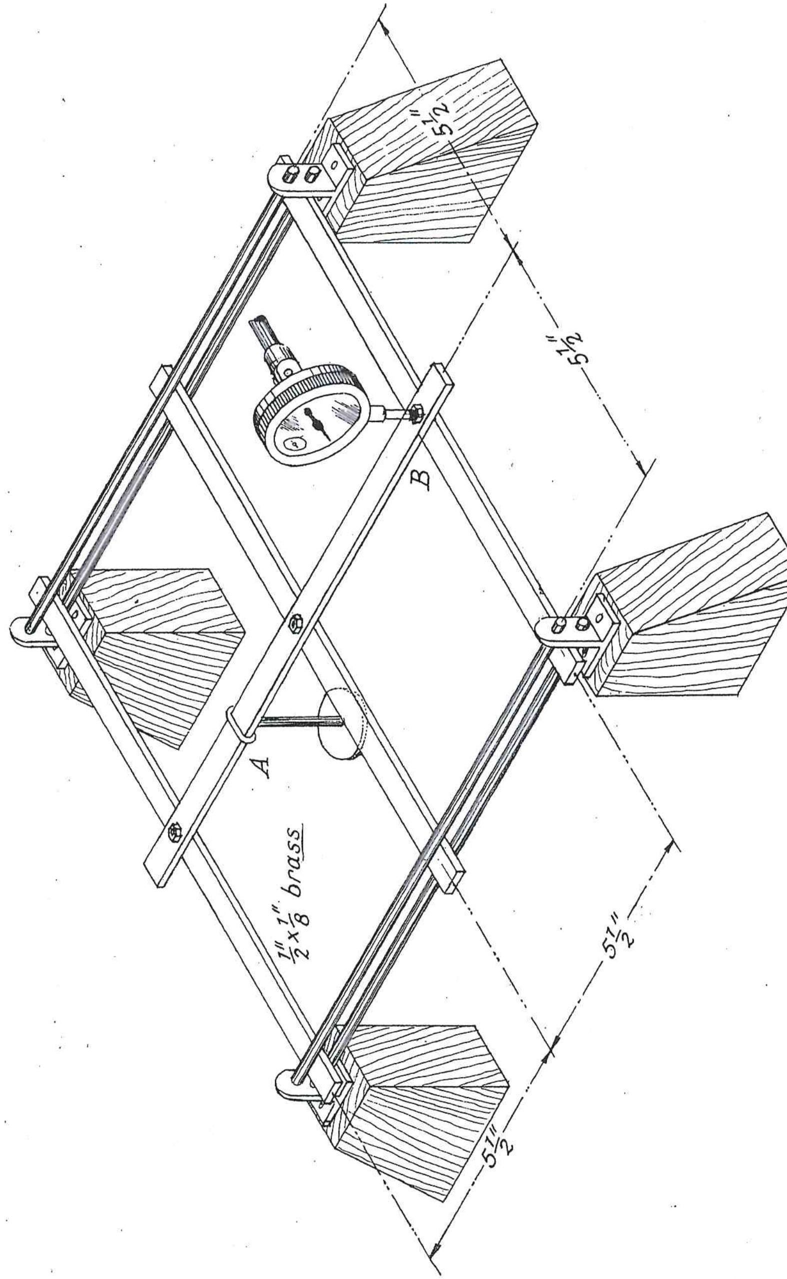


Fig. 4.14

Experiment Sheet No. 14

STRESSES AND DISPLACEMENTS IN THREE-WIRE SUSPENSION

Instructions

Using the apparatus shown in Fig. 4.3 determine experimentally and analytically the forces in the three suspension members and the component displacements of the loaded point under a vertical load of 10 lb.

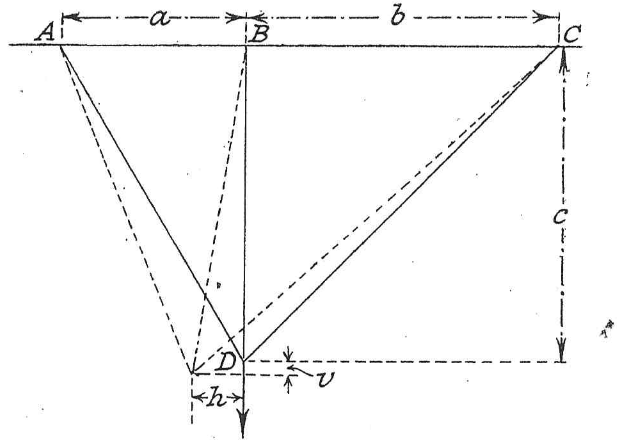
Results

1. *Experimental.*

AE values : AD$\longrightarrow\infty$ lb. Lengths : AD, 20·8 in.
BD=138·5 lb. BD, 18·0 in.
CD=124·2 lb. CD, 25·4 in.

T_{AD} not measured ; T_{BD}=2·70 lb. ; T_{CD}=3·31 lb.

Vertical displacement=0·35 in. ; horizontal displacement=0·62 in.



2. *Analytical.*

Let the vertical displacement of D=v
and the horizontal displacement =h.

If t is the tension coefficient in a member and Ω is AE/L^3 for any member

$$t_{AD}=(cv-ah)\Omega_{AD}$$
$$t_{CD}=(cv+bh)\Omega_{CD}$$
$$t_{BD}=cv\Omega_{BD}.$$

Equilibrium conditions at D give

$$at_{AD}-bt_{CD}=0,$$
$$c(t_{AD}+t_{CD}+t_{BD})=W,$$

and substituting for the tension coefficients the values above

$$v=\frac{W}{c^2}\left\{\frac{a^2\Omega_{AD}+b^2\Omega_{CD}}{(a+b)^2\Omega_{AD}\Omega_{CD}+\Omega_{BD}(a^2\Omega_{AD}+b^2\Omega_{CD})}\right\}$$

$$h=\frac{W}{c}\left\{\frac{a\Omega_{AD}-b\Omega_{CD}}{(a+b)^2\Omega_{AD}\Omega_{CD}+\Omega_{BD}(a^2\Omega_{AD}+b^2\Omega_{CD})}\right\}$$

In present case $\Omega_{AD}=\infty$; $\Omega_{BD}=0{\cdot}02375$; $\Omega_{CD}=0{\cdot}00755$; hence, $v=0{\cdot}38$ in.; $h=0{\cdot}66$ in.; $T_{AD}=5{\cdot}16$ lb.; $T_{BD}=2{\cdot}97$ lb.; $T_{CD}=3{\cdot}64$ lb.

Experiment Sheet No. 15

BEAM ON MULTIPLE SUPPORTS

Instructions

The top beam of the apparatus shown in Fig. 4.5 is rigidly clamped and a weight of 1 lb. is hung midway between the left-hand and middle suspension rods. Calculate the loads in all suspension rods and verify your results experimentally.

Results

Let the flexural rigidity of the beam be EI and the AE values for the rods be $(AE)_1$, $(AE)_2$ and $(AE)_3$, the load being midway between rods 1 and 2.

Using ordinary strain-energy methods of analysis it is found that the load in the centre rod, taken as the redundant element, is

$$R_2=\frac{\dfrac{11L^2}{96EI}+\dfrac{3}{8(AE)_1}+\dfrac{1}{8(AE)_3}}{\dfrac{L^2}{6EI}+\dfrac{1}{4(AE)_1}+\dfrac{1}{(AE)_2}+\dfrac{1}{4(AE)_3}}$$

where 2L is the total length of the beam and L is the length of each rod. The load in rod 1 is therefore $\frac{3}{4}-\frac{R_2}{2}$ and in rod 3 it is $\frac{1}{4}-\frac{R_2}{2}$.

The values of $(AE)_1$, $(AE)_2$ and $(AE)_3$ are found by direct calibration to be 88·3 lb., 59·2 lb. and 58·3 lb. respectively. EI can be determined experimentally or by measurement and is 1,220 lb.-in.^{2} Substitution gives the calculated values of the loads as

$$R_1=0{\cdot}522 \text{ lb.};\quad R_2=0{\cdot}456 \text{ lb.};\quad R_3=0{\cdot}022 \text{ lb.}$$

The corresponding values by experiment are

$$R_1=0{\cdot}515 \text{ lb.};\quad R_2=0{\cdot}460 \text{ lb.};\quad R_3=0{\cdot}020 \text{ lb.}$$

Notes

The total energy in the structure is the sum of the bending energy in the beam and that due to the extensions of the bars so

$$\frac{dU}{dR_2}=\frac{1}{EI}\int M\frac{dM}{dR_2}dx+\Sigma\frac{P_0L}{AE}\,\frac{dP_0}{dR_2}$$

where M is the bending moment at any point in the beam and P_0 is the load in any rod.

Experiment Sheet No. 16

BEAMS COUPLED BY ELASTIC CONNEXIONS

Instructions

The apparatus shown in Fig. 4.5 is loaded as in Experiment Sheet No. 15, but the central support of the top beam is completely freed.

Calculate the loads in all rods and verify your results experimentally. The top and bottom beams have the same values of EI.

Results

The load in the central rod is now found by strain-energy analysis to be

$$R_2 = \frac{\frac{11L^2}{96EI} + \frac{3}{8(AE)_1} + \frac{1}{8(AE)_3}}{\frac{L^2}{3EI} + \frac{1}{4(AE)_1} + \frac{1}{(AE)_2} + \frac{1}{4(AE)_3}}$$

and

$$R_1 = \frac{3}{4} - \frac{R_2}{2},$$

$$R_3 = \frac{1}{4} - \frac{R_2}{2}.$$

The calculated values are $R_1 = 0{\cdot}593$, $R_2 = 0{\cdot}315$, $R_3 = 0{\cdot}092$.

and the experimental values are $R_1 = 0{\cdot}589$, $R_2 = 0{\cdot}320$, $R_3 = 0{\cdot}061$.

Notes

The strain energy of the top beam must now be included; this beam is centrally loaded by a force R_2 and the total energy in it is $\left(\frac{R_2L^3}{6EI} \times \frac{R_2}{2}\right)$. Hence $\frac{dU}{dR_2}$ for the top beam is $\frac{R_2L^3}{6EI}$, and this term must be added to the expression given in Experiment Sheet No. 15.

Experiment Sheet No. 17

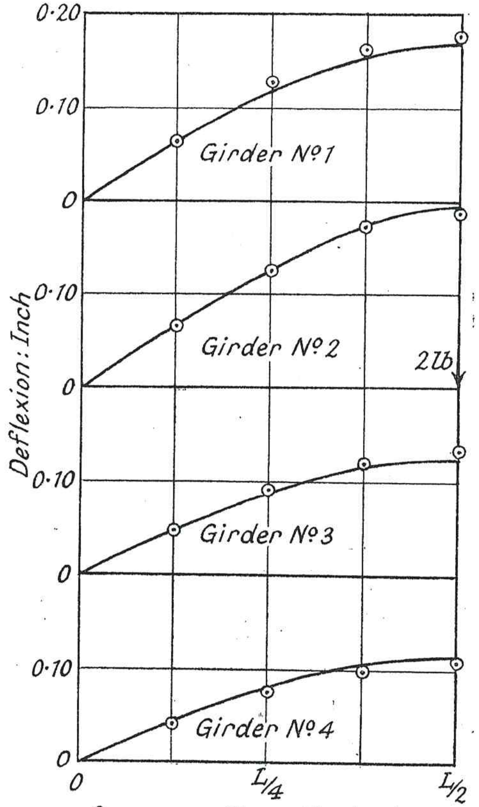
INTERCONNECTED BRIDGE GIRDERS

Instructions

Using the apparatus shown in Figs. 4.11 and 4.12 determine the deflected form of all four girders when a load of 2 lb. is placed at the centre point of one of the two middle girders.

Results

The experimental points are shown below together with the calculated values of the deflexions.



Curves are theoretical values.
Points are experimental values.

Notes

The calculations may be rather involved for undergraduate students, but the necessary equations are given in the paper under Reference 15.

CHAPTER 5

THE EXPERIMENTAL STUDY OF ARCHES

The Linear Arch. The theoretical study of the arch, whether it is a continuous rib or of the voussoir type, may be approached in two ways. The first or direct method is to emphasize the essential difference between a curved beam which has statically determinate reactions and the same beam which is prevented either entirely or partly from spreading by the provision of redundant reactions at the abutments. These redundant reactions introduce bending moments of opposite sign to those produced in the simple beam and the resultant numerical value of the moment at the critical section is thus reduced.

The second method of approach is by analogy from the flexible cable loaded similarly to the arch. A cable carrying suspended weights is in stable equilibrium and the external loads are resisted by direct tensions in the sections of the cable. If the configuration of the cable is inverted the linear arch is obtained and if its sections are replaced by pin-jointed compression members the system is in equilibrium (but now unstable) and the external loading is carried by a system of rods in direct compression. There is no bending at any section and so the best theoretical configuration is obtained.

The simplest case is that of a uniformly distributed load covering the whole span of a flexible cable which is easily proved to assume a parabolic shape. The inversion of this to form a parabolic arch results in a rib which is everywhere in compression and free from bending action.

Both these lines of approach are instructive and should be followed and experiments to illustrate each of them are desirable.

The experimental approach to the direct method is by the measurement of the thrust in a rib by allowing one end to slide or roll freely when loads are applied and to restore it to its original position by the application of a measurable force (preferably by dead weights). This is simply done and the apparatus

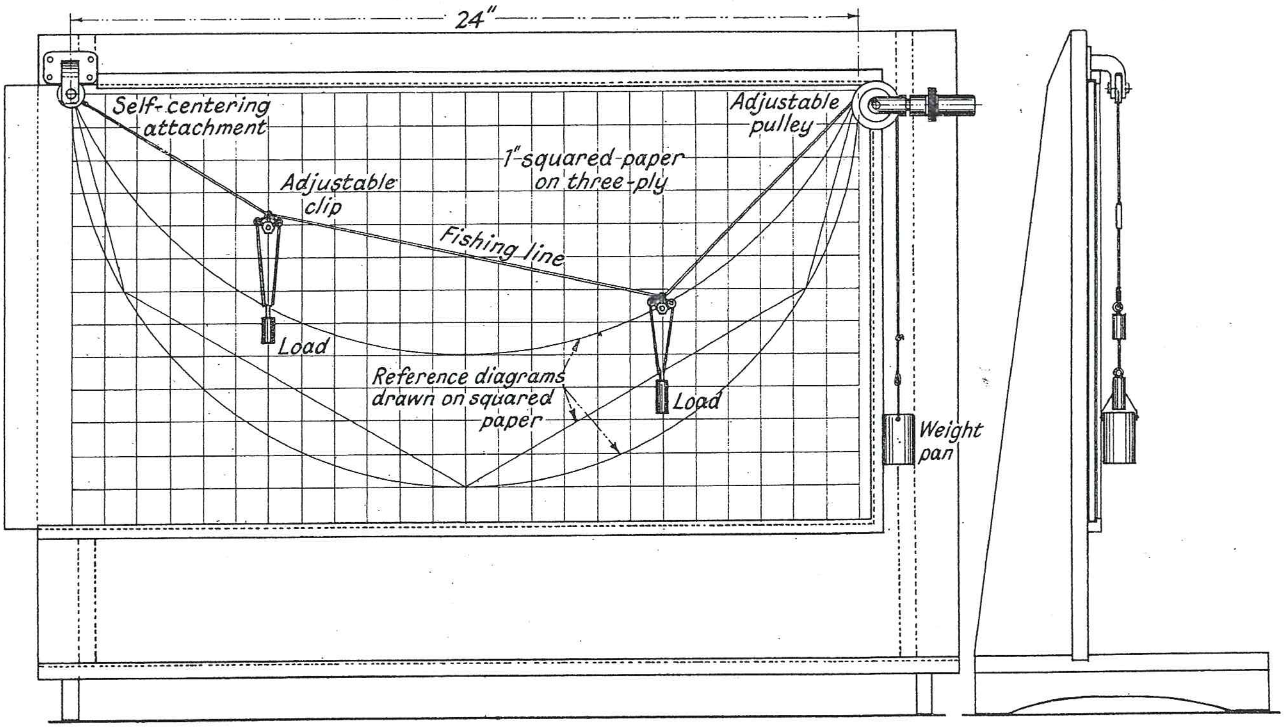

24"
Self-centering attachment
Adjustable clip
1" squared-paper on three-ply
Fishing line
Adjustable pulley
Load
Load
Reference diagrams drawn on squared paper
Weight pan

Fig. 5.1

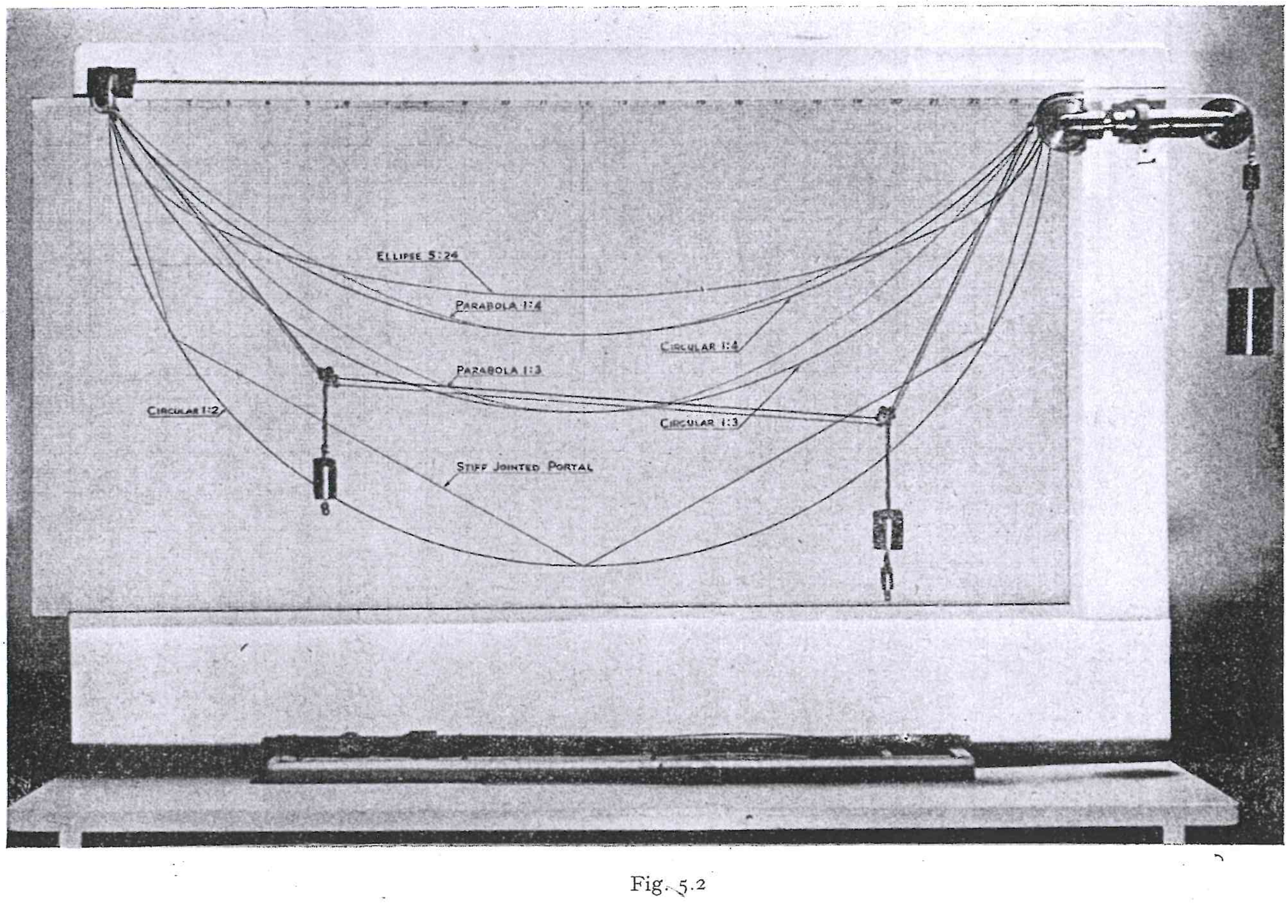

Ellipse 5:24
Parabola 1:4
Circular 1:4
Parabola 1:3
Circular 1:2
Circular 1:3
Stiff Jointed Portal

Fig. 5.2

can be made to a small scale, but it is easier to obtain accurate results on a larger arch as small absolute errors in the final position of the support are less important. In the Imperial College laboratory, therefore, this experiment is made on a rib of 6 ft. span which yields excellent agreement between theoretical and experimental results.

The second method of approach, *i.e.* by means of the linear arch, is, however, very suitable for small scale apparatus and the model shown in Figs. 5.1 and 5.2 has been designed for this purpose.

A length of fishing line is attached at one support and is led over a frictionless pulley (shown in Fig. 5.3) at the other to a weight pan (the photograph in Fig. 5.2 shows two pullies; the second was originally introduced for a special purpose and is not required for the present experiment). If loads are hung at any points on the string and a load is placed in the pan the string will assume its equilibrium position and will be the linear arch for that system of loading. If the balance weight is changed a new equilibrium position will follow and if the loads on the string are moved so as to act in the same vertical lines as before, the two configurations will represent two linear arches for the same applied loads but with different values of horizontal thrust. To enable the loads on the string to be adjusted in position they are carried on small hangers threaded on the string as shown in Fig. 5.3. In the model shown the hangers are made to weigh exactly 1 gm. The weights used were specially made and can

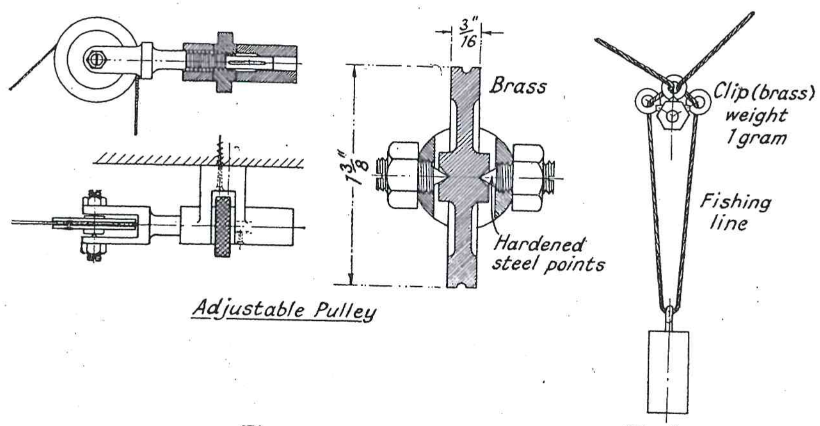


Fig. 5.3

be added in a chain as shown in Fig. 5.2. The balance weight is provided by shot carried in a small container so that the exact weight required can be obtained.

A sheet of squared paper mounted on three-ply is carried in slides on the back board of the model and on this paper, which serves as a reference sheet, a number of arch ribs are drawn in outline. The supports of these arches must coincide with the ends of the linear arch as given by the string and the pulley over which the string is led is adjustable horizontally to make this possible.

Students are given for analysis a particular arch rib, *e.g.* a parabolic outline of 1 : 3 rise to span ratio, carrying point loads at specified horizontal distances from one end. The horizontal thrust is calculated and the bending-moment diagram is drawn. Since the vertical reaction V and horizontal thrust H at each end of the arch are known the tensions in the end sections of the linear arch, $\sqrt{V^2+H^2}$, can be calculated.

The apparatus is now used as follows: loads proportional to those assumed in the calculation are attached to the hangers and roughly adjusted for position. The appropriate value of $\sqrt{V^2+H^2}$ is carefully weighed with lead shot in the pan. The string then assumes its equilibrium position and the loads are adjusted by trial and error until they are exactly in the specified positions; the string then gives the correct linear arch for the assumed conditions and can be transferred to the students' note-books by using the squared paper background; the forces in the different sections are then easily obtained by calculation. The distance between the linear arch thus drawn and the centre line of the rib at any section when multiplied by the appropriate force in the linear arch is the bending-moment at that section. This, when plotted, should be compared with the diagram already obtained by calculation. The simplest experiment with this apparatus and one that is useful for ensuring that it is working correctly without undue friction, is to take the case of an arch pinned at the abutments and also at any other point. This is statically determinate and the balance weight is very quickly determined. When the string is properly loaded it should pass through the position of the third pin since there is no bending moment at this point.

Examples of the use of the apparatus are given in Experiment Sheets Nos. 18 and 19.

The Voussoir Arch. Experimental studies of the behaviour of the voussoir arch have been fully described elsewhere.[16] The models used for this work ranged from a span of 4 ft. to one of 10 ft. and are too large and too expensive for many laboratories. Reasonably good quantitative results can, however, be obtained by the use of the small-scale models shown in Figs. 5.4, 5.5 and 5.6. The voussoirs are made of bearing metal and are cast as solid wedges. The arch represented is supposed to carry a filling to make up to a horizontal road-way and this means a heavier load on the haunch voussoirs than on those at the crown. The voussoirs are adjusted by lightening holes as shown, so that the weight of any one of them represents the total weight of filling supported on it together with its own weight ; the haunch voussoirs are solid and the lightening holes increase in diameter as the crown is approached.

To stabilize the arch against lateral collapse small holes are drilled in the bearing faces of all voussoirs and short lengths of circular twisted gut are used as loose dowels between adjacent voussoirs ; these do not affect the behaviour of the arch in its own plane and in addition to the primary object of lateral stabilization they make the erection of the model very simple. A light guard rail below the soffit of the arch prevents the complete disassembly of the model when failure occurs.

Two models are shown : one is provided with pin-joints at the springing and the other with skew-backs as in an actual structure. The supports are adjusted to give the exact span by means of a screw which enables one of them to be moved horizontally through a small distance. Small eyes are screwed into the intrados of each voussoir so that a load container can be hung to any one of them. The load is applied to the selected voussoir by lead shot which may be poured from a can.

The arch is correctly assembled and shot is slowly poured into the container. At a certain load, which can be calculated, the voussoir to which the container is attached and the one next to it towards the crown will be found to have separated at the intrados and be bearing on each other at the extrados, thus forming a virtual pin. As the load is increased, a second virtual pin will form at the intrados somewhere in the other haunch. If the model used is that with the pin supports, the structure has now been transformed into a three-bar mechanism and will collapse ; this is the true instability load of the arch. If, however,

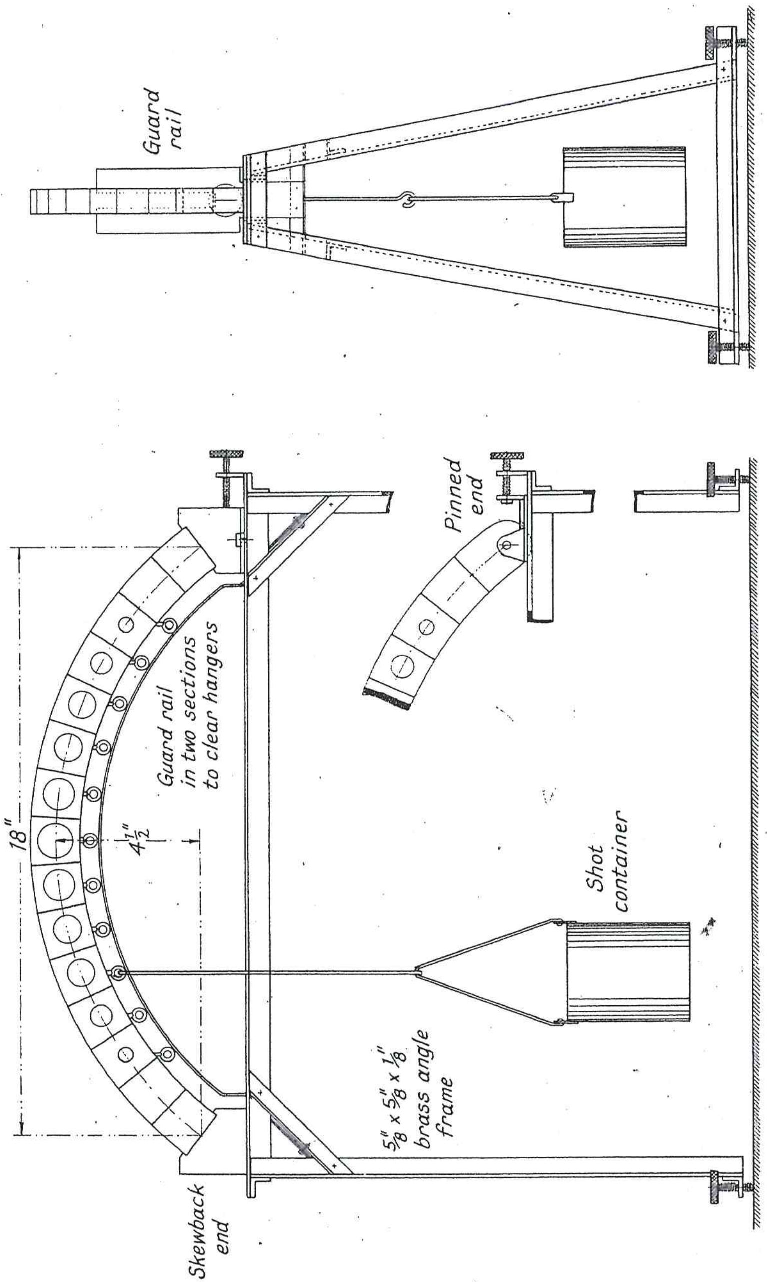


Fig. 5.4

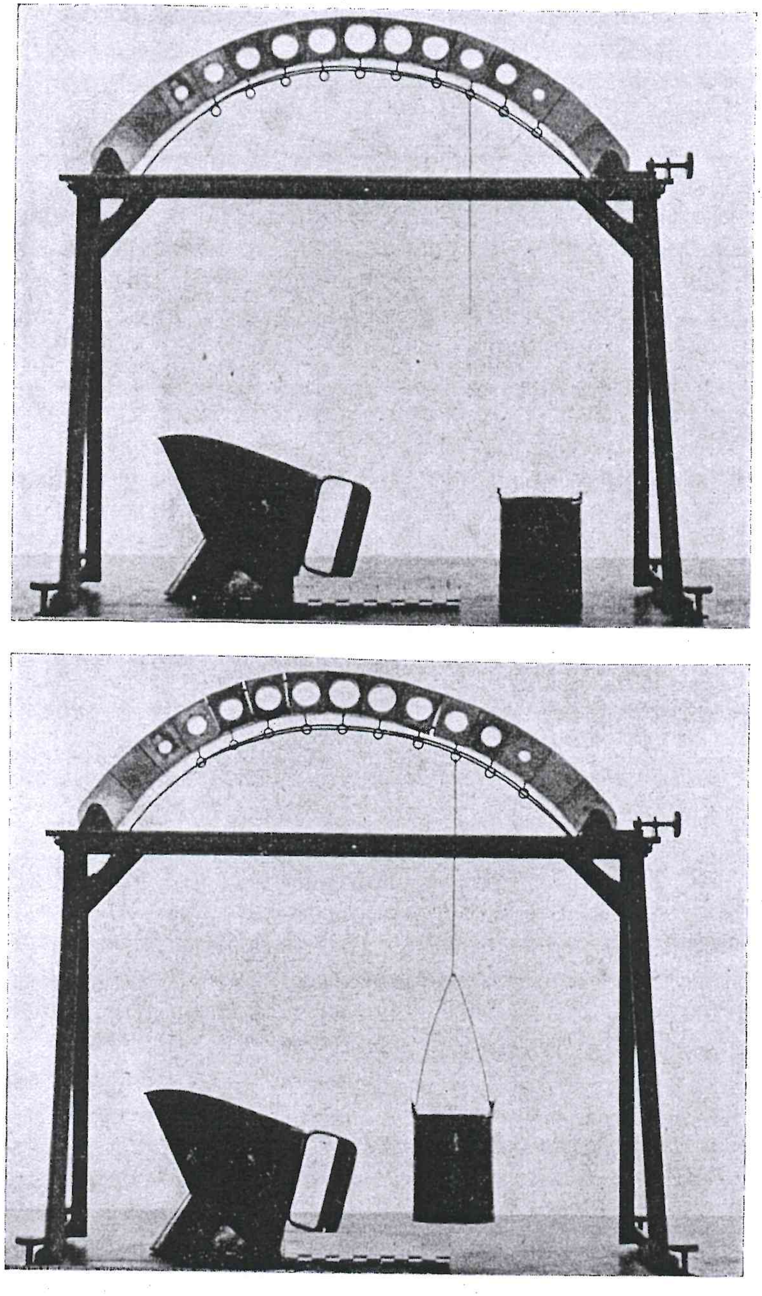


Fig. 5.5

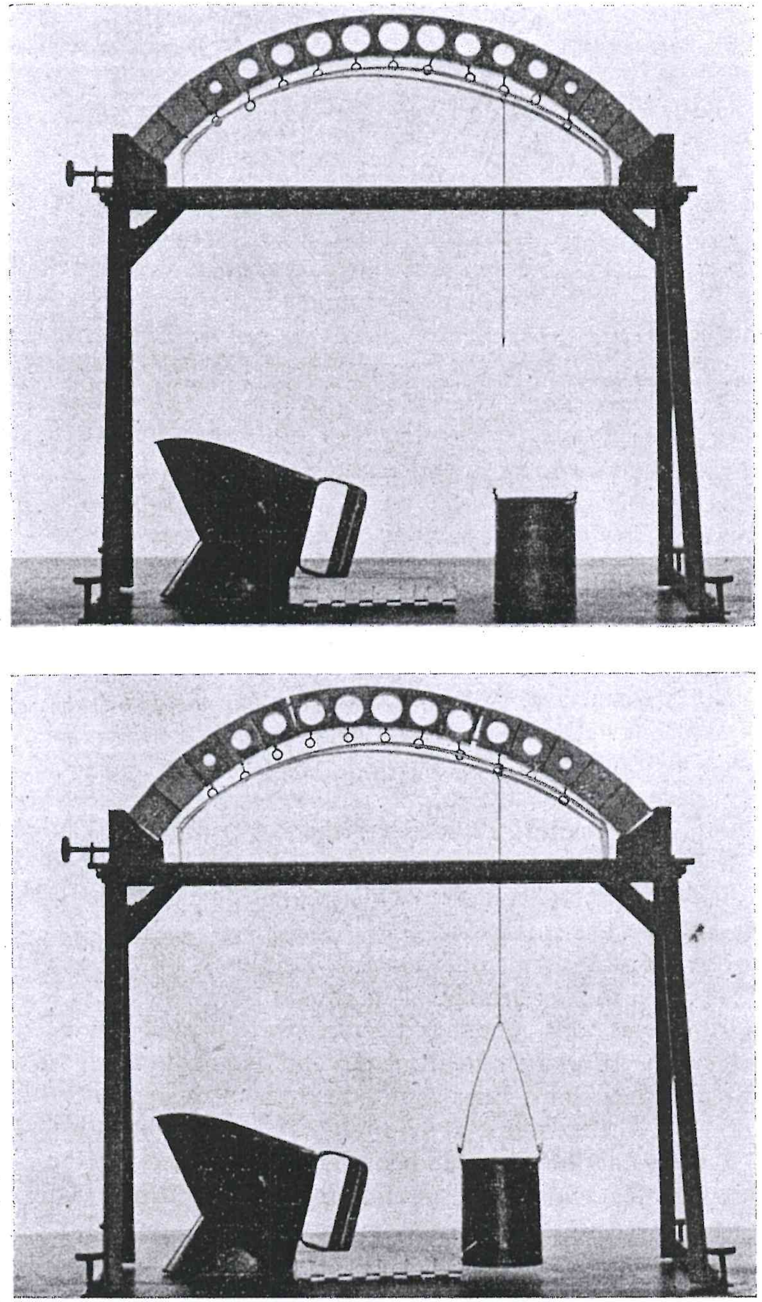


Fig. 5.6

the skew-back supports are being used the structure consists of a pair of cantilever portions, one from each skew-back, pin-connected by a central curved member. This is a perfectly satisfactory theoretical structure and will carry still more load. Ultimately, however, the voussoirs at the springings will hinge on the skew-backs and a mechanism will develop which will collapse. The various loads and positions of pins can be calculated and compared with those obtained experimentally.

These models show the first pin very clearly, but instead of a single second pin appearing it is found that two or three develop at adjacent joints practically simultaneously. The explanation for this is readily seen if the linear arch is drawn, as it will be found to follow closely the intrados of the arch for a distance of two or three voussoirs, showing that pin joints do actually tend to form simultaneously in this region.

This model, even if used only qualitatively, emphasizes the different stages in the behaviour of the voussoir arch:

(*a*) It is initially a solid arch rib.

(*b*) It consists of two curved cantilevers pin connected (one pin).

(*c*) It consists of two curved cantilevers pin connected by a curved member (two pins).

(*d*) It becomes a mechanism and collapses.

There is theoretically another stage between (*c*) and (*d*) when the arch consists of a cantilever section and two pinned sections. This is so near to complete collapse, however, that it is not detectable on the small model.

A comparison of calculated and failing loads for the pin-ended arch is given in Experiment Sheet No. 20.

A simple wooden model for demonstration purposes is shown in Fig. 5.7. It consists of three circular segments made of wood with imaginary joint lines painted on them. These are hinged together at the extrados at one joint and at the intrados at the other, the joints being made complete by fasteners on the opposite face. The ends of the model are pinned. If one fastener is released the arch is a two-member structure pinned to fixed supports, theoretically sound and statically determinate. If the other fastener is released the model collapses as shown in the figure.

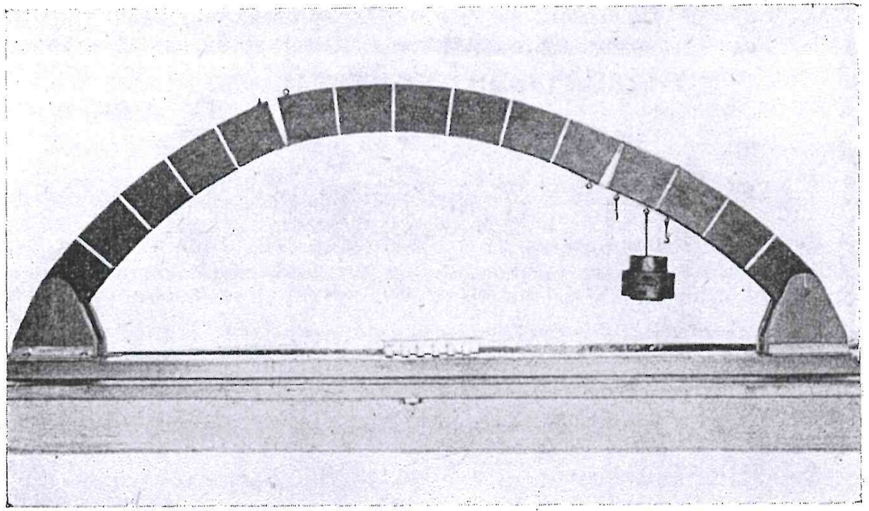


Fig. 5.7

The linear arch model previously described can be used to illustrate the behaviour of the voussoir arch, but it is necessary to use a weighted string to represent the weight of voussoirs and fill instead of the light line which is sufficient for an arch rib. A correctly graded weighting is not easy to achieve, but a chain of uniform weight will serve to show the main principles.

Experiment Sheet No. 18

THREE-PINNED SEGMENTAL ARCH

Instructions

A segmental arch rib pinned at the supports and crown has a span L and a rise L/4. Equal vertical loads are carried at distances a and b from the left-hand support. Calculate the magnitude of the resultant reactions at the right-hand support and check them experimentally, using the model shown in Fig. 5.1, for the following cases:—

(1) $a=0{\cdot}25L$; $b=0{\cdot}75L$.

(2) $a=0{\cdot}125L$; $b=0{\cdot}625L$.

Results

By taking moments about the left-hand support and about the crown pin the vertical and horizontal components of the required reaction are found to be

$$V=\frac{W}{L}(a+b),$$

$$H=\frac{2W}{L}(L+a-b).$$

The resultant reaction is then $\sqrt{H^2+V^2}$.

This reaction is equal and opposite to the force in the last link of the linear arch and is therefore the load which should theoretically be placed in the pan of the model.

The string was loaded with two equal weights approximately in the specified positions; shot was then placed in the pan and the loads correctly adjusted until the experimental linear arch intersected at mid-span the line diagram of the rib drawn on the squared paper reference sheet. This indicated zero bending moment at that point and the load in the pan was therefore the experimental value of the reaction. The results were as given below.

Experiment	W gm.	a/L	b/L	Reaction	
				Experimental	Calculated
1	31	0·25	0·75	44	43·8
2	41	0·25	0·75	58	58·0
3	51	0·25	0·75	72	72·1
4	31	0·125	0·625	39	38·8

Experiment Sheet No. 19

TWO-PINNED PARABOLIC ARCH

Instructions

A two-pinned parabolic arch rib having a rise to span ratio of 1/3 carries a central concentrated load. Using the apparatus shown in Fig. 5.1 determine the points in the rib where the bending moment is zero. Verify your result by calculation.

Results

By the usual methods of analysis the magnitude of the horizontal thrust H is found to be $\frac{75}{128}$W where W is the central load on the rib. Since the vertical reaction V is 0·5W the resultant reaction at each support, $\sqrt{V^2+H^2}$, is 0·77W.

A load of 52 gm. was hung at mid-span on the string of the apparatus and 0·77×52 gm., *i.e.* 40 gm. was placed in the balance pan. The resulting linear arch intersected the line diagram of the rib, drawn on the back board, at 3·5 in. from each side of the mid-span. The experimental points of zero bending moment are therefore 8·5 in. from the supports, *i.e.* at 8·5/24 or 0·354 of the span.

The equation for the bending moment in the rib measuring x from one support is

$$M=\frac{Wx}{32L}(9L-25x)$$

where L is the span.

This is zero when $x=0{\cdot}36L$ and the experimental result is thus practically exact.

Experiment Sheet No. 20

TESTS ON VOUSSOIR ARCH

Instructions

Using the pin-ended voussoir arch shown in Fig. 5.4, determine the magnitude of the loads which must be applied to each of the voussoirs in turn to cause collapse of the structure. Compare your experimental results with theoretical values.

Results

The voussoirs are numbered for reference to left and right of the keystone respectively. Their weights in ounces were found to be as follows :—

7L	6L	5L	4L	3L	2L	1L	C	1R	2R	3R	4R	5R	6R	7R
6·75	6·8	6·4	5·65	5·15	4·65	4·25	3·8	4·20	4·65	5·20	5·70	6·55	6·80	6·90

The span was adjusted to 18 in. with a rise of 4·5 in. The full depth of the voussoirs was 1·45 in., but to allow for the effects of wear on the edges an effective depth of 1·40 in. was assumed.

The failing loads in ounces were found for each voussoir loaded in turn and the mean values for corresponding points are tabulated below with the calculated values and the positions of the virtual pins.

Load at voussoir ...	1	2	3	4	5
Experimental failing load	98·0	58·5	47·5	40·5	40
Calculated failing load ...	128·0	65·5	48·0	41·5	42
Pins formed	0–1R 3–4L	1–2R 3–4L	2–3R 2–3L	3–4R 2–3L	4–5R 2–3L

Notes

This arch cannot collapse through instability when loaded at the keystone, but it might fail by slipping at a very large load. When the load is applied to either of the voussoirs 1 or 2, slip occurs and premature instability is produced at a much lower load than calculated on the basis of the formation of virtual pins. The method of calculation is given in the second paper of Reference No. 16.

CHAPTER 6

EXPERIMENTS WITH SAND

The Sand Table. The classical theories of earth pressure are based on the assumptions that earth is granular and non-cohesive. Actual earth seldom displays these characteristics, but dry sand approximates to the theoretical requirements if dilatancy and

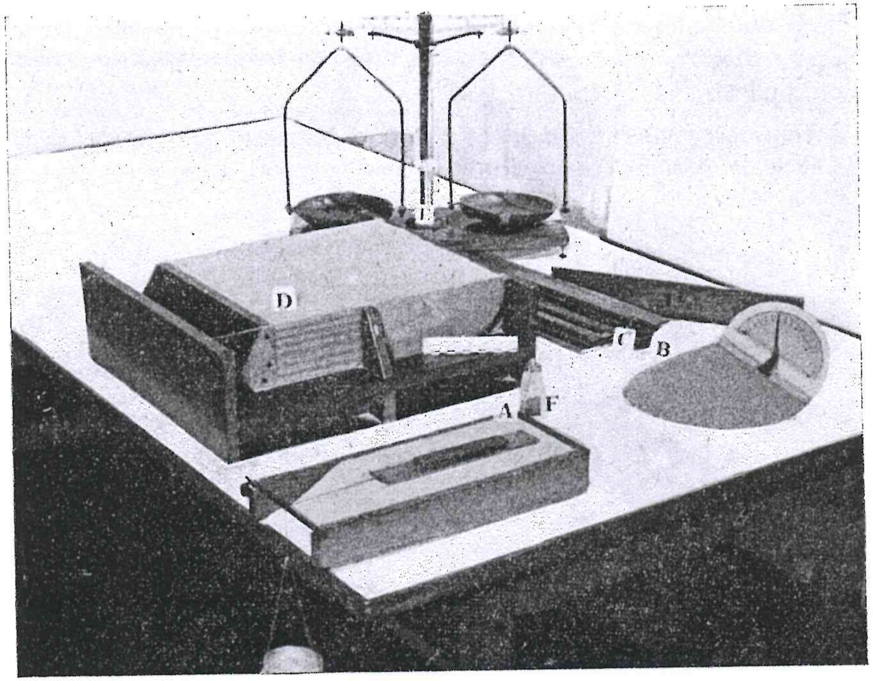


Fig. 6.1

arching are excluded and may be used for the experimental study of behaviour. Professor C. F. Jenkin[17] in the course of a research upon earth pressures developed simple apparatus which is described in the present chapter ; it is easy to make and is instructive.

Both Rankine's and the wedge theory of earth pressure applied to the design of retaining walls require knowledge of the coefficient of internal friction of the material and the latter also includes a term for the coefficient of friction of the material against the surface of the wall. The experimental determination of these coefficients forms a useful exercise for the student in addition to the actual calculation of pressure on the back of a wall.

The experiments are best carried out on a special table provided with a ledge on all sides to prevent the sand being spilt on the floor. This ledge is finished on the inner side with a leather " coving " so that the table can be swept clean. A hole in the middle of the table is closed by a plug ; this can be withdrawn and the spilt sand swept through the hole into a receptacle below. These may seem small matters, but it is of some importance to keep the sand from being spread to other apparatus in the laboratory. A general view of the sand table with the different pieces of apparatus is shown in Fig. 6.1.

Determination of Angle of Friction between Wall and Sand. The wall in the experimental apparatus is made of wood and a

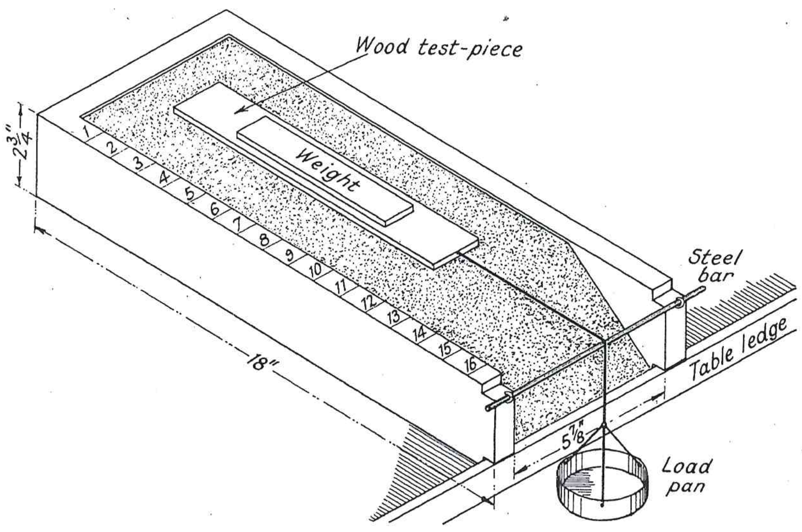


Fig. 6.2

similar piece of wood is provided for this determination. A flat sand surface is built up in a box as shown in Fig. 6.2, and the piece of wood, to which is attached a string and container passing over a metal bar, is laid on the top and then lightly loaded. Lead

shot is slowly poured into the container until slipping of the wood occurs ; the container and shot are then weighed.

If W is this weight and w is that of the wood and its superposed weight, the angle of friction $\psi = \tan^{-1} W/w$. This experiment should be repeated several times in order to obtain a good average value.

Determination of Coefficient of Internal Friction. This angle is measured directly from the angle of repose of the material by piling the sand in a heap until the critical slope is reached ; the angle is then measured by a simple inclinometer as shown in Fig. 6.3.

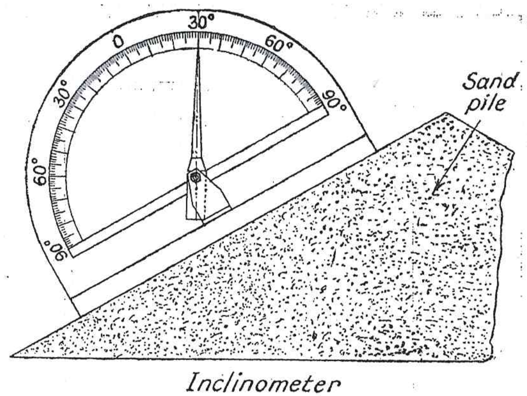


Fig. 6.3

Determination of Angle of Rupture. The model retaining wall is shown in Fig. 6.4 and consists of a box of which the front and back are made of ¾-in. wood and the two sides of ¼-in. plate glass. The " wall " is made of ¾-in. wood and is adjustable in its position by means of thumb-screws as shown. Clamps are provided to fix it firmly in any desired position. The box behind the wall is filled with sand in layers ½ in. thick. Each layer is levelled by wooden screeds which rest on the tops of the glass sides of the box. At every ½-in. level a thin layer of coloured sand is placed at the ends of the wall to form division lines 3 in. long between the layers. The box is filled to the top in this way and the wall is then very slowly moved away from the sand by means of the adjusting nuts until the plane of rupture is formed. This is very clearly traced out by the breaks in the coloured lines as shown in Fig. 6.5. The cross-section of the wedge producing pressure on the wall is thus determined and its weight is found from the measured density of the sand. This

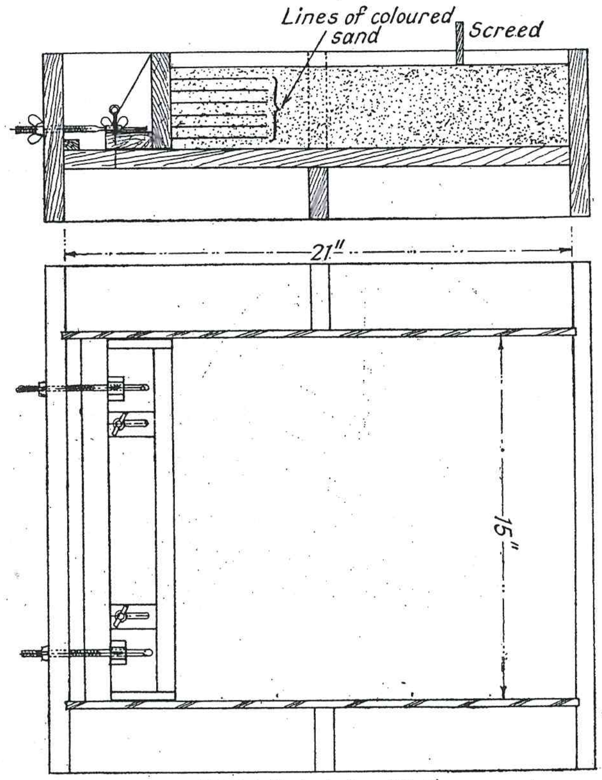


Fig. 6.4

can also be determined theoretically, graphically or analytically, by the wedge theory and the results compared.

Experiment Sheet No. 21 gives actual results obtained in this way.

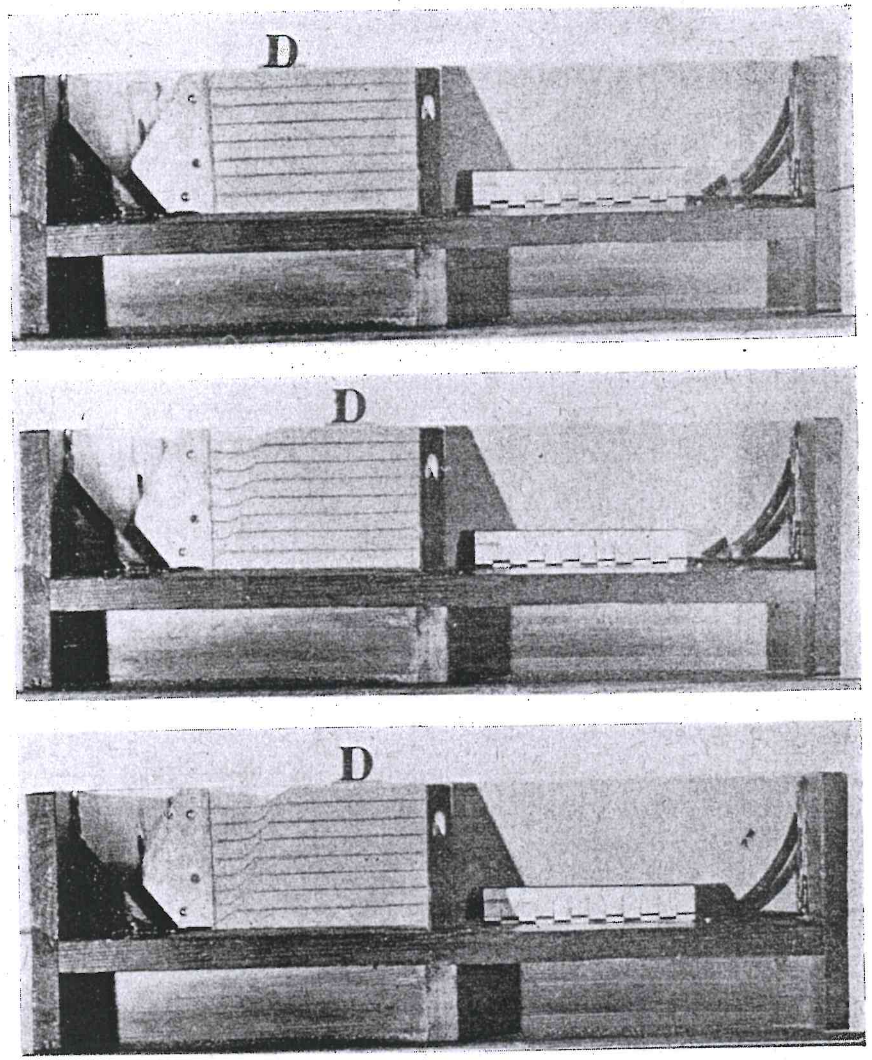


Fig. 6.5

Experiment Sheet No. 21

PRESSURE ON A RETAINING WALL

Instruction

Determine the coefficient of friction between sand and wood and the angle of repose of sand. Find the angle of rupture of the sand fill behind the model retaining wall and so calculate the actual pressure of the wedge on the wall. Using the data obtained for the angles of friction, calculate or determine graphically the theoretical pressure and compare with your experimental value.

Results

(*a*) *Coefficient of Friction.* Tests were made as described in this chapter, using the apparatus shown in Fig. 6.2. The average of five tests gave $\psi=39{\cdot}5°$.

(*b*) *Angle of Repose.* Five tests of the type shown in Fig. 6.3 gave the average value $\phi=34°$.

(*c*) *Angle of Rupture.* Five tests on the retaining wall gave an average angle of rupture of 25·0° to the vertical. The density of the sand was found to be 0·053 lb. per cu. in., and the pressure from the wedge was found to be 0·105 lb. per in. of wall.

(*d*) *Theoretical Value of Pressure.* Using the standard graphical construction given in any text-book for a wall 4 in. high the area of the pressure triangle was found to be 2·14 sq. in., and from this the pressure is $2{\cdot}14\times0{\cdot}053=0{\cdot}113$ lb. per in. of wall.

Notes

The methods are sufficiently clear from the text of this chapter. The graphical solution is well known and can be found in any standard text-book.

APPENDIX

MISCELLANEOUS EQUIPMENT

The efficient use of the model-structures apparatus described demands a certain amount of auxiliary equipment; the exact requirements will very soon become apparent, but a few of the more important items are listed here.

Lighting. The laboratory should have a good general lighting system but in addition a certain number of table lamps are necessary to give a local intensity of some brilliance. In working with celluloid models for example, the displacements are small even when the deformeter is not used and accurate readings are only obtainable in a good adjustable light. If the Beggs' method is used this is absolutely essential.

A useful lamp for this purpose is the Terry angle-poise, which can be seen in Fig. 1.2. This is easily placed in any desired position and has proved very satisfactory. Other adjustable lamps may be more readily available and can, of course, be used; the type is immaterial provided that they can be quickly arranged to throw light from the direction required.

Micrometer Microscopes. These are somewhat expensive items of equipment, but one or two at least are necessary. That shown in Fig. 3.3 was made by Messrs. R. & J. Beck, Ltd., London, and is arranged to cover a model about 12 in. diameter. The microscope is mounted on a rigid metal arm attached to a heavy base and a scale is contained in the eye-piece which is fitted with a traversing cross-wire controlled by a micrometer head working on a goniometer and reading to $\frac{1}{10,000}$ in., so that imposed displacements are read directly. The magnification is about 40 diameters.

Balance. An ordinary chemical balance is necessary for weighing shot for the linear-arch apparatus, for the voussoir arch experiment and for other purposes. Any good standard type will be satisfactory for this.

Celluloid. A supply of sheet celluloid is essential. This is supplied by the British Xylonite Company in sheets 55 in. × 24 in. and useful thicknesses are 0·080 in. and 0·180 in. It is sold by weight, the present cost being about 4*s*. 9*d*. per lb.

Fretsaw. A fretsaw is necessary for making the models. A hand-saw will suffice, but a small electrically-driven saw has been found to give the best results. The models should be finished with a fine file.

Scales, Squares, etc. Scales of various lengths and divisions are needed. It is clearly unimportant what type is used but a few metre rods are useful and also a number of 12-in. drawing scales. For "large displacement" experiments it has been found that $\frac{1}{48}$ in. is a good unit and this is obtained from a $\frac{1}{4}$ in.-to-the-foot scale divided into "inches." Boxwood is the most durable and satisfactory material; cardboard scales have a short life. An assortment of set squares of various sizes should also be provided; these are required in a number of experiments, *e.g.* for measuring displacements of celluloid models, for obtaining readings on the squared grid in the linear arch experiment and for other purposes. Work needing T-squares and drawing-boards is best done in the drawing office.

Drawing-boards, however, are required for mounting celluloid models and Imperial is a convenient size.

Wedges for Deformeter. It is advisable to supply small wedges for separating the two bars of deformeters in order that the plugs can be inserted easily. Duralumin is best for this purpose as it is softer than the steel of which the instrument is made and therefore does not damage it. It is easier to make new wedges than a new deformeter. One of these wedges can be seen in the photograph of Fig. 3.3.

Miscellaneous. Drawing pins, ordinary pins, needles, small round nails, small tack hammers, etc., are indispensable for celluloid model work and must be in plentiful supply.

A good supply of weights from 1 oz. to 2 lb. are also required for a variety of purposes.

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